Higher genus knot contact homology and recursion for the colored HOMFLY polynomial

Tobias Ekholm

Uppsala University and Institut Mittag-Leffler, Sweden

String Math, Hamburg, July 24–28, 2017
Plan of talk:

- Background
- Augmentation varieties and Gromov-Witten disk potentials
- Legendrian SFT and quantization of the augmentation variety
- Recursive construction of the wave function
$K = K_1 \cup \cdots \cup K_k \subset S^3$ a $k$-component link.
\[ K = K_1 \cup \cdots \cup K_k \subset S^3 \text{ a } k\text{-component link.} \]

\[ L_K \subset T^*S^3 \text{ the Lagrangian conormal of } K: \]
\[ L_K = \{(q, p) \in T^*S^3 : q \in K, p|_{TK} = 0\}, \]
components of \( L_K \) are \( \approx S^1 \times \mathbb{R}^2 \).
• $K = K_1 \cup \cdots \cup K_k \subset S^3$ a $k$-component link.

• $L_K \subset T^* S^3$ the Lagrangian conormal of $K$:

$$L_K = \{(q, p) \in T^* S^3 : q \in K, p|_{TK} = 0\},$$

components of $L_K$ are $\approx S^1 \times \mathbb{R}^2$.

• $\Lambda_K \subset ST^* S^3$ the Legendrian conormal of $K$:

$$ST^* S^3 = \{(q, p) \in T^* S^3 : |p| = 0\}, \quad \Lambda_K = L_K \cap ST^* S^3,$$

components of $\Lambda_K$ are $\approx S^1 \times S^1$. 
Let $P_{m_1,\ldots,m_k}(K)$ denote the (un-normalized) HOMFLY-polynomial of $K$ with $K_j$ colored by the $m_j^{th}$ symmetric representation $s_{m_j}$ (1-row Young diagram with $m$ boxes).

$$P_{m_1,\ldots,m_k}(K) = \int DA \ e^{\frac{ik}{4\pi}CS(A)} \prod_{j=1}^{k} \text{tr}_{s_{m_j}}(\text{Hol}(K_j)),$$

where the path integral is over gauge orbits of $U(N)$-connections $A$. 
Let $P_{m_1,\ldots,m_k}(K)$ denote the (un-normalized) HOMFLY-polynomial of $K$ with $K_j$ colored by the $m_j^{th}$ symmetric representation $s_{m_j}$ (1-row Young diagram with $m$ boxes).

$$P_{m_1,\ldots,m_k}(K) = \int DA e^{ik\frac{j}{4\pi}CS(A)} \prod_{j=1}^k \text{tr}_{s_{m_j}}(\text{Hol}(K_j)),$$

where the path integral is over gauge orbits of $U(N)$-connections $A$.

Define the HOMFLY wave function:

$$\Psi_K(q, Q, x) = \sum_{m_1,\ldots,m_k} P_{m_1,\ldots,m_k}(K)e^{-m_1x_1} \ldots e^{-m_kx_k},$$

where $q = e^{\frac{2\pi i}{k+N}}$ and $Q = q^N$. 

Chern-Simons, large N, and topological string
The resolved conifold $X$ is the total space of $\mathcal{O}(-1)^{\otimes 2} \to \mathbb{C}P^1$. 
The resolved conifold $X$ is the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{C}P^1$.

Both $X$ and $T^*S^3$ are resolutions of a quadratic cone in $\mathbb{C}^4$. Topologically the same outside 0-sections. Symplectically asymptotic at $\infty$. 
The resolved conifold $X$ is the total space of $\mathcal{O}(-1)^{\oplus 2} \to \mathbb{C}P^1$.

Both $X$ and $T^* S^3$ are resolutions of a quadratic cone in $\mathbb{C}^4$. Topologically the same outside 0-sections. Symplectically asymptotic at $\infty$.

