Aspects of Calabi-Yau variety and SCFTs in various dimensions

Shing-Tung Yau

String Math 2017
University of Hamburg
24-28 July, 2017

0. Outline

- Differential equations of period integrals of algebraic manifolds, and some applications. These are related to the computation of observables of 2d $(2,2)$ SCFT if the manifolds are Calabi-Yau manifolds.
- Three dimensional canonical singularity and 4d $\mathcal{N}=2$, 5d $\mathcal{N}=1$ and 4d $\mathcal{N}=1$ superconformal field theories. Canonical singularity naturally appears in the degeneration limit of Calabi-Yau manifold.

1. **Differential equations of period integrals of algebraic manifolds, and some applications**

Based on joint works with
S. Bloch (Chicago & Tsinghua MSC)
A. Huang (Harvard)
B. Lian (Brandeis University)
V. Srinivas (Tata)

A study on the interplay between

SPECIAL FUNCTIONS $\leftrightarrow$ COMPLEX GEOMETRY

3. Introduction & History

Computing period of integrals is a very important component of computations in mirror symmetry as was pioneered by candelas et al: When they computed the partition functions for type B theory of the mirror of the CY quintic. The most important quantity is the following integral

$$\int_{\gamma} Res \frac{1}{f(\psi)} \sum_{i=0}^{4} (-1)^{i+1} x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_4,$$

of a holomorphic 3-form on the mirror quintic, where

$$Y_{\psi} : \quad f(\psi) := x_0^5 + \cdots + x_4^5 + 5\psi x_0 \cdots x_4 = 0, \ \psi \in B \subset \mathbb{C}$$

is the Dwork family of quintic hypersurfaces, and γ is a locally constant 3-cycle on the mirror quintic.

Candelas spent a great deal of effort to calculate the period for the one parameter family of CY threefolds. However, it is certainly desirable to understand the mirror theory when there is more than one parameter. This was developed by two groups of authors : Hosono-Klemm-lian-Yau, Candelas-Ossa-Font-Katz-Morrison.

4. Introduction & History

- The computation is very complicated when the number of parameters is getting bigger. But most important invariants of mirror geometry depends on the deep understanding of the partition functions and the mirror maps. They depend on the periods of the integral, where the major tool to understand them is through the Picard-Fuchs equations.

- The derivation of such equations becomes very nontrivial for general CY manifolds. The major part of this talk is devoted to studying such equations which we generalized to cover periods of algebraic manifolds beyond toric varieties. Historically, after the development by Euler, Gauss, Riemann, there were works of Picard, Fuchs and also GKZ (Gelfand-Kapranov-Zelevinsky). Since GKZ is not adequate to cover non-toric situation, Bong lian and I with An Huang and others started to develop the theory of tautological system. We shall discuss them here.

5. Euler-Gauss hypergeometric functions

The EG hypergeometric equation is the ODE defined on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$:

$$z(1-z)\frac{d^2}{dz^2} + [c - (a+b+1)z]\,\frac{d}{dz} - ab = 0$$

where $a, b, c \in \mathbb{C}$ are fixed parameters.

Every second-order linear ODE on $\mathbb{P}^1$ with three regular singular points can be transformed into this equation.

A EG hypergeometric function is a local solution to this equation. For $c \notin \mathbb{Z}_{\leq 0}$, around $z = 0$, it has a power series solution of the form

$${}_2F_1(a, b, c;\, z) := \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with radius of convergence 1. Here $(\alpha)_n = \prod_{k=0}^{n-1}(\alpha + k) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.

6. From complex geometry to EG functions

The Legendre family of elliptic curves:

$$Y_\lambda : \quad y^2 = x(x-1)(x-\lambda), \quad (x,y) \equiv [x,y,1] \in \mathbb{P}^2$$

parameterized by $\lambda \in B := \mathbb{C} - \{0,1\}$.

For $\lambda \in B$,

$$Y_\lambda \simeq^{homeo.} T^2.$$

For a given $\lambda_0 \in B$, we also have canonical identification

$$H^1(Y_\lambda, \mathbb{C}) \equiv H^1(Y_{\lambda_0}, \mathbb{C}) \equiv H^1(T, \mathbb{C}) \cong \mathbb{C}^2$$

if λ varies in any contractible neighborhood U of λ_0.

The 1-form

$$\omega_\lambda := \frac{dx}{y}$$

is holomorphic on Y_λ, so it is d-closed and defines a cohomology class on $[\omega_\lambda] \in H^1(T, \mathbb{C}) \equiv \mathbb{C}^2$. This vector varies holomorphically with $\lambda \in U$.

7. Period integrals

Fix a basis $\gamma_1, \gamma_2 \in H_1(T, \mathbb{Z}) = H^1(T, \mathbb{Z})^*$. Then

$$[\omega_\lambda] = \gamma_1^* \langle \gamma_1^*, \omega_\lambda \rangle + \gamma_2^* \langle \gamma_2^*, \omega_\lambda \rangle = \gamma_1^* \int_{\gamma_1} \omega_\lambda + \gamma_2^* \int_{\gamma_2} \omega_\lambda.$$

The coefficient functions $\int_{\gamma_i} \omega_\lambda \in \mathcal{O}_B(U)$ are called period integrals of the family Y_λ.

Remark: Even though they are defined locally, these period integrals admit (multi-valued) analytic continuations along any path in B. Therefore the period integrals generate a **local system** on B.

8. Differential equations for period integrals

Proposition: The period integrals are precisely the solutions to the EG equation (for $a = b = \frac{1}{2}$, $c = 1$):

$$\mathcal{L}\varphi := \lambda(1-\lambda)\frac{d^2}{d\lambda^2}\varphi + (1-2\lambda)\frac{d}{d\lambda}\varphi - \frac{1}{4}\varphi.$$

Proof. Check that

$$\mathcal{L}\omega_\lambda = \left(\frac{\partial}{\partial x}\frac{(x-1)^2x^2}{2y^3}\right)dx$$

Right side is an exact 1-form on Y_λ-finite set.

It follows that

$$\mathcal{L}\int_{\gamma_i}\omega_\lambda = \int_{\gamma_i}\mathcal{L}\omega_\lambda = 0$$

by Stoke's theorem. □

9. Computing period integrals

Remarks: This effectively reduces the task of computing each integral $\int_{\gamma_i} \omega_\lambda$ to one of determining two constants in the general solution to an ODE.

For example, at $\lambda = 0$, the curve Y_λ develops a node. With a little more work – basically by studying how the form ω_λ develops a pole when $\lambda = 0$, we can determine those constants.

10. Computing period integrals

If γ_1 is the basic 1-cycle on Y_0 that avoids the node, then

$$\int_{\gamma_1} \omega_\lambda = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \lambda).$$

If γ_2 is the basic 1-cycle that runs through the node, then

$$\int_{\gamma_2} \omega_\lambda = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, \lambda) \log \lambda + g_1(\lambda)$$

where $g_1(\lambda)$ is a unique power series determined by the EG equation.

