Higher genus knot contact homology and recursion for the colored HOMFLY polynomial

Tobias Ekholm

Uppsala University and Institut Mittag-Leffler, Sweden

String Math, Hamburg, July 24–28, 2017

Plan

String Math 2019, July 01–05, Uppsala, Sweden
for info, see http://stringmath2019.uu.se

Plan of talk:

- Background
- Augmentation varieties and Gromov-Witten disk potentials
- Legendrian SFT and quantization of the augmentation variety
- Recursive construction of the wave function

Chern-Simons, large N, and topological string

- $K = K_1 \cup \cdots \cup K_k \subset S^3$ a k-component link.

- $K = K_1 \cup \cdots \cup K_k \subset S^3$ a k-component link.
- $L_K \subset T^*S^3$ the Lagrangian conormal of K:

$$L_K = \{(q,p) \in T^*S^3 \colon q \in K, p|_{TK} = 0\},$$

components of L_K are $\approx S^1 \times \mathbf{R}^2$.

- $K = K_1 \cup \cdots \cup K_k \subset S^3$ a k-component link.
- $L_K \subset T^*S^3$ the Lagrangian conormal of K:

$$L_K = \{(q,p) \in T^*S^3 \colon q \in K, p|_{TK} = 0\},$$

components of L_K are $\approx S^1 \times \mathbf{R}^2$.

- $\Lambda_K \subset ST^*S^3$ the Legendrian conormal of K:

$$ST^*S^3 = \{(q,p) \in T^*S^3 \colon |p| = 0\}, \quad \Lambda_K = L_K \cap ST^*S^3,$$

components of Λ_K are $\approx S^1 \times S^1$.

- Let $P_{m_1,\dots,m_k}(K)$ denote the (un-normalized) HOMFLY-polynomial of K with K_j colored by the m_j^{th} symmetric representation s_{m_j} (1-row Young diagram with m boxes).

$$P_{m_1,\dots,m_k}(K) = \int \mathcal{D}A\, e^{\frac{ik}{4\pi} CS(A)} \prod_{j=1}^{k} \mathrm{tr}_{s_{m_j}}(\mathrm{Hol}(K_j)),$$

where the path integral is over gauge orbits of $U(N)$-connections A.

- Let $P_{m_1,\dots,m_k}(K)$ denote the (un-normalized) HOMFLY-polynomial of K with K_j colored by the m_j^{th} symmetric representation s_{m_j} (1-row Young diagram with m boxes).

$$P_{m_1,\dots,m_k}(K) = \int \mathcal{D}A \, e^{\frac{ik}{4\pi} CS(A)} \prod_{j=1}^{k} \mathrm{tr}_{s_{m_j}}(\mathrm{Hol}(K_j)),$$

  where the path integral is over gauge orbits of $U(N)$-connections A.

- Define the HOMFLY wave function:

$$\Psi_K(q, Q, x) = \sum_{m_1,\dots,m_k} P_{m_1,\dots,m_k}(K) e^{-m_1 x_1} \dots e^{-m_k x_k},$$

  where $q = e^{\frac{2\pi i}{k+N}}$ and $Q = q^N$.

- The resolved conifold X is the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbf{C}P^1$.

- The resolved conifold X is the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbf{C}P^1$.
- Both X and T^*S^3 are resolutions of a quadratic cone in $\mathbf{C}^4$. Topologically the same outside 0-sections. Symplectically asymptotic at ∞.

- The resolved conifold X is the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbf{C}P^1$.
- Both X and T^*S^3 are resolutions of a quadratic cone in $\mathbf{C}^4$. Topologically the same outside 0-sections. Symplectically asymptotic at ∞.
- By a non-exact shift along a closed 1-form in a neighborhood of $K \subset S^3$, L_K can be shifted off the 0-section and becomes a Lagrangian in X asymtotic to $[T, \infty) \times \Lambda_K$ at ∞.

Chern-Simons, large N, and topological string

- Witten relates $U(N)$ Chern-Simons theory in S^3 to A-model open topological string in T^*S^3 with N branes on S^3.

Chern-Simons, large N, and topological string

- Witten relates $U(N)$ Chern-Simons theory in S^3 to A-model open topological string in T^*S^3 with N branes on S^3.