By a non-exact shift along a closed 1-form in a neighborhood of $K \subset S^3$, $L_K$ can be shifted off the 0-section and becomes a Lagrangian in $X$ asymptotic to $[T, \infty) \times \Lambda_K$ at $\infty$. 
Witten relates $U(N)$ Chern-Simons theory in $S^3$ to A-model open topological string in $T^*S^3$ with $N$ branes on $S^3$. 

Gopakumar-Vafa and Ooguri-Vafa relates A-model open topological string in $T^*S^3$ with $N$ branes on $S^3$ to closed strings $X$, and open strings connecting these $N$ branes to branes on $L_K$ to strings in $X$ with endpoint only on $L_K$. 
Witten relates $U(N)$ Chern-Simons theory in $S^3$ to A-model open topological string in $T^*S^3$ with $N$ branes on $S^3$.

Gopakumar-Vafa and Ooguri-Vafa relates A-model open topological string in $T^*S^3$ with $N$ branes on $S^3$ to closed strings $X$, and open strings connecting these $N$ branes to branes on $L_K$ to strings in $X$ with endpoint only on $L_K$. 
Combining these results gives

\[ \psi_K \left( x, Q, g_s = \frac{2\pi i}{k + N} \right) = Z_{GW}(X; L_K) \]

\[ = \exp \left( \sum_{\chi, r, n} C_{\chi, r, l} g_s^{-\chi} Q^r e^{nx} \right), \]

where \( Z_{GW} \) is the Gromov-Witten partition function counting holomorphic curves in \( X \) with boundary on \( L_K \),
\( t = \log Q = Ng_s \) is the area of \( \mathbb{C}P^1 \).
Quantizing strings connecting $L_K$ to itself in the same spirit upgrades generators $(x_j, p_j)$ of $H_1(\Lambda_K)$ to operators, $p_j = g_s \frac{\partial}{\partial x_j}$. Acting on other string states:

$$p_j \psi_K(x) = g_s \frac{\partial}{\partial x} \psi_K.$$
• Quantizing strings connecting $L_K$ to itself in the same spirit upgrades generators $(x_j, p_j)$ of $H_1(\Lambda_K)$ to operators, $p_j = g_s \frac{\partial}{\partial x_j}$. Acting on other string states:

$$p_j \psi_K(x) = g_s \frac{\partial}{\partial x} \psi_K.$$ 

• Short wave asymptotics give

$$\psi_K(x) = \exp \left( g_s^{-1} \int p dx + \ldots \right) = \exp \left( g_s^{-1} W_K(x) \ldots \right),$$

where $W_K(x)$ is the GW disk potential.
From the recursion relation for $\Psi_K$ we find that

$$p_j = \frac{\partial W_K}{\partial x_j},$$

is a Lagrangian variety $V_K$ in $(\mathbb{C}^*)^{2k}$. 


From the recursion relation for $\Psi_K$ we find that

$$p_j = \frac{\partial W_K}{\partial x_j},$$

is a Lagrangian variety $V_K$ in $(\mathbb{C}^*)^{2k}$.

The variety $V_K$ is closely related to the augmentation variety of knot contact homology. The relation gives a way to calculate GW disk potentials via much easier curve counts at infinity. We will explain this next.
Knot contact homology is a Floer-type theory (the Chekanov-Eliashberg algebra of $\Lambda_K$) associated to the contact action functional:

$$c: [0, 1] \rightarrow ST^*S^3, \; c(0), c(1) \in \Lambda_K, \quad c \mapsto \int_c pdq,$$

where $pdq$ is the contact 1-form.
Knot contact homology is a Floer-type theory (the Chekanov-Eliashberg algebra of $\Lambda_K$) associated to the contact action functional:

$$c: [0, 1] \to ST^* S^3, \ c(0), c(1) \in \Lambda_K, \quad c \mapsto \int_c pdq,$$

where $pdq$ is the contact 1-form.