Thus we have effectively solved an integration problem – elliptic integrals – by relating it to the geometry of curves. This is the spirit in which we proceed to study higher dimensional analogues of elliptic integrals.

11. Higher dimensional analogues: Period sheaves

Let B connected complex manifold (parameter space).

Let $E \to B$ be a vector bundle equipped with a flat connection

$$\nabla : \mathcal{O}(E) \to \mathcal{O}(E) \otimes \Omega^1_B.$$

Let

$$\langle \; , \; \rangle : \mathcal{O}(E) \otimes \mathcal{O}(E^*) \to \mathcal{O}_B$$

be the usual pairing.

Fix global section $s^* \in \Gamma(B, E^*)$.

Definition: The **period sheaf**

$$\mathbf{\Pi} \equiv \mathbf{\Pi}(E, s^*) \subset \mathcal{O}_B$$

is the image of the map

$$\mathcal{O}(E) \supset \ker \nabla \to \mathcal{O}_B, \quad \gamma \mapsto \langle \gamma, s^* \rangle.$$

12. **Period sheaves from Complex Geometry**

Let $\pi : \mathcal{Y} \to B$ be a family of d-dimensional compact complex manifolds, with $Y_b := \pi^{-1}(b)$.

From topology: cohomology groups of fibers $H^k(Y_b, \mathbb{C})$ form a vector bundle $E^* := R^k\pi_*\mathbb{C}$ over B; dual bundle $E = E^{**}$ has fibers $H_k(Y_b, \mathbb{C})$, and

$$\langle\ ,\ \rangle : \mathcal{O}(E) \otimes \mathcal{O}(E^*) \to \mathcal{O}_B$$

is the Poincaré pairing; E is equipped with a canonical flat (Gauss-Manin) connection ∇.

Fix $s^* \in \Gamma(B, E^*)$, and represent $s^*(b) \in H^k(Y_b, \mathbb{C})$ by a closed form on Y_b. Represent section $\gamma \in \ker \nabla$ by cycle on Y_b. So, a local section $f \in \mathbf{\Pi}(U)$ becomes an integral

$$f(b) = \langle \gamma, s^*(b) \rangle = \int_\gamma s^*(b).$$

We call this a **period integral** of $\mathcal{Y}$ with respect to s^*.

13. **Problem**

Fix a compact Kähler manifold X^{d+1}, and assume

$$\pi : \mathcal{Y} \to B$$

is a family of smooth Calabi-Yau hypersurfaces (complete intersections) in X. Consider the associated flat bundle $E^* = R^d \pi_* \mathbb{C}$.

The subspaces

$$\Gamma(Y_b, K_{Y_b}) \subset H^d(Y_b, \mathbb{C}).$$

form a subbundle $\mathrm{H}^{top} \subset E^*$.

14. Problem

Key Fact [Lian-Yau]: The line bundle H^{top} admits a canonical trivialization in terms of an explicit family version of the Poincare residue map

$$\omega := Res\frac{\Omega}{f},$$

where Ω is a holomorphic $(d+1)-$form on certain principal bundle over X, and f is the universal section defining the hypersurface family.

This is essentially a consequence of Poincaré sequence.

The Riemann-Hilbert Problem for Period Integrals:
Construct a complete system of partial differential equations for the period integrals in $\Pi(E, \omega)$.

Goal: To study the explicit solutions and monodromy of this local system.

15. What's known: hypersurfaces in $X = \mathbb{P}^{d+1}$

Dwork-Griffiths' reduction-of-pole method can (in principle) be used to derive differential equations; usually works for **one-parameter** families only.

Example. For the Legendre family, this method yields precisely the EG equation

$$\lambda(1-\lambda)\frac{d^2}{d\lambda^2}\varphi + (1-2\lambda)\frac{d}{d\lambda}\varphi - \frac{1}{4}\varphi = 0.$$

Once an ODE is found, one can apply standard techniques to solve them.

16. What's known: hypersurfaces in a toric manifold

Let X^{d+1} be a toric manifold with respect to torus T, Assume $c_1(X) \geq 0$, and assume that generic CY hypersurface in X is smooth. Consider the family $\pi : \mathcal{Y} \to B$ of all such hypersurfaces.

Let $\hat{\mathfrak{t}}$ be the Lie algebra of $T \times \mathbb{C}^\times$. Then T induces a linear action on $H^0(-K_X)$, and $\mathbb{C}^\times$ acts by scaling. So, we have a Lie algebra action

$$\hat{\mathfrak{t}} \to End\ H^0(-K_X), \quad y \mapsto Z_y.$$

Let $\beta : \hat{\mathfrak{t}} \to \mathbb{C}$ be a character which takes zero on T, and takes 1 on the Euler operator, as a generator of the Lie algebra of $\mathbb{C}^\times$.

Each section $f \in H^0(-K_X)$ restricted to $T \subset X$ is a Laurent polynomial. In fact, the restriction of $H^0(-K_X)$ has a basis of Laurent monomials x^{μ_i} in $x_0, .., x_d$ – coordinates on $T = (\mathbb{C}^\times)^{d+1}$.

17. Toric hypersurfaces: differential equations

Proposition: *The period integrals of the family $\mathcal{Y}$ of CY hypersurfaces in X satisfy the PDE system*

$$\Box_l \varphi = 0, \quad (Z_y + \beta(y))\varphi = 0, \qquad y \in \hat{\mathfrak{t}}$$

where the l are integral vectors such that $\sum_i l_i \mu_i = 0$, $\sum_i l_i = 0$, and

$$\Box_l := \prod_{l_i > 0} \left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_i < 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i}$$

This system is called a **GKZ hypergeometric system.**

Remark: A theorem of GKZ says that solution space of this system is finite dim. However, this system is never complete – there are always more solutions than period integrals. But there is a conjectural way to pick out the period integrals among solutions.

18. **Beyond Toric**

There were a few more isolated examples on the RH problem for period integrals beyond toric hypersurfaces between 1996-2010.

For example, the problem was open even for the case of hypersurfaces in a flag variety (i.e. GL_n/P).

We'll now discuss a partial solution to this problem for a large class of manifolds including flag varieties.

19. **Tautological Systems**

Consider the case of a general projective manifold X.

Data & notations:
X: projective manifold
G: complex algebraic group (a group defined by algebraic equations), with Lie algebra $\mathfrak{g}$
$G \times X \to X, \ \ (g, x) \mapsto gx$, a group action
L: an equivariant base-point-free line bundle on X
$V := H^0(X, L)^*$

$\phi : X \to \mathbb{P}V$ the corresp. equivariant map
I_ϕ: the ideal of $\phi(X)$
$\langle , \rangle$: natural symplectic pairing on $TV^* = V \times V^*$
D_{V^*}: the ring of polynomial differential operators on V^*

20. **Example to keep in mind**

$X = \mathbb{P}^2$
$G = PSL_3$
$L = O(3)$
$V^* = Sym^3\ \mathbb{C}^3$

$\phi : X \hookrightarrow \mathbb{P}V$ is the Segre embedding, $[z_0, z_1, z_2] \mapsto [z_0^3, z_0^2 z_1, z_0^2 z_2, .., z_2^3]$.
I_ϕ=the quadratic ideal generated by the Veronese binomials.
D_{V^*}= the Weyl algebra $\mathbb{C}[a_0, ..., a_9, \frac{\partial}{\partial a_0}, .., \frac{\partial}{\partial a_9}]$.