- Gopakumar-Vafa and Ooguri-Vafa relates A-model open topological string in T^*S^3 with N branes on S^3 to closed strings X, and open strings connecting these N branes to branes on L_K to strings in X with endpoint only on L_K.

- Combining these results gives

$$\Psi_K\left(x, Q, g_s = \frac{2\pi i}{k+N}\right) = Z_{GW}(X; L_K)$$
$$= \exp\left(\sum_{\chi,r,n} C_{\chi,r,l} g_s^{-\chi} Q^r e^{nx}\right),$$

where Z_{GW} is the Gromov-Witten partition function counting holomorphic curves in X with boundary on L_K,
$t = \log Q = N g_s$ is the area of $\mathbf{C}P^1$.

- Quantizing strings connecting L_K to itself in the same spirit upgrades generators (x_j, p_j) of $H_1(\Lambda_K)$ to operators, $p_j = g_s \frac{\partial}{\partial x_j}$. Acting on other string states:

$$p_j \Psi_K(x) = g_s \frac{\partial}{\partial x} \Psi_K.$$

- Quantizing strings connecting L_K to itself in the same spirit upgrades generators (x_j, p_j) of $H_1(\Lambda_K)$ to operators, $p_j = g_s \frac{\partial}{\partial x_j}$. Acting on other string states:

$$p_j \Psi_K(x) = g_s \frac{\partial}{\partial x} \Psi_K.$$

- Short wave asymptotics give

$$\Psi_K(x) = \exp\left(g_s^{-1} \int p dx + \dots\right) = \exp\left(g_s^{-1} W_K(x) \dots\right),$$

where $W_K(x)$ is the GW disk potential.

- From the recursion relation for Ψ_K we find that

$$p_j = \frac{\partial W_K}{\partial x_j},$$

is a Lagrangian variety V_K in $(\mathbf{C}^*)^{2k}$.

- From the recursion relation for Ψ_K we find that

$$p_j = \frac{\partial W_K}{\partial x_j},$$

is a Lagrangian variety V_K in $(\mathbf{C}^*)^{2k}$.

- The variety V_K is closely related to the augmentation variety of knot contact homology. The relation gives a way to calculate GW disk potentials via much easier curve counts at infinity. We will explain this next.

Knot contact homology

- Knot contact homology is a Floer-type theory (the Chekanov-Eliashberg algebra of Λ_K) associated to the contact action functional:

$$c\colon [0,1] \to ST^*S^3,\ c(0), c(1) \in \Lambda_K, \qquad c \mapsto \int_c pdq,$$

where pdq is the contact 1-form.

Knot contact homology

- Knot contact homology is a Floer-type theory (the Chekanov-Eliashberg algebra of Λ_K) associated to the contact action functional:

$$c\colon [0,1] \to ST^*S^3,\ c(0), c(1) \in \Lambda_K, \qquad c \mapsto \int_c pdq,$$

where pdq is the contact 1-form.

- Critical paths of positive action are *Reeb chords*, flow lines of R with $d(pdq)(R,\cdot) = 0$, $pdq(R) = 1$.

Knot contact homology

- Knot contact homology is a Floer-type theory (the Chekanov-Eliashberg algebra of Λ_K) associated to the contact action functional:

$$c\colon [0,1] \to ST^*S^3,\ c(0), c(1) \in \Lambda_K, \qquad c \mapsto \int_c pdq,$$

  where pdq is the contact 1-form.

- Critical paths of positive action are *Reeb chords*, flow lines of R with $d(pdq)(R, \cdot) = 0$, $pdq(R) = 1$.

- The knot contact homology algebra $\mathcal{A}_K$ is $CE(\Lambda_K)$, the free unital (non-commutative) graded algebra

$$\begin{aligned}\mathcal{A}_K &= \mathbb{C}[H_2(ST^*S^3, \Lambda_K)]\langle \text{ Reeb chords } \rangle \\ &= \mathbb{C}[e^{\pm x_j}, e^{\pm p_j}, Q^{\pm 1}]_{j=1}^{k}\langle \text{ Reeb chords } \rangle\end{aligned}$$

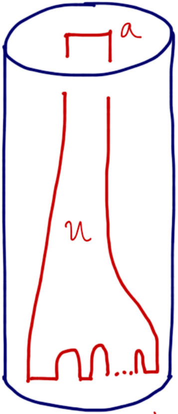
- The grading $|c|$ of a Reeb chord is defined by a Maslov index. For Λ_K, Reeb chords correspond to oriented binormal geodesics on K with grading equal to the Morse index (in an $\mathbf{R}^3$-patch, min $= 0$, sad $= 1$, max $= 2$).