Critical paths of positive action are *Reeb chords*, flow lines of $R$ with $d(pdq)(R, \cdot) = 0$, $pdq(R) = 1$. 
Knot contact homology is a Floer-type theory (the Chekanov-Eliashberg algebra of $\Lambda_K$) associated to the contact action functional:

$$c: [0, 1] \rightarrow ST^* S^3, \quad c(0), c(1) \in \Lambda_K, \quad c \mapsto \int_c pdq,$$

where $pdq$ is the contact 1-form.

Critical paths of positive action are Reeb chords, flow lines of $R$ with $d(pdq)(R, \cdot) = 0, \quad pdq(R) = 1$.

The knot contact homology algebra $A_K$ is $CE(\Lambda_K)$, the free unital (non-commutative) graded algebra

$$A_K = \mathbb{C}[H_2(ST^* S^3, \Lambda_K)]\langle \text{Reeb chords} \rangle$$

$$= \mathbb{C}[e^{\pm x_j}, e^{\pm p_j}, Q^{\pm 1}]_{j=1}^k \langle \text{Reeb chords} \rangle$$
The grading \(|c|\) of a Reeb chord is defined by a Maslov index. For \(\Lambda_K\), Reeb chords correspond to oriented binormal geodesics on \(K\) with grading equal to the Morse index (in an \(\mathbb{R}^3\)-patch, \(\text{min} = 0\), \(\text{sad} = 1\), \(\text{max} = 2\)).
The grading $|c|$ of a Reeb chord is defined by a Maslov index. For $\Lambda_K$, Reeb chords correspond to oriented binormal geodesics on $K$ with grading equal to the Morse index (in an $\mathbb{R}^3$-patch, min = 0, sad = 1, max = 2).

The differential $d : \mathcal{A}_K \rightarrow \mathcal{A}_K$ is linear, satisfies Leibniz rule, and is defined on generators through a holomorphic curve count. The dg-algebra $(\mathcal{A}_K, d)$ is invariant under deformations up to homotopy and in particular up to quasi-isomorphism.
Knot contact homology

\[\nu: (D, \partial D) \to (\mathbb{R} \times \mathbb{Y}, \mathbb{R} \times \Lambda),\]

\[dn + J \cdot dn \cdot i = 0.\]

\[\partial a = \sum \left| M_A(a; b) \right| e^A b \]

\[|a| - |b| = 1\]

\[b_1 \ b_2 \ldots \ b_K ; \quad b = b_1 \ldots b_K\]
\[ z^2 = 0 \]

\[ \partial \left( \left\langle \text{untwist} \right\rangle \right) = \left\langle \text{untwist} \right\rangle \]

In particular,

\[ \left\langle \frac{1}{1} \right\rangle = \left\langle \frac{1}{1} \right\rangle \]
In general, the knot contact homology can be explicitly computed from a braid presentation of a link. For a braid on $n$ strands the algebra has $n(n-1)$ generators in degree 0, $n(2n-1)$ in degree 1, and $n^2$ in degree 2.
In general, the knot contact homology can be explicitly computed from a braid presentation of a link. For a braid on $n$ strands the algebra has $n(n-1)$ generators in degree 0, $n(2n-1)$ in degree 1, and $n^2$ in degree 2.

The unknot

\[ \mathcal{A}_U = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^\pm 1] \langle c, e \rangle, \quad |c| = 1, \quad |e| = 2, \]
\[ \partial e = c - c = 0, \quad \partial c = 1 - e^x - e^p + Qe^x e^p \]
The trefoil $T$ (differential in degree 1):

$$A_T = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}]\langle a_{12}, a_{21}, b_{12}, b_{21}, c_{ij}, e_{ij}\rangle_{i,j\in\{1,2\}},$$