21. **Group actions**

Define a Lie algebra map (Fourier transform):

$$V^* \to Der\ Sym(V), \quad \zeta \mapsto \partial_\zeta, \quad \partial_\zeta a := \langle a, \zeta \rangle.$$

The linear action $G \to Aut\ V$ induces Lie algebra map

$$\mathfrak{g} \to Der\ Sym(V), \quad x \mapsto Z_x.$$

Let a_i and ζ_i be any dual bases of V, V^*. Then $\partial_{\zeta_i} = \frac{\partial}{\partial a_i}$.

22. **Tautological systems**

Definition: Fix $\beta \in \mathbb{C}$. Let $\tau(X, L, G, \beta)$ be the left ideal in D_{V^*} generated by the following differential operators:
$\{p(\partial_\zeta)|p(\zeta) \in I_\phi\}$, (polynomial operators)
$\{Z_x|x \in \mathfrak{g}\}$, ($G$ operators)
$\varepsilon_\beta := \sum_i a_i \frac{\partial}{\partial a_i} + \beta$, (Euler operator.)
We call this system of differential operators a **tautological system**.

23. **Regularity & Holonomicity**

Theorem: [Lian-Song-Yau] Suppose X has only finite number of G orbits. Then the tautological system $\tau(X, L, G, \beta)$ is **regular holonomic**. Moreover, the solution rank is bounded above by the degree of $X \mapsto \mathbb{P}V$ if the $\mathbb{C}[X]$ is Cohen-Macaulay.

Corollary: Any formal power series solution is **analytic**; the sheaf of solutions is a locally constant sheaf of **finite rank** on some open $V^*_{gen} \subset V^*$.

24. From complex geometry to special functions

Let X be a compact complex G-manifold such that $-K_X$ is base point free. Consider the family $\mathcal{Y}$ of all CY hypersurfaces in X.

Theorem: [Lian-Yau] The period integrals of the family $\mathcal{Y}$

$$\int_\gamma \omega$$

are solutions to the tautological system $\tau(X, -K_X, G, 1)$.

Remark: The special case when $X = F(d_1, .., d_r; n)$ and $G = SL_n$, this was a result of L-S-Y.

25. Solution rank of τ – special case

Consider the family of CY hypersurfaces Y_σ in X, and write $\tau \equiv \tau(X, -K_X, G, 1)$ for the corresponding tautological system.

Theorem: [Bloch-Huang-Lian-Srinivas-Yau] Let G be a semisimple group and X^n a projective homogeneous G-space (i.e. G/P), such that $\mathfrak{g} \otimes \Gamma(X, K_X^{-r}) \twoheadrightarrow \Gamma(X, TX \otimes K_X^{-r})$. Then the solution rank of τ at any point σ is $\dim H^n(X - Y_\sigma)$.

Remark: (1) It was conjectured that the statement is true without the surjectivity assumption. The latter seems difficult to check in general.

(2) The proof uses a method of Dimca to interpret the de Rham cohomology of the complement and the Lie algebra homology group of certain $\mathfrak{g}$-module.

26. Solution rank of τ & the completeness problem

Theorem: [Huang-Lian-Zhu] Geometric rank formula. Let G be a semisimple group and X^n a projective homogeneous G-space. Then the solution rank of τ at any point σ is $\dim H^n(X - Y_\sigma)$.

Recall that the period sheaf $\mathbf{\Pi} \equiv \mathbf{\Pi}(E, \omega) \subset \mathcal{O}_B$ is the image of the map $\mathcal{O}(E) \supset \ker \nabla \to \mathcal{O}_B, \quad \gamma \mapsto \langle \gamma, \omega \rangle$,

and that $\mathrm{rk}\, \Pi(E, \omega) \leq$ solution rk of τ. When is this an equality, i.e. when is τ **complete**?

Corollary: Suppose X is a projective homogeneous space. Then the tautological system τ is complete iff the primitive cohomology $H^n(X)_{prim} = 0$.

27. Solution rank of τ & the completeness problem

Corollary: For $X = \mathbb{P}^{n-1}$, $G = PSL_n$, the system τ is complete.

Remark: This was conjectured by Hosono-Lian-Yau (1995).

Remark: The geometric rank formula is proved using the Riemann-Hilbert correspondence [Kashiwara, Mebkhout].

28. **Algebraic rank formula**

Theorem: [Bloch-Huang-Lian-Srinivas-Yau] Let G be a semisimple group and X a projective homogeneous G-space. Then the solution rank of τ at any point σ is $\dim H_0(\mathfrak{g} \oplus \mathbb{C}, \oplus_{j=0}^{\infty} \Gamma(X, K_X^{-j}) e^{\sigma})$.

Example: $G = PSL_n$, and $X = \mathbb{P}^{n-1}$. Then $K_X^{-1} = \mathcal{O}(n)$. Let $x_1, ..., x_n$ be the standard homogeneous coordinates of X. Our module is the span of he^{σ}, where h is a monomial of degree divisible by n. $\mathfrak{g} \oplus \mathbb{C}$ acts on this space as operators $x_i \frac{\partial}{\partial x_j}$, $1 \leq i \neq j \leq n$, and $x_i \frac{\partial}{\partial x_i} + 1$, $1 \leq i \leq n$.

29. **Large complex structure limit candidate**

It is of particular interest to find points where the solution rank of τ is exactly 1, i.e. by the rank thoerem points σ such that $\dim H^n(X - Y_\sigma) = 1$: these are candidates of the so-called **large complex structure limit** (LCSL) in mirror symmetry.

For the Grassmannian $G(2, n)$, a candidate is given by $x_{12}x_{23}...x_{n1}$ [B-H-L-S-Y], where x_{ij} are Plücker coordinates. The construction is generalized to all projective homogenous spaces G/P in [H-L-Z], based on a classical multiplicity theorem on Verma modules [B-G-G]

Very recently, it appears that mod p solutions to tautological systems give strong constraints on mod p behavior of the unique holomorphic period integral at any large complex structure limit point, for almost all primes p. These constraints may be enough to provide complete information about the period. Together with a Torelli type lemma, this may be used to prove that the large complex structure limit is unique (if it exists) in the moduli space in certain important situations, including hypersurfaces in P^n, and more generally in homogeneous varieties.