Knot contact homology

- The grading $|c|$ of a Reeb chord is defined by a Maslov index. For Λ_K, Reeb chords correspond to oriented binormal geodesics on K with grading equal to the Morse index (in an $\mathbf{R}^3$-patch, min $= 0$, sad $= 1$, max $= 2$).

- The differential $d\colon \mathcal{A}_K \to \mathcal{A}_K$ is linear, satisfies Leibniz rule, and is defined on generators through a holomorphic curve count. The dg-algebra $(\mathcal{A}_K, d)$ is invariant under deformations up to homotopy and in particular up to quasi-isomorphism.

Knot contact homology



$b_1 \quad b_2 \ldots \, b_k \; ; \; \underline{b} = b_1 \ldots b_k$

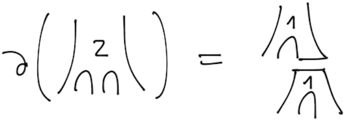
$u : (D, \partial D) \to (\mathbb{R} \times Y, \mathbb{R} \times \Lambda),$

$du + J \circ du \circ i = 0 .$

$$\partial a = \sum_{|a| - |\underline{b}| = 1} |\mathcal{M}_A(a; \underline{b})| \, e^A \, \underline{b}$$

Knot contact homology

$\underline{\partial^2 = 0:}$



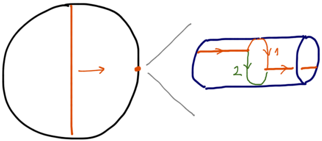
In particular,

Knot contact homology

- In general, the knot contact homology can be explicitly computed from a braid presentation of a link. For a braid on n strands the algebra has $n(n-1)$ generators in degree 0, $n(2n-1)$ in degree 1, and n^2 in degree 2.

Knot contact homology

- In general, the knot contact homology can be explicitly computed from a braid presentation of a link. For a braid on n strands the algebra has $n(n-1)$ generators in degree 0, $n(2n-1)$ in degree 1, and n^2 in degree 2.
- The unknot



$$\mathcal{A}_U = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle c, e\rangle, \quad |c| = 1, \ |e| = 2,$$
$$\partial e = c - c = 0, \quad \partial c = 1 - e^x - e^p + Qe^x e^p$$

- The trefoil T (differential in degree 1):

$$\mathcal{A}_T = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle a_{12}, a_{21}, b_{12}, b_{21}, c_{ij}, e_{ij}\rangle_{i,j\in\{1,2\}},$$
$$|a_{ij}| = 0,\ |b_{ij}| = |c_{ij}| = 1,\ |e_{ij}| = 2,$$

$$\begin{aligned}
\partial b_{12} &= e^{-x}a_{12} - a_{21},\\
\partial b_{21} &= e^{x}a_{21} - a_{12},\\
\partial c_{11} &= e^{p}e^{x} - e^{x} - (2Q - e^{p})a_{12} - Qa_{12}^2 a_{21},\\
\partial c_{12} &= Q - e^{p} + e^{p}a_{12} + Qa_{12}a_{21},\\
\partial c_{21} &= Q - e^{p} + e^{p}e^{x}a_{21} + Qa_{12}a_{21},\\
\partial c_{22} &= e^{p} - 1 - Qa_{21} + e^{p}a_{12}a_{21},
\end{aligned}$$

Augmentation variety

- Consider $\mathcal{A}_K$ as a family over $(\mathbf{C}^*)^{2k+1}$ of $\mathbf{C}$-algebras, where points in $(\mathbf{C}^*)^{2k+1}$ correspond to values of coefficients (e^{x_j}, e^{p_j}, Q).

Augmentation variety

- Consider $\mathcal{A}_K$ as a family over $(\mathbf{C}^*)^{2k+1}$ of $\mathbf{C}$-algebras, where points in $(\mathbf{C}^*)^{2k+1}$ correspond to values of coefficients (e^{x_j}, e^{p_j}, Q).