$$|a_{ij}| = 0, \quad |b_{ij}| = |c_{ij}| = 1, \quad |e_{ij}| = 2,$$

$$\partial b_{12} = e^{-x}a_{12} - a_{21},$$
$$\partial b_{21} = e^{x}a_{21} - a_{12},$$
$$\partial c_{11} = e^{p}e^{x} - e^{x} - (2Q - e^{p})a_{12} - Qa_{12}^{2}a_{21},$$
$$\partial c_{12} = Q - e^{p} + e^{p}a_{12} + Qa_{12}a_{21},$$
$$\partial c_{21} = Q - e^{p} + e^{p}e^{x}a_{21} + Qa_{12}a_{21},$$
$$\partial c_{22} = e^{p} - 1 - Qa_{21} + e^{p}a_{12}a_{21},$$
Consider $\mathcal{A}_K$ as a family over $(\mathbb{C}^*)^{2k+1}$ of $\mathbb{C}$-algebras, where points in $(\mathbb{C}^*)^{2k+1}$ correspond to values of coefficients $(e^{x_j}, e^{p_j}, Q)$. 

An augmentation of $\mathcal{A}_K$ is a chain map $\epsilon: \mathcal{A}_K \to \mathbb{C}$, $\epsilon \circ \partial = 0$, of unital dg-algebras ($\mathbb{C}$ lives in degree 0 and has trivial differential).

The augmentation variety $V_K$ is the algebraic closure of $\{ (e^{x_j}, e^{p_j}, Q) \in (\mathbb{C}^*)^{2k+1} : \mathcal{A}_K \text{ has augmentation} \}$. 
Consider $A_K$ as a family over $(\mathbb{C}^*)^{2k+1}$ of $\mathbb{C}$-algebras, where points in $(\mathbb{C}^*)^{2k+1}$ correspond to values of coefficients $(e^{x_i}, e^{p_j}, Q)$.

An augmentation of $A_K$ is a chain map

$$\epsilon: A_K \to \mathbb{C}, \quad \epsilon \circ \partial = 0,$$

of unital dg-algebras ($\mathbb{C}$ lives in degree 0 and has trivial differential).
Consider $\mathcal{A}_K$ as a family over $(\mathbb{C}^*)^{2k+1}$ of $\mathbb{C}$-algebras, where points in $(\mathbb{C}^*)^{2k+1}$ correspond to values of coefficients $(e^{x_j}, e^{p_j}, Q)$.

An augmentation of $\mathcal{A}_K$ is a chain map

$$\epsilon: \mathcal{A}_K \to \mathbb{C}, \quad \epsilon \circ \partial = 0,$$

of unital dg-algebras ($\mathbb{C}$ lives in degree 0 and has trivial differential).

The augmentation variety $V_K$ is the algebraic closure of

$$\left\{ (e^{x_j}, e^{p_j}, Q) \in (\mathbb{C}^*)^{2k+1} : \mathcal{A}_K \text{ has augmentation} \right\}.$$
For the unknot $U$:

$$A_U(e^x, e^p, Q) = 1 - e^x - e^p + Qe^x e^p.$$
Augmentation variety

- For the unknot $U$:

\[ A_U(e^x, e^p, Q) = 1 - e^x - e^p + Qe^x e^p. \]

- For the trefoil $T$:

\[
A_T(e^x, e^p, Q) = Q^3 - Q^3 e^x - Q^2 e^p + Q^2 e^x e^p \\
- 2Qe^x e^{2p} + 2Q^2 e^x e^{2p} + Qe^x e^{3p} \\
- e^{2x} e^{3p} - Qe^x e^{4p} + e^{2x} e^{4p}.
\]
Exact Lagrangian fillings $L$ of $\Lambda_K$ in $T^*S^3$ induces augmentations by

$$\epsilon_L(a) = \sum_{|a|=0} |M_A(a)| e^A.$$

The map on coefficients are just the induced map on homology.
There are two natural exact fillings of $\Lambda_K$ in $T^* S^3 \ (Q = 1)$: $L_K$ and $M_K \approx S^3 - K$. Thus, $e^p = 1$ and $e^x = 1$ belong to $V_K|_{Q=1}$ for any $K$. 