30. **Chain integral solutions**

- The injective map $H_n(X - Y_*) \to Hom_D(\tau, \mathcal{O}^{an})$ (local solution space of τ) is not surjective in general, (e.g. X being toric) where $Hom_D(\tau, \mathcal{O}^{an})$ is in general given as compactly supported middle cohomology of a perverse sheaf.
- **Remark**: This generalizes the famous GKZ formula for toric X, giving the generic rank of the GKZ system τ as the volume of a convex polytope in $\mathbb{R}^n$: when τ is a GKZ system, the geometric rank formula reproduces this GKZ volume rank formula.
- **Proposition**: More concretely, denote $U_b := X - V(b)$, and $\cup D$ the union of all G-invariant divisors in X (which may be empty), for any relative cycle $C \in H_n(U_b, U_b \cap (\cup D))$, the chain integral $\int_C \frac{\Omega}{f_b}$ is a local analytic solution to τ at b.
- We understand the above *chain integral map* from $H_n(U_b, U_b \cap (\cup D))$, to the space of local holomorphic solutions of τ at b, in several interesting cases.

31. Chain integral solutions

- **Theorem**: [Huang,Lian,Yau,Zhu] Let X be a smooth toric variety, and take $G = T$ to be the torus acting on X with the open dense orbit, then the chain integral map is an isomorphism.
- Proof is based on a general geometric formula for τ due to Huang,Lian,Zhu, and some local Weyl algebra computation in the toric case.
- **Remark:** Note that τ in this case reduces to a GKZ system. This theorem gives a canonical geometric construction for all solutions to this GKZ system. For e.g. $X = \mathbb{P}^2$, this was explicitly realized by physicists Avram et al (who call these chain integrals "semi-periods").
- **Theorem**: [Huang,Lian,Yau,Zhu] Suppose $X = G/B$ is a complete flag variety, and take the group in defining τ to be a Borel subgroup B, then the chain integral map is an isomorphism.
- There are generalizations of these results to τ associated to other groups. The chain integral map is not surjective in general, for interesting geometric reasons: there could be lower dimensional chains invariant under the group action, that can contribute to the solutions of τ. On the other hand, there is a direct generalization of these results to the general type case.

32. The Hyperplane Conjecture

- *Mirror symmetry for toric CY hypersurfaces [Batyrev][Batyrev-Borisov]:*
  - $\Delta, \Delta^\vee$: reflexive pair of dual polytopes in $\mathbb{R}^n$
  - $\Sigma, \Sigma^\vee$: respective fans over their faces
  - $X = X_\Sigma$, $X^\vee = X_{\Sigma^\vee}$: associated Δ-regular Fano toric varieties, i.e. generic $Y_b \subset X$ intersects each T-orbit transversally.

  Then a crepant resolution $\widetilde{X} \to X$ gives a crepant resolution $\widetilde{Y_b} \to Y_b$.

- Assume $\Delta, \Delta^\vee$ admit regular projective triangulations, i.e. a unimodular triangulation with each n-simplex having a vertex at 0.
  - $\widetilde{X} \to X$, $\widetilde{X^\vee} \to X^\vee$: the corresponding projective resolutions.
  - $D_1, .., D_p \in H^2(\widetilde{X}, \mathbb{Z})$: the T-invariant divisors in $\widetilde{X}$.

    Put $D_0 := -\sum_{i=1}^p D_i = -[\widetilde{Y}]$.
  - $M_+ \subset H_2(\widetilde{X}, \mathbb{Z})$: the set of integral points in the Mori cone of $\widetilde{X}$.

- **Theorem:** [GKZ '90][Adolphson '97] *For the universal CY family in* $\widetilde{X^\vee}$

$$\operatorname{rk} sol(\tau_{GKZ}) = n!\ vol(\Delta).$$

- *Solutions to* τ_{GKZ} *[HLY '95].* Put $\mathcal{U} := \{|a_0| >> 0\}$, and introduce the '*B-series*'

$$B_{\widetilde{X}} : \mathcal{U} \to H^\bullet(\widetilde{X}, \mathbb{C})$$

$$B_{\widetilde{X}}(a) = \frac{1}{a_0} \sum_{\ell=(\ell_0,..,\ell_p)\in M_+} \frac{\Gamma(-D_0 - \ell_0 + 1)}{\prod_{i=1}^p \Gamma(D_i + \ell_i + 1)} a^{\ell+D}.$$

33. The Hyperplane Conjecture

- **Theorem:** *[HLY '95]* $\Pi(\omega) \subsetneq sol(\tau_{GKZ}) = \langle B_{\widetilde{X}} \rangle$.

- **Hyperplane Conjecture:** *[HLY 95'] The period sheaf $\Pi_{\widetilde{Y}^\vee} \equiv \Pi(\omega)$ of the universal family of mirror CYs $\widetilde{Y}^\vee$ is generated by the functional components of the cup product $[\widetilde{Y}] \cup B_{\widetilde{X}}$:*

$$\boxed{\Pi_{\widetilde{Y}^\vee} = \langle [\widetilde{Y}] \cup B_{\widetilde{X}} \rangle.}$$

Thus, the cup product sheaf is independent of the choice of $\widetilde{X} \to X$.

- **Example:** Take

$$X = \mathbb{P}^4, \quad X^\vee = \mathbb{P}^4/(\mathbb{Z}_5)^3.$$

Then we have the well-known formula of Candelas et al:

$$B_{\widetilde{X}}(a) = \frac{1}{a_0} \sum_{d \geq 0} \frac{\Gamma(5H + 5d + 1)}{\Gamma(H + d + 1)^5} z^{d+H}, \quad z = -\frac{a_1 \cdots a_5}{a_0^5}.$$

The cup product sheaf $\langle 5H \cup B_{\widetilde{X}} \rangle$ agrees with the rank 4 period sheaf $\Pi_{\widetilde{Y}^\vee}$ of **mirror quintic** threefolds, as predicted by the hyperplane conjecture.

- **Example:** Now take $X = \mathbb{P}^4/(\mathbb{Z}_5)^3$, $X^\vee = \mathbb{P}^4$.
Then the hyperplane conjecture describes the rank 204 period sheaf $\Pi_{\widetilde{Y}^\vee}$ of **quintics** in $\mathbb{P}^4$. But proving the hyperplane conjecture becomes harder.

34. The Hyperplane Conjecture

- **Main difficulties:**
  • Many different crepant resolutions $\widetilde{X} \to X$, each involving quite complicated combinatorial choices.
  • Usually none of the crepant resolution are Fano, making it hard to compute cup product sheaf.
  • The period sheaf $\Pi_{\widetilde{Y}^\vee}$ was not well understood – even for the quintics – until three years ago.

- **Strategy:** *Blow down $\widetilde{X}$ to a minimal canonical model that is Fano forgetting all the combinatorial choices. Use resolution-independent way to describe the cup product sheaf.*

- **Basic ideas:** To explain the main ideas, we'll mostly focus on the case

$$X^\vee = \mathbb{P}^n \text{ and } X = \text{the mirror } \mathbb{P}^n.$$

General case is a bit more involved.