- An augmentation of $\mathcal{A}_K$ is a chain map

$$\epsilon\colon \mathcal{A}_K \to \mathbf{C}, \quad \epsilon \circ \partial = 0,$$

of unital dg-algebras ($\mathbf{C}$ lives in degree 0 and has trivial differential).

Augmentation variety

- Consider $\mathcal{A}_K$ as a family over $(\mathbf{C}^*)^{2k+1}$ of $\mathbf{C}$-algebras, where points in $(\mathbf{C}^*)^{2k+1}$ correspond to values of coefficients (e^{x_j}, e^{p_j}, Q).
- An augmentation of $\mathcal{A}_K$ is a chain map
$$\epsilon\colon \mathcal{A}_K \to \mathbf{C}, \quad \epsilon \circ \partial = 0,$$
of unital dg-algebras ($\mathbf{C}$ lives in degree 0 and has trivial differential).
- The augmentation variety V_K is the algebraic closure of
$$\left\{(e^{x_j}, e^{p_j}, Q) \in (\mathbf{C}^*)^{2k+1} \colon \mathcal{A}_K \text{ has augmentation}\right\}.$$

Augmentation variety

- For the unknot U:

$$A_U(e^x, e^p, Q) = 1 - e^x - e^p + Qe^x e^p.$$

Augmentation variety

- For the unknot U:

$$A_U(e^x, e^p, Q) = 1 - e^x - e^p + Qe^x e^p.$$

- For the trefoil T:

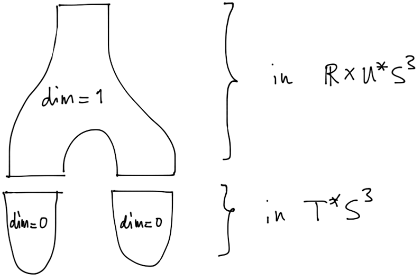
$$\begin{aligned} A_T(e^x, e^p, Q) &= Q^3 - Q^3 e^x - Q^2 e^p + Q^2 e^x e^p \\ &- 2Qe^x e^{2p} + 2Q^2 e^x e^{2p} + Qe^x e^{3p} \\ &- e^{2x} e^{3p} - Qe^x e^{4p} + e^{2x} e^{4p}. \end{aligned}$$

Augmentations and exact Lagrangian fillings

- Exact Lagrangian fillings L of Λ_K in T^*S^3 induces augmentations by

$$\epsilon_L(a) = \sum_{|a|=0} |\mathcal{M}_A(a)| e^A.$$

The map on coefficients are just the induced map on homology.



Augmentations and exact Lagrangian fillings

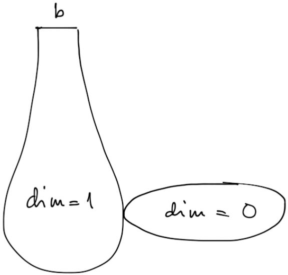
- There are two natural exact fillings of Λ_K in T^*S^3 $(Q = 1)$: L_K and $M_K \approx S^3 - K$. Thus, $e^p = 1$ and $e^x = 1$ belong to $V_K|_{Q=1}$ for any K.

Augmentations and exact Lagrangian fillings

- There are two natural exact fillings of Λ_K in T^*S^3 $(Q = 1)$: L_K and $M_K \approx S^3 - K$. Thus, $e^p = 1$ and $e^x = 1$ belong to $V_K|_{Q=1}$ for any K.

- For the unknot $A_U(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)$.

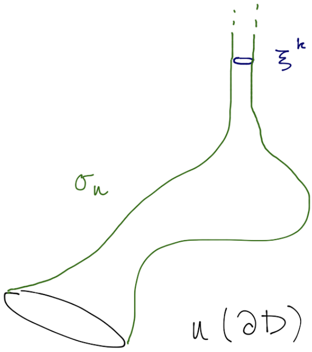
- In contrast to the exact case $L_K \subset X$ supports closed holomorphic disks and the previous definition of a chain map does not work because of new boundary phenomena.



Compare the family of real curves in $\mathbf{C}^2$, $xy = \epsilon$, $\epsilon \to 0$.