For the unknot $A_U(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)$. 

 augmentation and exact Lagrangian fillings
There are two natural exact fillings of $\Lambda_K$ in $T^*S^3 (Q = 1)$: $L_K$ and $M_K \approx S^3 - K$. Thus, $e^p = 1$ and $e^x = 1$ belong to $V_K|_{Q=1}$ for any $K$.

For the unknot $A_U(e^x, e^p, Q = 1) = (1 - e^x)(1 - e^p)$. 
In contrast to the exact case $L_K \subset X$ supports closed holomorphic disks and the previous definition of a chain map does not work because of new boundary phenomena.

Compare the family of real curves in $\mathbb{C}^2$, $xy = \epsilon$, $\epsilon \to 0$. 
We resolve this problem by using Fukaya-Oh-Ohta-Ono obstruction chains: fix a chain $\sigma_D$ for each rigid disk $D$ that connects its boundary in $L_K$ to a multiple of a standard homology generator at infinity.
We introduce quantum corrected holomorphic disks with punctures: these are ordinary holomorphic disks with all possible insertions of $\sigma$ along the boundary. In the moduli space $\mathcal{M}_A(a; \sigma)$ of quantum corrected disks, boundary bubbling becomes interior points.
Analyzing the boundary then shows that

$$\epsilon_L(a) = \sum_{|a| = 1} M_A(a; \sigma) e^A$$

is a chain map provided $p = \frac{\partial W_K}{\partial x}$. (This substitution counts quantum corrected disks at infinity.)
Augmentations and non-exact Lagrangian fillings

- Analyzing the boundary then shows that

\[ \epsilon_L(a) = \sum_{|a|=1} M_A(a; \sigma)e^A \]

is a chain map provided \( p = \frac{\partial W_K}{\partial x} \). (This substitution counts quantum corrected disks at infinity.)

- We find that \( p = \frac{\partial W_K}{\partial x} \) parameterizes a branch of the augmentation variety.
We next consider the full quantization. This involves generalizing $A_K$ to all genus and corresponds to quantizing $V_K$. We call this theory Legendrian SFT. It requires a framework involving bounding chains so that there is no boundary splittings in more complicated holomorphic curves with boundary. Before going into detail, the structure of the theory would then be the following:
We next consider the full quantization. This involves generalizing $\mathcal{A}_K$ to all genus and corresponds to quantizing $V_K$. We call this theory Legendrian SFT. It requires a framework involving bounding chains so that there is no boundary splittings in more complicated holomorphic curves with boundary. Before going into detail, the structure of the theory would then be the following:

There is an SFT-potential $F = F(e^x, Q, g_s)$ that counts configurations of rigid holomorphic curves in $X$ with boundary on $L_K$, bounding chains, and positive punctures. Note that curves contributing to $F$ must have all positive punctures of degree 0. We have

$$F = F_0 + F_1 + F_2 + \ldots,$$

where $F_j$ counts curves with several positive punctures.
There is similarly a Hamiltonian $H = H(e^x, e^p, Q, g_s)$ that counts rigid 1-parameter families of curves at infinity.
There is similarly a Hamiltonian $H = H(e^x, e^p, Q, g_s)$ that counts rigid 1-parameter families of curves at infinity.