35. Idea of proof: 1. D-module description of period sheaf

- First, we want to construct a *complete* D-module τ for $\Pi_{\widetilde{Y}^\vee}$ such that

$$\boxed{\Pi_{\widetilde{Y}^\vee} = sol(\tau) \supset \langle [\widetilde{Y}] \cup B_{\widetilde{X}} \rangle.}$$

- Let $\tau_{eGKZ} = \tau(\widetilde{X^\vee}, \mathrm{Aut}(\widetilde{X^\vee}))$, i.e. the *tautological system* obtained from τ_{GKZ} by enlarging the symmetry group from T to $\mathrm{Aut}(\widetilde{X^\vee})$.

- For $X^\vee = \mathbb{P}^n$, $PH^n(X^\vee) = 0$. Then the *Completeness Theorem* implies τ_{eGKZ} is a complete D-module for $\Pi_{\widetilde{Y}^\vee}$, so the equality above holds in this case. Hence

$$\mathrm{rk}\, \Pi_{\widetilde{Y}^\vee} = \mathrm{rk}\, sol(\tau_{eGKZ})$$

- Next, to show the inclusion above:
Theorem: *[Lian, M.Zhu '17] For any mirror pair $\widetilde{X}, \widetilde{X^\vee}$, the cup product sheaf is $\mathrm{Aut}(\widetilde{X^\vee})$-invariant. In other words, the cup product sheaf is a subsheaf of $\Pi_{\widetilde{Y}^\vee}$.*

36. Idea of proof: 2. Compare $H^\bullet(\widetilde{X})$ and $H^\bullet(X)$

- For reverse inclusion $\langle [\widetilde{Y}] \cup B_{\widetilde{X}} \rangle \supset \Pi_{\widetilde{Y}^\vee}$, it reduces to proving

$$\boxed{\mathrm{rk}\ [\widetilde{Y}]\cup = \mathrm{rk}\,\Pi_{\widetilde{Y}^\vee}.}$$

- If $[\widetilde{Y}]$ is ample, we can use Lefschetz:

$$\mathrm{rk}(\text{cup product sheaf}) = \mathrm{rk}[\widetilde{Y}]\cup = \dim H^\bullet(\widetilde{X}) / \dim PH^\bullet(\widetilde{X}).$$

  But $[\widetilde{Y}]$ is only semi-ample in general, hence Lefschetz can fail.

- **Idea:** *Compare $H^\bullet(\widetilde{X})$ and $H^\bullet(X)$ under $\pi : \widetilde{X} \to X$, and express $[\widetilde{Y}]\cup$ in terms of $[Y]\cup$ on the singular space X. Then apply hard Lefschetz to compute* $\mathrm{rk}[Y]\cup$ *in each degree on X=the minimal Fano blow-down of $\widetilde{X}$.*

37. Second application of the RH problem – monodromy invariant differential zero loci

- As before, begin with a Fano G-variety X with finite number of orbits, and consider the period sheaf $\Pi(\omega)$ of the universal CY family in X. We shall assume that the tautological system $\tau = \tau(X, G)$ is complete:

$$\Pi(\omega) = sol(\tau).$$

- **Problem:** *Fix a differential operator $\delta \in D_{V^\vee}$ which is homogeneous under $\mathbb{G}_m \curvearrowright V^\vee$. Describe the zero set cut out by the local system $\delta\Pi(\omega)$ on $B \subset V^\vee$:*

$$\mathcal{N}(\delta) := \{b \in B \mid \delta\mathfrak{s}(b) = 0, \ \ \forall \mathfrak{s} \in \Pi(\omega)\}.$$

- **Motivation:** Describing the zeros of special functions has been an old question in mathematics beginning from the time of Riemann. Investigations of zeros of classical hypergeometric functions began as early as the 1920s by E. Hille and his contemporaries. More recent work in this direction involving generalized hypergeometric functions has been carried out by Ki, Kim, Duke, Imamoglu, Eichler and Zagier.

- We shall consider the projectivization of $\mathcal{N}(\delta)$ and its closure

$$\overline{\mathcal{N}(\delta)} \subset \mathbb{P}V^\vee.$$

38. Projectivity of $\overline{\mathcal{N}(\delta)}$

- **Proposition:** *[Chen, Huang, Lian, Yau '17] Let $\delta \in D_{V^\vee}$ be a differential operator which is homogeneous under $\mathbb{G}_m \curvearrowright V^\vee$. Then $\overline{\mathcal{N}(\delta)}$: the closure of the projectivization of the zero locus of the $\delta-$derivative of periods is a projective variety.*

- $\mathcal{N}(\delta)$ can be empty in general.
  Conjecture: *If $\delta \in \mathbb{C}[\partial]$ is homogeneous under $\mathbb{G}_m \curvearrowright V^\vee$ with order at least 1, then $\mathcal{N}(\delta) \neq \emptyset$.*

- **Example:** Take $X = \mathbb{P}^1$. Then $\Pi(\omega) = \langle \Delta^{-1/2} \rangle$ where $\Delta = 4a_1a_2 - a_0^2$. For generic $\delta \in \mathbb{C}[\partial]$ homogeneous under $\mathbb{G}_m \curvearrowright V^\vee$ with order d, $\overline{\mathcal{N}(\delta)} \subset \mathbb{P}V^\vee = \mathbb{P}^2$ is a curve of degree d.

39. Vanishing criterion

- **Theorem:(Differential zero criterion)** *[CHLY '17]*
  Let $\delta \in D_{V^\vee}$ *and* $b \equiv \sum b_i a_i^\vee \in B$*. Then*

$$\boxed{b \in \mathcal{N}(\delta) \Longleftrightarrow \delta e^f(b) \in Z_{\hat{\mathfrak{g}}}^\vee Re^{f(b)}.}$$

  i.e. b is a point in the zero locus, if and only if $\delta e^f(b)$ vanishes in the Lie algebra coinvariant space in the algebraic rank formula.

- This gives a linear algebraic way to determine which $\delta \in D_{V^\vee}$ kills $sol(\tau)_b$, say by writing a basis of the space $Z_{\hat{\mathfrak{g}}}^\vee Re^{f(b)}$.

- The proof relies on an Lie algebraic description of τ [BHLSY '14].

40. Hypergeometric functions – the case $X = \mathbb{P}^2$

- **Classical theory:** Periods of elliptic curves form a rank 2 local system on the j-plane $\mathbb{P}^1_j$. It is governed by the *Weierstrass PF equation:*

$$\varphi'' + \frac{1}{j}\varphi' + \frac{31j-4}{144j^2(1-j)^2}\varphi = 0.$$

Note: Under a change of variable this is Gauss hypergeometric equation for ${}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; z)$.

- **Lifting to τ_{eGKZ}:** We can lift the local system to the section space $V^\vee = \Gamma(\mathbb{P}^2, \mathcal{O}(3))$, and consider the universal family of cubic curves in $\mathbb{P}^2$. By the *Completeness Theorem*, $\Pi(\omega)$ is governed by $\tau_{eGKZ} = \tau(\mathbb{P}^2, \mathbb{P}GL_3)$. We can view τ_{eGKZ} as the lifting of the ODE under the base change:

$$B \ni b = (b_0, .., b_9) \mapsto j = \frac{g_2(b)^3}{g_2(b)^3 - 27g_3(b)^2} \in \mathbb{P}^1_j.$$

The cubic curve Y_b then corresponds to the elliptic curve $y^2 = 4x^3 - g_2(b)x - g_3(b)$, where $g_2(a), g_3(a) \in \mathbb{C}[a]^{\mathbb{P}GL_3}$ are the classical *Aronhol invariant polynomials* of degree 4 and 6.