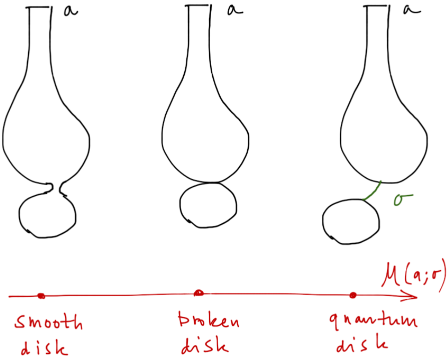
Augmentations and non-exact Lagrangian fillings

- We resolve this problem by using Fukaya-Oh-Ohta-Ono obstruction chains: fix a chain σ_D for each rigid disk D that connects its boundary in L_K to a multiple of a standard homology generator at infinity.



Augmentations and non-exact Lagrangian fillings

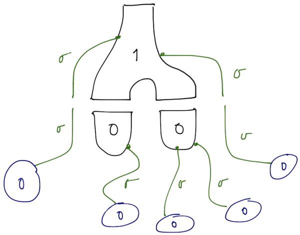
- We introduce quantum corrected holomorphic disks with punctures: these are ordinary holomorphic disks with all possible insertions of σ along the boundary. In the moduli space $\mathcal{M}_A(a;\sigma)$ of quantum corrected disks, boundary bubbling becomes interior points.



- Analyzing the boundary then shows that

$$\epsilon_L(a) = \sum_{|a|=1} \mathcal{M}_A(a;\sigma)e^A$$

is a chain map provided $p = \frac{\partial W_K}{\partial x}$. (This substitution counts quantum corrected disks at infinity.)

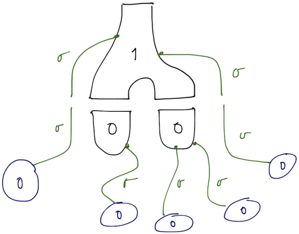


Augmentations and non-exact Lagrangian fillings

- Analyzing the boundary then shows that

$$\epsilon_L(a) = \sum_{|a|=1} \mathcal{M}_A(a;\sigma)e^A$$

is a chain map provided $p = \frac{\partial W_K}{\partial x}$. (This substitution counts quantum corrected disks at infinity.)



- We find that $p = \frac{\partial W_K}{\partial x}$ parameterizes a branch of the augmentation variety.

- We next consider the full quantization. This involves generalizing $\mathcal{A}_K$ to all genus and corresponds to quantizing V_K. We call this theory Legendrian SFT. It requires a framework involving bounding chains so that there is no boundary splittings in more complicated holomorphic curves with boundary. Before going into detail, the structure of the theory would then be the following:

Legendrian SFT

- We next consider the full quantization. This involves generalizing $\mathcal{A}_K$ to all genus and corresponds to quantizing V_K. We call this theory Legendrian SFT. It requires a framework involving bounding chains so that there is no boundary splittings in more complicated holomorphic curves with boundary. Before going into detail, the structure of the theory would then be the following:

- There is an SFT-potential $F = F(e^x, Q, g_s)$ that counts configurations of rigid holomorphic curves in X with boundary on L_K, bounding chains, and positive punctures. Note that curves contributing to F must have all positive punctures of degree 0. We have

$$F = F_0 + F_1 + F_2 + \dots,$$

where F_j counts curves with several positive punctures.

- There is similarly a Hamiltonian $H = H(e^x, e^p, Q, g_s)$ that counts rigid 1-parameter families of curves at infinity.

- There is similarly a Hamiltonian $H = H(e^x, e^p, Q, g_s)$ that counts rigid 1-parameter families of curves at infinity.
- The boundary of the 1-dimensional moduli spaces then gives the equation

$$e^{-F} H e^F = 0, \text{ or simply } He^F = 0.$$

Here we need only consider broken curves with one positive degree 1 chord and the rest degree 0.

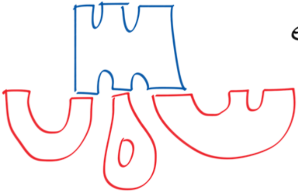
Legendrian SFT

H counts

in $(\mathbb{R}\times ST^*S^3, \mathbb{R}\times\Lambda_K)$

F counts

in (X, L)



$e^{-F} H e^{F} = 0$,

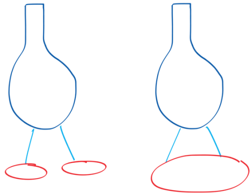
corresponds

to

∂(1-dim mfd)

Legendrian SFT

- Consider next the counterpart of the substitution $p = \frac{\partial W}{\partial x}$. When counting arbitrary curves we can make any insertions. A coefficient e^p in H then contributes
$$e^{-F} e^{g_s \frac{\partial}{\partial x}} e^F$$
which means we should set $p = g_s \frac{\partial}{\partial x}$ in H.