The boundary of the 1-dimensional moduli spaces then gives the equation

$$e^{-F} He^F = 0,$$

or simply

$$He^F = 0.$$

Here we need only consider broken curves with one positive degree 1 chord and the rest degree 0.
Legendrian SFT

\[ H \text{ counts} \]

\[ \text{in } (\mathbb{R} \times S^1, \mathbb{R} \times \Lambda K) \]

\[ F \text{ counts} \]

\[ \text{in } (X, L) \]

\[ \text{e}^{-F} H e^F = 0, \text{ corresponds to } \]  
\[ \varphi(1-\text{dim mfd}) \]
Consider next the counterpart of the substitution $p = \frac{\partial W}{\partial x}$. When counting arbitrary curves we can make any insertions. A coefficient $e^p$ in $H$ then contributes

$$e^{-F} e^{gs \frac{\partial}{\partial x} e^F}$$

which means we should set $p = gs \frac{\partial}{\partial x}$ in $H$. 

\[ \frac{1}{2} gs \left( \frac{\partial F}{\partial x} \right)^2, \quad \frac{1}{2} gs^2 \left( \frac{\partial^2 F}{\partial x^2} \right), \quad \text{etc.} \]

$$e^{-F} e^{gs \frac{\partial}{\partial x} e^F}$$

counts all insertions.
Note that $\Psi_K(x) = e^{F_0}$. Thus using elimination theory in the non-commutative setting $e^p e^x = e^{gs} e^x e^p$ we should find an operator equation

$$\hat{A}_K(e^x, e^p, Q)\Psi_K(x) = 0,$$

which gives the recursion for the colored HOMFLY.
We sketch how to define the Legendrian SFT in a way that should lead to a calculation of \( H \).
We sketch how to define the Legendrian SFT in a way that should lead to a calculation of $H$.

Let the degree 1 Reeb chords be denoted $b_1, \ldots, b_m$ and the degree 0 Reeb chords $a_1, \ldots, a_n$.
We sketch how to define the Legendrian SFT in a way that should lead to a calculation of $H$.

Let the degree 1 Reeb chords be denoted $b_1, \ldots b_m$ and the degree 0 Reeb chords $a_1, \ldots, a_n$

Additional data: a Morse function $f$ on $L_K$ which gives obstruction chains. A 4-chain $C_K$ for $L_K$ with $\partial C_K = 2[L_K]$ which looks like $\pm J\nabla f$ near the boundary.
Legendrian SFT

\[ \frac{1}{2} \times \frac{1}{2} \times \times \times \]

\[ m \text{ odd} \]

\[ \times - \times = e^{\frac{1}{2}g_s} - e^{-\frac{1}{2}g_s} \]

\[ u \cap C_k \]

\[ \varphi \rightarrow \partial \nu \rightarrow \partial \nu \rightarrow \{ \cdot \} \]
We define the GW-potential $e^F$ as the generating function of oriented graphs with holomorphic curves at vertices, and intersections with chains at the edges weighted by $\pm \frac{1}{2}$:

$$F = \sum C_{\chi, m, l} g_s^{-\chi} e^{m x} a_{i_1} \ldots a_{i_r}$$
Let $H(b_j)$ denote the count of rigid holomorphic curves in the symplectization with a positive puncture at $b_j$:

$$H(b_j) = \sum C_{\chi,m,n,l,J} g_s^{-\chi} e^{m \chi} e^{n \rho} a_{i_1} \ldots a_{i_r} g_s^l \frac{\partial}{\partial a_{j_1}} \ldots \frac{\partial}{\partial a_{j_l}}.$$

Then if $p = g_s \frac{\partial}{\partial x}$, $e^{-F} H(b_j) e^F$ counts ends of a 1-dimensional moduli space and in particular:

$$H(b_j) e^F = 0$$

as desired.
For the unknot there are no (formal) higher genus curves and the operator equation is

\[ \hat{A}_U(e^x, e^p, Q) = (1 - e^x - e^p - Qe^x e^p)\psi_U = 0. \]
Legendrian SFT

For the unknot there are no (formal) higher genus curves and the operator equation is

\[ \hat{A}_U(e^x, e^p, Q) = (1 - e^x - e^p - Qe^x e^p)\psi_U = 0. \]