- **Proposition:** *Under the base change, the functions φ and $\mathfrak{s} \in sol(\tau_{eGKZ})$ are related by*

$$\varphi(j) = \varphi\left(\frac{g_2^3}{g_2^3 - 27g_3^2}\right) = g_2(b)^{1/4}\mathfrak{s}(b).$$

41. Differential zero locus – cubic curve periods

Next, we give the first nontrivial case of the conjecture on non-emptyness of $\overline{\mathcal{N}(\delta)}$:

- **Corollary:** *[CHLY '17] For generic $\delta \in \mathbb{C}[\partial]_1$, the projective variety $\overline{\mathcal{N}(\delta)} \subset \mathbb{P}V^\vee = \mathbb{P}^9$ is nonempty and has dimension 7.*

- **Proof:** Put $G = \mathbb{P}GL_3$, $\mathfrak{g} = LieG$, and consider $G \curvearrowright V^\vee \equiv \mathbb{C}[\partial]_1$. We argue that $\mathcal{N}(\delta) \neq \emptyset$ if δ lies in a *stable* G-orbit. It is well-known that stable G-orbits are exactly the dense set $B \subset V^\vee$ consisting of smooth curves.

- First, the *Differential Zero Criterion* above, together with Hodge theory in terms of Jacobi rings, implies

$$b \in \mathcal{N}(\delta) \Longleftrightarrow \delta = Z_x^\vee \cdot f(b), \ \ \exists x \in \mathfrak{g} \Longleftrightarrow \delta g_2(b) = \delta g_3(b) = 0.$$

  Fix $b \in B$. Then $\{Z_x^\vee \cdot f(b) \mid x \in \mathfrak{g}\} \subset V^\vee$ has dimension 8. The orbit $G \cdot \delta$ is closed in $\mathbb{P}V^\vee$ and has dimension 8, so they intersect in $\mathbb{P}V^\vee$.

- Thus $\exists g \in G$, $x \in \mathfrak{g}$, such that $g \cdot \delta = Z_x^\vee \cdot f(b)$. Hence for some $x' \in \mathfrak{g}$

$$\delta = Z_{x'}^\vee \cdot f(g^{-1}b), \ \ i.e. \ g^{-1}b \in \mathcal{N}(\delta).$$

  Thus if $G \cdot \delta$ is stable, then $\mathcal{N}(\delta)$ intersects *every* stable orbit $G \cdot b$.

42. Equations for $\overline{\mathcal{N}(\delta)}$

We can turn the proof into a set of equations for the projective variety $\overline{\mathcal{N}(\delta)}$.

- Fix bases $(a_i^\vee)$ of $V^\vee$, and (x_j) of $\mathfrak{g}$. Write

$$\delta = \sum_i \lambda_i \partial_{a_i} \equiv \sum_i \lambda_i a_i^\vee, \quad b = \sum_i b_i a_i^\vee, \quad x = \sum_j c_j x_j.$$

  Then the differential zero criterion $\boxed{\delta = Z_x^\vee \cdot f(b), \quad \exists x \in \mathfrak{g}}$ takes the form of a linear equation

$$\Lambda \equiv (\lambda_1, \lambda_2, \ldots)^t = M(b)(c_1, c_2, \ldots)^t$$

  having solution.

- This is equivalent to

$$\boxed{\mathrm{rk}\, M(b) = \mathrm{rk}(M(b)|\Lambda).}$$

  This gives an effective way to compute the equations for the differential zeros of $\Pi(\omega)$, together with a stratification of this zero loci.

- Translating back into the Gauss hypergeometric function φ, we obtain
  $\boxed{\delta(g_2^{-1/4}\varphi\left(\frac{g_2^3}{g_2^3-27g_3^2}\right))(b) = 0 \text{ iff } b \text{ satisfies the above algebraic equation.}}$

43. Three dimensional canonical singularity and superconformal field theories

Joint work with Dan Xie (Harvard).

44. Motivation

- Two dimensional ADE singularity is very useful in studying superconformal field theory (SCFT): It leads to the classification of 6d $(2, 0)$ SCFT (Witten, 1995); Branes probing these singularities give another way of engineering SCFT (Douglas-Moore, 1996).
- The main purpose of a series of papers (DX, Yau 15-17) is to use three dimensional analog of 2d ADE singularity to study SCFT in $d \geq 4$.
- The main advantage of this approach is that the classification and the description of quantum moduli space are reduced to much simpler geometric problems.

45. Canonical singularity

A canonical singularity X is defined as follows (Reid 81):

- The Weil divisor K_X is Q-Cartier, i.e. there is an integer r such that rK_X is a Cartier divisor.
- For any resolution of singularity $f : Y \to X$, with exceptional divisors $E_i \in Y$, we have

$$K_Y = f^* K_X + \sum_i a_i E_i, \tag{0.1}$$

  with $a_i \geq 0$. r is called index of the singularity. If $a_i > 0$ for all exceptional divisors, it is called **terminal** singularity.

Two dimensional canonical singularity has a ADE classification:

$$\begin{aligned} A_n &: \quad x^2 + y^2 + z^n = 0 \\ D_n &: \quad x^2 + y^{n-1} + zy^2 = 0 \\ E_6 &: \quad x^2 + y^3 + z^4 = 0 \\ E_7 &: \quad x^2 + y^3 + yz^3 = 0 \\ E_8 &: \quad x^2 + y^3 + z^5 = 0 \end{aligned} \tag{0.2}$$

46. Canonical singularity

Three dimensional canonical singularity has following properties:

- There exists a cyclic cover of the index r canonical singularity by index 1 singularity. The index 1 singularity is called rational Gorenstein singularity.
- There exists a **crepant** partial resolution $f : Y \to X$, i.e.

$$K_Y = f^* K_X \tag{0.3}$$

  such that Y is Q-factorial and has only terminal singularity. Such resolutions are not unique, but the number of crepant divisor is the same!
- The index one terminal singularity is classified by following isolated cDV singularity:

$$f(x, y, z) + tg(x, y, z, t) = 0. \tag{0.4}$$

  here $f(x, y, z)$ is the 2-dimensional ADE singularity.

47. Canonical singularity

There are no complete classifications of 3d canonical singularity, but we have a huge class of examples:

- Quotient singularity C^3/G, with G a finite group and $G \in SL(3)$.
- Toric Gorenstein singularity.
- Quasi-homogeneous Hypersurface singularity $f(z_1, z_2, z_3, z_4)$ satisfying the condition
$$f(\lambda^{q_i} z_i) = \lambda f(z_i), \quad \sum q_i > 1. \tag{0.5}$$

We are going to use three dimensional canonical singularity X to study four dimensional $\mathcal{N} = 2$ SCFT, five dimensional $\mathcal{N} = 1$ SCFT, and four dimensional $\mathcal{N} = 1$ SCFT. We will focus on the constraint on X from superconformal invariance, and the description of quantum moduli space.