$\frac{1}{2} g_s^2 \left(\frac{\partial F}{\partial x}\right)^2$, $\frac{1}{2} g_s^2 \left(\frac{\partial^2 F}{\partial x^2}\right)$, etc

$e^{-F} e^{g_s \partial/\partial x} e^F$

counts all insertions.

- Note that $\Psi_K(x) = e^{F_0}$. Thus using elimination theory in the non-commutative setting $e^p e^x = e^{g_s} e^x e^p$ we should find an operator equation

$$\hat{A}_K(e^x, e^p, Q)\Psi_K(x) = 0,$$

which gives the recursion for the colored HOMFLY.

Legendrian SFT

- We sketch how to define the Legendrian SFT in a way that should lead to a calculation of H.

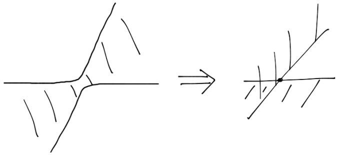
Legendrian SFT

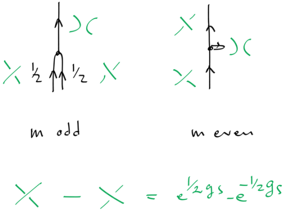
- We sketch how to define the Legendrian SFT in a way that should lead to a calculation of H.
- Let the degree 1 Reeb chords be denoted $b_1, \dots b_m$ and the degree 0 Reeb chords $a_1, \dots, a_n$

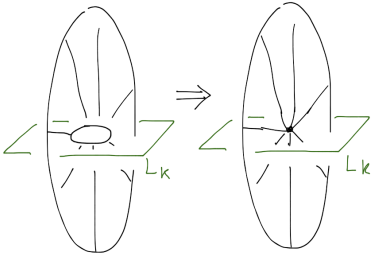
Legendrian SFT

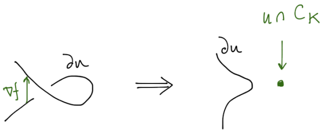
- We sketch how to define the Legendrian SFT in a way that should lead to a calculation of H.
- Let the degree 1 Reeb chords be denoted $b_1, \ldots b_m$ and the degree 0 Reeb chords $a_1, \ldots, a_n$
- Additional data: a Morse function f on L_K which gives obstruction chains. A 4-chain C_K for L_K with $\partial C_K = 2[L_K]$ which looks like $\pm J \nabla f$ near the boundary.

Legendrian SFT



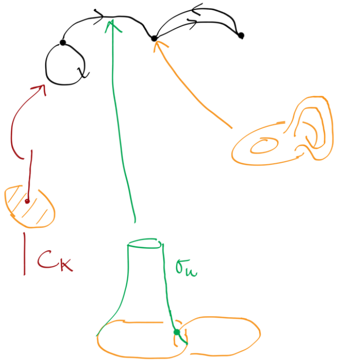






- We define the GW-potential e^F as the generating function of oriented graphs with holomorphic curves at vertices, and intersections with chains at the edges weighted by $\pm\frac{1}{2}$:

$$F = \sum C_{\chi,m,l}\, g_s^{-\chi} e^{mx} a_{i_1} \dots a_{i_r}$$



- Let $H(b_j)$ denote the count of rigid holomorphic curves in the symplectization with a positive puncture at b_j:

$$H(b_j) = \sum C_{\chi,m,n,I,J}\, g_s^{-\chi} e^{mx} e^{np}\, a_{i_1} \ldots a_{i_r} g_s^l \frac{\partial}{\partial a_{j_1}} \cdots \frac{\partial}{\partial a_{j_l}}.$$

- Then if $p = g_s \frac{\partial}{\partial x}$, $e^{-F} H(b_j) e^F$ counts ends of a 1-dimensional moduli space and in particular:

$$H(b_j)e^F = 0$$

as desired.