For the Hopf link \( L \) Reeb chord generators are as for the trefoil. The relevant parts for the operator \( H \) is as follows:

\[
\begin{align*}
H(c_{11}) &= (1 - e^{x_1} - e^{p_1} + Qe^{x_1} e^{p_1}) + g_s^2 \partial_{a_{12}} \partial_{a_{21}} + O(a), \\
H(c_{22}) &= (1 - e^{x_2} - e^{p_2} + Qe^{x_2} e^{p_2}) + Qe^{x_2} e^{p_2} g_s^2 \partial_{a_{12}} \partial_{a_{21}} + O(a), \\
H(c_{12}) &= (e^{p_2} e^{-p_1} - Qe^{x_2} e^{p_2}) g_s \partial_{a_{12}} \\
&\quad + g_s^{-1}(e^{-g_s} - 1)(1 - e^{x_2})a_{21} + O(a^2), \\
H(c_{21}) &= (Qe^{x_1} e^{p_1} - e^{g_s} e^{p_1} e^{-p_2}) g_s \partial_{a_{21}} \\
&\quad + g_s^{-1} ((e^{g_s} (e^{g_s} - 1) - e^{2g_s} (e^{g_s} - 1) e^{x_1} )e^{p_1} e^{-p_2} \\
&\quad + (e^{g_s} - 1)Qe^{x_1} e^{p_1} g_s^2 \partial_{a_{12}} \partial_{a_{21}} ) a_{12} + O(a^2).
\end{align*}
\]
After the change of variables,

\[ e^{x_1'} = e^{g_s} e^{x_1}, \quad e^{p_1'} = e^{g_s} e^{p_1}; \]
\[ e^{x_2'} = Q^{-1} e^{-x_2}, \quad e^{p_2'} = e^{-g_s} Q^{-1} e^{-p_2}; \]
\[ Q' = e^{g_s} Q, \quad g_s' = -g_s, \]

we find the D-module ideal generators

\[ \hat{A}_L^1 = (e^{x_1} - e^{x_2}) + (e^{p_1} - e^{p_2}) - Q(e^{x_1} e^{p_1} - e^{x_2} e^{p_2}) \]
\[ \hat{A}_L^2 = (1 - e^{-g_s} e^{x_1} - e^{p_1} + Q e^{x_1} e^{p_1})(e^{x_1} - e^{p_2}) \]
\[ \hat{A}_L^3 = (1 - e^{-g_s} e^{x_2} - e^{p_2} + Q e^{x_2} e^{p_2})(e^{x_2} - e^{p_1}), \]

in agreement with HOMFLY.
Similarly, for the trefoil we get the D-module ideal generator

\[ \hat{A}_T = e^{5g_s} Q^3 (Q - e^{2g_s} e^p) (Q - e^{g_s} e^{2p}) \]
\[ + (e^{3g_s} (Q - e^{g_s} e^{2p}) (Q - e^{2g_s} e^{2p}) (Q - e^{3g_s} e^{2p}) \]
\[ + e^{3g_s} Q e^{2p} (Q - e^{3g_s} e^{2p}) (Q - e^{g_s} e^{p}) \]
\[ - e^{3g_s} Q^2 \mu (1 - e^{g_s} e^{p}) (Q - e^{g_s} e^{2p}) e^x \]
\[ - e^{3p} (1 - e^p) (Q - e^{3g_s} e^{2p}) e^{2x}. \]

in agreement with recursion for colored HOMFLY.
We next turn to finding the wave function $\Psi_K(x)$ recursively.
We next turn to finding the wave function $\Psi_K(x)$ recursively.