48. 4d $\mathcal{N} = 2$ SCFT

Here are some basic facts of 4d $\mathcal{N} = 2$ SCFT:

- 4d $\mathcal{N} = 2$ SCFT has a $SO(2,4) \times SU(2)_R \times U(1)_R$ symmetry group.
- It can have a Coulomb branch where generically the low energy theory is described by abelian gauge theory, and $U(1)_R$ symmetry acts on this branch. It is an interesting and challenging problem to find the Seiberg-Witten solution which describes the low energy effective theory on the Coulomb branch.

One can get 4d $\mathcal{N} = 2$ SCFT by putting type IIB string theory on the following background (Shapere, Vafa, 99; DX, S.T. Yau, 15):

$$R^{1,3} \times X. \tag{0.6}$$

Here X is a 3-fold isolated canonical singularity with the following constraint:

- X admits a C^* action ($U(1)_R$ symmetry).

49. 4d $\mathcal{N} = 2$ SCFT

The Coulomb branch solution is given by the **mini-versal** deformation of singularity X, i.e.

$$F(x, S) = 0, \quad F(x, 0) = X. \tag{0.7}$$

with S parameterizing the Coulomb branch. There is a Kodaira-Spencer map (isomorphism):

$$KS: \quad T_{S,0} \to T^1 \tag{0.8}$$

T^1 is a vector space characterizing the infinitesimal deformations and we can read the Coulomb branch spectrum from it.
Remark: In general, there is a non-empty vector space T^2 which implies that there is non-trivial chiral ring relation for Coulomb branch spectrum.
Example: The miniversal deformation for hypersurface singularity is:

$$F(\lambda, z) = f(z) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha \tag{0.9}$$

ϕ_α is the monomial basis of Jacobian algebra $J_f = \frac{C[z_1, \ldots, z_4]}{(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_4})}$.

50. 5d $\mathcal{N} = 1$ SCFT

Here are some basic facts about 5d $\mathcal{N} = 1$ SCFT:

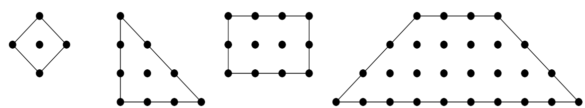
- It has a $SO(2,5) \times SU(2)_R$ symmetry group.
- It can have a Coulomb branch which is parameterized by real numbers.

5d $\mathcal{N} = 1$ SCFT can be engineered by putting M theory on following background:

$$R^{1,4} \times X. \tag{0.10}$$

Here X is a three dimensional canonical singularity. The Coulomb branch is described by the crepant resolution of the singularity.

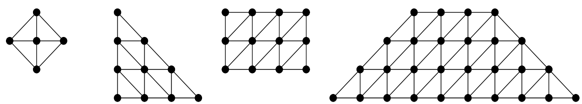
Example: The three dimensional toric Gorenstein singularity is described by a two dimensional convex polygon:



51. 5d $\mathcal{N} = 1$ SCFT

The crepant resolution is achieved by finding the unimodular lattice triangulation of the polygon:

- Many different crepant resolutions: each crepant resolution gives a chamber of Coulomb branch. One can also compute the prepotential in each chamber.



52. 4d $\mathcal{N} = 1$ SCFT

Here are some basic facts of 4d $\mathcal{N} = 1$ SCFT:

- It has a $SO(2,4) \times U(1)_R$ symmetry group.
- The chiral operators form a chiral ring, and $U(1)_R$ symmetry group acts on these operators. The moduli space can be read from the chiral ring.

Consider type IIB string theory on the following background

$$R^{1,3} \times X, \tag{0.11}$$

where X is a three dimensional canonical singularity. One gets four dimensional $\mathcal{N} = 1$ theory by putting N D3 branes on the tip of X. To get a 4d $\mathcal{N} = 1$ SCFT, we need to impose the following conditions on X:

- There is an effective C^* action on X.
- X has to be K stable (Collins, Xie, Yau, 16), where K stability is interpreted as the stability of the chiral ring of field theory.

53. 4d $\mathcal{N} = 1$ SCFT

If the field theory on D brane is superconformal, then in the large N limit, it is argued that the SCFT is dual to type IIB string theory on the background $AdS_5 \times K$. Here K is defined as the five-dimensional link of X and K admits a Sasaki-Einstein metric.

So in this case, the stability of the chiral ring is equivalent to the existence of Sasaki-Einstein metric on the link K, or equivalently the existence of Ricci-flat conic metric on X:

$$ds^2 = dr^2 + r^2 dg_K^2.$$

54. 4d $\mathcal{N} = 1$ SCFT

The existence of Ricci-flat conic metric on X is conjectured by Yau to be related to the stability of X. Donaldson ('02) has made it more precise by introducing the notion of K-stability (K stability conjecture has recently been proven by Chen, Donaldson, Sun). Let's start with a polarized ring (X, ζ) where ζ is a $U(1)_R$-like symmetry. K-stability has the following ingredients (Collins, Szkelyhidi, '15):

- Test configuration: one use a one parameter group η acting on X, then take a flat limit to get a new ring X_0 (this is the candidate chiral ring of a SCFT).
- One define a Futaki invariant $F(X, X_0, \zeta, \eta)$, and X_0 is said to destabilize X if $F \leq 0$ for $X \neq X_0$.

X is called K-stable if there is no destabilizing test configuration.

55. 4d $\mathcal{N} = 1$ SCFT

- The test configuration has one more symmetry group generated by η.
- The Futaki invariant is defined as follows. One can compute the central charge by calculating the Hilbert series of X with respect to symmetry ζ, i.e.

$$F(\zeta, t) = \sum_{\alpha} dim\, H_\alpha t^\alpha = \frac{a_0(\zeta)}{s^3} + \frac{a_1(\zeta)}{s^2} + \ldots, \quad t = exp(-s).$$

  Here H_α is a subspace of affine ring of X, and it has charge α under the symmetry ζ. a_0 is inversely proportional to trial central charge a. If X is stable, the true $U(1)_R$ symmetry and the central charge can be computed by minimizing a_0 (Martelli-Sparks-Yau, '05).

56. 4d $\mathcal{N} = 1$ SCFT

Now for a test configuration generated by a symmetry η, one can form a one-dimensional family $U(1)_R$-like symmetry on X_0 parameterized by a **positive** real number p. One can similarly consider the Hilbert series for X_0 with respect to this one parameter family of symmetries:

$$F(\zeta, \eta, p, exp(-s)) = \frac{a_0(p)}{s^3} + \frac{a_1(p)}{s^2} + \ldots$$

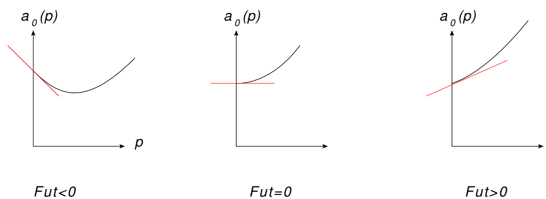
The Futaki invariant is defined as

$$Fut = D_p a_0(p)|_{p=0}.$$

This simply-looking Futaki invariant is the same as the one defined by Donaldson, but it has a direct physical meaning as I will explain below.