Legendrian SFT

- For the unknot there are no (formal) higher genus curves and the operator equation is

$$\widehat{A}_U(e^x, e^p, Q) = (1 - e^x - e^p - Qe^x e^p)\Psi_U = 0.$$

- For the unknot there are no (formal) higher genus curves and the operator equation is

$$\widehat{A}_U(e^x, e^p, Q) = (1 - e^x - e^p - Qe^x e^p)\Psi_U = 0.$$

- For the Hopf link L Reeb chord generators are as for the trefoil. The relevant parts for the operator H is as follows:

$$\begin{aligned}
H(c_{11}) &= (1 - e^{x_1} - e^{p_1} + Qe^{x_1}e^{p_1}) + g_s^2 \partial_{a_{12}}\partial_{a_{21}} + \mathcal{O}(a),\\
H(c_{22}) &= (1 - e^{x_2} - e^{p_2} + Qe^{x_2}e^{p_2}) + Qe^{x_2}e^{p_2} g_s^2 \partial_{a_{12}}\partial_{a_{21}} + \mathcal{O}(a),\\
H(c_{12}) &= (e^{p_2}e^{-p_1} - Qe^{x_2}e^{p_2}) g_s \partial_{a_{12}}\\
&\qquad + g_s^{-1}(e^{-g_s} - 1)(1 - e^{x_2}) a_{21} + \mathcal{O}(a^2),\\
H(c_{21}) &= (Qe^{x_1}e^{p_1} - e^{g_s}e^{p_1}e^{-p_2}) g_s \partial_{a_{21}}\\
&\qquad + g_s^{-1}\left((e^{g_s}(e^{g_s} - 1) - e^{2g_s}(e^{g_s} - 1)e^{x_1})e^{p_1}e^{-p_2} \right.\\
&\qquad\qquad \left. + (e^{g_s} - 1) Qe^{x_1}e^{p_1} g_s^2 \partial_{a_{12}}\partial_{a_{21}} \right) a_{12} + \mathcal{O}(a^2).
\end{aligned}$$

Legendrian SFT

- After the change of variables,

$$\begin{aligned} e^{x_1'} &= e^{g_s} e^{x_1}, \quad e^{p_1'} = e^{g_s} e^{p_1}; \\ e^{x_2'} &= Q^{-1} e^{-x_2}, \quad e^{p_2'} = e^{-g_s} Q^{-1} e^{-p_2}; \\ Q' &= e^{g_s} Q, \quad g_s' = -g_s, \end{aligned}$$

we find the D-module ideal generators

$$\begin{aligned} \widehat{A}_L^1 &= (e^{x_1} - e^{x_2}) + (e^{p_1} - e^{p_2}) - Q(e^{x_1} e^{p_1} - e^{x_2} e^{p_2}) \\ \widehat{A}_L^2 &= (1 - e^{-g_s} e^{x_1} - e^{p_1} + Q e^{x_1} e^{p_1})(e^{x_1} - e^{p_2}) \\ \widehat{A}_L^3 &= (1 - e^{-g_s} e^{x_2} - e^{p_2} + Q e^{x_2} e^{p_2})(e^{x_2} - e^{p_1}), \end{aligned}$$

in agreement with HOMFLY.

Legendrian SFT

- Similarly, for the trefoil we get the D-module ideal generator

$$
\begin{aligned}
\widehat{A}_T = {} & e^{5g_s} Q^3 (Q - e^{2g_s} e^p)(Q - e^{g_s e^{2p}}) \\
& + \big(e^{3g_s}(Q - e^{g_s} e^{2p})(Q - e^{2g_s} e^{2p})(Q - e^{3g_s} e^{2p}) \\
& \quad + e^{3g_s} Q e^{2p} (Q - e^{3g_s} e^{2p})(Q - e^{g_s} e^p) \\
& \quad - e^{3g_s} Q^2 \mu (1 - e^{g_s} e^p)(Q - e^{g_s} e^{2p})\big) e^x \\
& - e^{3p}(1 - e^p)(Q - e^{3g_s} e^{2p}) e^{2x}.
\end{aligned}
$$

in agreement with recursion for colored HOMFLY.

Recursive calculation of the wave function

- We next turn to finding the wave function $\Psi_K(x)$ recursively.