Note first that $W_K(x)$ is given by solving an algebraic equation and hence an analytic function.
Recursive calculation of the wave function

Using this and the curve counting isomorphism map

\[ CH_{\text{lin}}(\Lambda_K) \oplus C_*(K) \rightarrow \text{Cone}(C_*(\Omega(K, K), K) \rightarrow C_*(K)) \]

we find that for generic points in \( V_K \)

\[ \text{rank}(CH_{\text{lin}}^0) = 0, \quad \text{rank}(CH_{\text{lin}}^1) = 1, \quad \text{rank}(CH_{\text{lin}}^2) = 1 \]
Recursive calculation of the wave function

- Using this and the curve counting isomorphism map

\[ CH^\text{lin}(\Lambda_K) \oplus C_*(K) \to \text{Cone}(C_*(\Omega(K, K), K) \to C_*(K)) \]

we find that for generic points in \( V_K \)

\[ \text{rank}(CH^\text{lin}_0) = 0, \quad \text{rank}(CH^\text{lin}_1) = 1, \quad \text{rank}(CH^\text{lin}_2) = 1 \]

- Furthermore, if \( c \) generates \( CH^\text{lin}_1 \) then the count of disks at infinity with positive puncture \( c \) passing through the reference curve \( \xi \) is generically non-zero.
Recursive calculation of the wave function

We illustrate the principle of the recursion in the first step, for the annulus.

\[ c \text{ generates } CH_1 \]

\[ 1 = \text{dim } 1, \quad 1 = \text{dim } 0 \]

\[ \partial c = 0 \]

\[ \partial U = \]

\[ \frac{\partial}{\partial \varphi} \neq 0 \]
This generalizes to and A-model topological recursion for all genera. At infinity there are only disks, all higher genus curves are formal and can be computed via these disks and linking numbers. (Note that the first step gives the annulus amplitude needed for usual B-model topological recursion on the spectral curve $V_K$.)
This generalizes to and A-model topological recursion for all genera. At infinity there are only disks, all higher genus curves are formal and can be computed via these disks and linking numbers. (Note that the first step gives the annulus amplitude needed for usual B-model topological recursion on the spectral curve $V_K$.)

We consider curves of index 0 and 1. A curve has type $(n, \chi)$ if it has $n$ positive degree 0 punctures and Euler characteristic $\chi$. An index 0 curve attached to an index 1 curve has attached type $(n_0, n_1, \chi)$ if it is attached via $n_0$ positive punctures and chain insertions and has $n_1$ free positive degree 0 punctures and Euler characteristic $\chi$. 
Recursive calculation of the wave function

- Assume inductively we know the counts of index 0 curves of type $(n, \chi)$ for $-\chi + n < r$. Pick a generator $b$ of $CH_{1}^{\text{lin}}$ and consider the boundary of index 1 curves of type $(0, r)$ with positive puncture at $b$. 

The broken curves in the boundary with attached curve of type $(1, 0, r)$ are all attached at an insertion (the ones attached at a chord do not contribute since $b$ is a cycle in $CH_{1}^{\text{lin}}$). The contribution is $B(e_{x}, Q) \cdot F_{r, 0, B} \neq 0$.

By the inductive assumption we can then solve for $F_{r, 0}$ in terms of earlier curves and curves at infinity.
Assume inductively we know the counts of index 0 curves of type $(n, \chi)$ for $-\chi + n < r$. Pick a generator $b$ of $CH_{lin}^1$ and consider the boundary of index 1 curves of type $(0, r)$ with positive puncture at $b$.

The broken curves in the boundary with attached curve of type $(1, 0, r)$ are all attached at an insertion (the ones attached at a chord do not contribute since $b$ is a cycle in $CH_{lin}$). The contribution is

$$B(e^x, Q) \cdot F_0^r, \quad B \neq 0.$$  

By the inductive assumption we can then solve for $F_0^r$ in terms of earlier curves and curves at infinity.
Recursive calculation of the wave function

For curves of type \((j, r - j), j > 0\) take a positive puncture at \(a_j\) and pick a primitive \(b_j\) of \(a_j\) in the linearized complex, study the boundary of index 1 curves of type \((j - 1, r - j)\) with positive puncture at \(b_j\) to see that we can express it in terms of less complex curves.