57. 4d $\mathcal{N} = 1$ SCFT

The meaning of Futaki invariant should be clear from the following graphs:



The destabilizing configuration gives less a_0 and therefore more central charge! So K stability is then interpreted as a generalized **a** maximization, and the new ingredient is that one has to consider a family of chiral rings to do a maximization.

58. More details

Definition

A test chiral ring $\mathcal{R}_0$ can be derived from $\mathcal{R}$ by using a symmetry generator η on $\mathcal{R}$ and taking a **flat** limit.

Let's discuss more precisely what this definition means. Assume that the chiral ring $\mathcal{R}$ is given by

$$R = \frac{C[x_0, x_1, \ldots, x_n]}{I}, \tag{0.12}$$

here $x_i, i = 0, \ldots, n$ are the generators of chiral ring and $I = (f_1, f_2, \ldots, f_m)$ is the ideal which gives the chiral ring relation among the generators. Now consider a one parameter subgroup $\eta(t)$ of C^{n+1} and define its action on the elements of idea I as

$$f(t) = \lambda(t) \cdot f = f(\lambda(t) \cdot (x_0, x_1, \ldots, x_n)). \tag{0.13}$$

So we have a family of rings $\mathcal{R}_t = \frac{C[x_0, x_1, \ldots, x_n]}{I_t}$ parameterized by t. The flat limit $I_0 = lim_{t \to 0} I_t$ is defined as follows. We can decompose any $f \in I$ as $f = f_1 + \ldots + f_k$ into elements in distinct weight spaces for the C^* action η on $C[x_0, ..., x_n]$. Let us write $in(f)$ for the element f_i with the smallest weight, which we can think of as the "initial term" of f. Then I_0 is the ideal generated by the set of initial terms $\{in(f) | f \in I\}$. The test chiral ring is defined as $\mathcal{R}_0 = \frac{C[x_0, x_1, \ldots, x_n]}{I_0}$.

59. More details

The test chiral ring has the following crucial proerties: a) The flat limit is the same if we use the symmetry generator $s\eta$ with $s > 0$; b) R_0 is invariant with respect to symmetries of R and η; c): The Hilbert series of $\mathcal{R}$ and $\mathcal{R}_0$ are the same for the symmetries of $\mathcal{R}$, this is the continuity condition on test configuration. Using above procedure, we can get infinite number of test chiral rings. The criteria for determining whether a test chiral ring $\mathcal{R}_0$ destabilizes a polarized ring $(\mathcal{R}, \zeta)$ is

Definition

A test chiral ring $\mathcal{R}_0$ destabilizes $(\mathcal{R}, \zeta)$ if $\mathcal{R}_0$ gives **no less** central charge a with respect to the space of possible $U(1)_R$ symmetries $a\zeta + s\eta,\ s \geq 0$.

It is crucial that $s \geq 0$ so we have the same test chiral ring using the symmetry generator $s\eta$ on $\mathcal{R}$. Now we state the definition of stable chiral ring:

Definition

A polarized chiral ring $(\mathcal{R}, \zeta)$ is called stable if there is no destabilizing test chiral ring.

60. The chiral ring

Consider a $\mathcal{N} = 1$ theory on world volume of N D3 branes probing a graded three dimensional normal, Kawamata log-terminal (klt), Gorenstein singularity X. The 3d singularity is defined by an affine ring

$$\mathcal{H}_X = \mathbb{C}[x_1, x_2, \ldots, x_r]/I, \tag{0.14}$$

here $\mathbb{C}[x_1, x_2, \ldots, x_r]$ is the polynomial ring and I is an ideal.

61. Hilbert series and the central charge a

Consider a possible $U(1)_R$ symmetry ζ which is realized as an automorphism of X. The trial central charge $a(\zeta)$ (of order N^2) of the field theory can be computed from the Hilbert series of the ring X (Bergman-Herzog '01, Martelli-Sparks-Yau '05, Martelli-Sparks-Yau '06, Eager '10). The Hilbert series of X with respect to ζ is defined by

$$Hilb(X, \zeta, t) = \sum (dim H_\alpha) t^\alpha; \tag{0.15}$$

Here H_α is the subspace of ring $\mathcal{H}_X$ with charge α under the action ζ. The Hilbert series has a Laurent series expansion around $t = 1$ obtained by setting $t = e^{-s}$ and expanding

$$Hilb(X, \zeta, e^{-s}) = \frac{a_0(\zeta)}{s^3} + \frac{a_1(\zeta)}{s^2} + \dots \tag{0.16}$$

The coefficients $(a_0(\zeta), a_1(\zeta))$ have following properties:

- a_0 is proportional to the volume of the link L_5 of the singularity, and the trial central charge $a(\zeta)$ (order N^2 term) is related to a_0 as

$$a(\zeta) = \frac{27N^2}{32} \frac{1}{a_0(\zeta)}. \tag{0.17}$$

- $a_0 = a_1$ which is due to the condition that Ω has charge 2.
- a_0 is convex function of the symmetry generators (Martelli-Sparks-Yau '06).

62. Hilbert series and the central charge a

For the singularity X, one can define a 5 dimensional link L_5 with Sasakian structure (boyer-Galicki, '08). If there is a Sasaki-Einstein metric on the link L_5, one can find the true $U(1)_R$ symmetry by minimizing a_0, and the field theory central charge is given by the formula (0.17). In the large N limit, the SCFT on D3 branes is dual to Type IIB string theory on the following geometry

$$AdS_5 \times L_5. \tag{0.18}$$

The existence of the SE metric on L_5 is also equivalent to the existence of a Ricci-flat conic metric on X.

Example: Consider the conifold singularity defined by the principal ideal $f(z) = z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$, and it is known that the link L_5 is the manifold $T^{1,1}$ and has a Sasaki-Einstein metric. There is a C^* action ζ on this singularity $f(\lambda^{q_i} z_i) = \lambda f(z_i)$ with weights $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The canonical three form is $\Omega = \frac{dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3}{dF}$. Ω has charge 1 under the symmetry ζ, so the possible $U(1)_R$ symmetry is actually $\zeta^{'} = 2\zeta$ in order to ensure Ω has charge two. The Hilbert series of X with respect to symmetry generator $\zeta^{'}$ is

$$Hilb(t) = \frac{(1-t^2)}{(1-t)^4}|_{t=e^{-s}} = \frac{2}{s^3} + \frac{2}{s^2} + \dots \tag{0.19}$$

Using formula 0.17, We find that the central charge is equal to $a = \frac{27}{64} N^2$ which agrees with the result derived from field theory (Klebanov-Witten, 1998).