- We next turn to finding the wave function $\Psi_K(x)$ recursively.
- Note first that $W_K(x)$ is given by solving an algebraic equation and hence an analytic function.

Recursive calculation of the wave function

- Using this and the curve counting isomorphism map

$$CH^{\mathrm{lin}}(\Lambda_K) \oplus C_*(K) \;\to\; \mathrm{Cone}(C_*(\Omega(K,K),K) \to C_*(K))$$

we find that for generic points in V_K

$$\mathrm{rank}(CH_0^{\mathrm{lin}}) = 0, \quad \mathrm{rank}(CH_1^{\mathrm{lin}}) = 1, \quad \mathrm{rank}(CH_2^{\mathrm{lin}}) = 1$$

Recursive calculation of the wave function

- Using this and the curve counting isomorphism map

$$CH^{\mathrm{lin}}(\Lambda_K) \oplus C_*(K) \;\to\; \mathrm{Cone}(C_*(\Omega(K,K),K) \to C_*(K))$$

  we find that for generic points in V_K

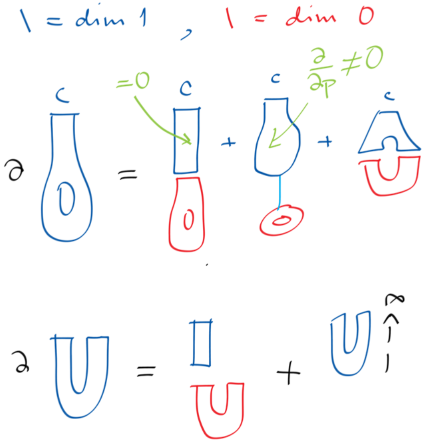
$$\mathrm{rank}(CH^{\mathrm{lin}}_0) = 0, \quad \mathrm{rank}(CH^{\mathrm{lin}}_1) = 1, \quad \mathrm{rank}(CH^{\mathrm{lin}}_2) = 1$$

- Furthermore, if c generates CH^{lin}_1 then the count of disks at infinity with positive puncture c passing through the reference curve ξ is generically non-zero.

Recursive calculation of the wave function

We illustrate the principle of the recursion in the first step, for the annulus.

c generates CH_1^{lin}



Recursive calculation of the wave function

- This generalizes to and A-model topological recursion for all genera. At infinity there are only disks, all higher genus curves are formal and can be computed via these disks and linking numbers. (Note that the first step gives the annulus amplitude needed for usual B-model topological recursion on the spectral curve V_K.)

Recursive calculation of the wave function

- This generalizes to and A-model topological recursion for all genera. At infinity there are only disks, all higher genus curves are formal and can be computed via these disks and linking numbers. (Note that the first step gives the annulus amplitude needed for usual B-model topological recursion on the spectral curve V_K.)

- We consider curves of index 0 and 1. A curve has type (n, χ) if it has n positive degree 0 punctures and Euler characteristic χ. An index 0 curve attached to an index 1 curve has attached type (n_0, n_1, χ) if it is attached via n_0 positive punctures and chain insertions and has n_1 free positive degree 0 punctures and Euler characteristic χ.

- Assume inductively we know the counts of index 0 curves of type (n, χ) for $-\chi + n < r$. Pick a generator b of CH_1^{lin} and consider the boundary of index 1 curves of type $(0, r)$ with positive puncture at b.

Recursive calculation of the wave function

- Assume inductively we know the counts of index 0 curves of type (n, χ) for $-\chi + n < r$. Pick a generator b of CH_1^{lin} and consider the boundary of index 1 curves of type $(0, r)$ with positive puncture at b.

- The broken curves in the boundary with attached curve of type $(1, 0, r)$ are all attached at an insertion (the ones attached at a chord do not contribute since b is a cycle in CH^{lin}). The contribution is

$$B(e^x, Q) \cdot F_0^r, \quad B \neq 0.$$

By the inductive assumption we can then solve for F_0^r in terms of earlier curves and curves at infinity.

Recursive calculation of the wave function

- For curves of type $(j, r-j)$, $j > 0$ take a positive puncture at a_j and pick a primitive b_j of a_j in the linearized complex, study the boundary of index 1 curves of type $(j-1, r-j)$ with positive puncture at b_j to see that we can express it in terms of less complex curves.