Moduli of Vacua and Integrability

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Outline

Defects and Moduli of Vacua

$\mathcal{N} = 4$ Super Yang-Mills

$\mathcal{N} = 4$ Hamiltonian Geometry

Integrating Hamiltonians

Defects

Study n-dimensional TFT $\mathcal{Z}$ through its defects and their OPE:

- Local operators $\mathcal{Z}(S^{n-1})$: commutative algebra[1]
  with Poisson bracket $(n = 3, 5, \dots)$
  or odd Poisson bracket $(n = 2, 4, \dots)$[2]
- Line operators $\mathcal{Z}(S^{n-2})$: tensor category[3]
- Surface operators $\mathcal{Z}(S^{n-3})$: tensor 2-category[4]
- $\dots$

[1] We will suppress gradings

[2] E_n algebra – cf. Beem, BZ, Bullimore, Dimofte, Neitzke

[3] E_{n-1} category

[4] E_{n-2} 2-category

Nekrasov Ω-background

Canonical deformation of algebra of defect operators:

study TFT $\mathcal{Z}(X)$ equivariantly for $U(1)$ acting by rotations[5] of spacetime X

$\rightsquigarrow$ deformation[6] of $\mathcal{Z}(X)$ over $H^*(BU(1)) = \mathbb{C}[\epsilon]$.

OPE of defects loses directions $\rightsquigarrow$ loses commutativity:

Shifted Poisson algebra of defects is deformation quantized

[5] Today: only 1-parameter version

[6] Note: ϵ graded parameter: all values $\epsilon \neq 0$ equivalent.

Moduli of vacua

BZ-Neitzke: Geometric model for TFT $\mathcal{Z}$ as maps into a [shifted] Poisson space[7], built from algebra of defects:

moduli stack of vacua $\mathfrak{M}_{\mathcal{Z}}$.

Measure with increasing resolution:

1. Local operators give $\Gamma(\mathfrak{M}, \mathcal{O})$

   $\rightsquigarrow$ realize affinization $\mathfrak{M} \to \mathfrak{M}^{loc} = Aff(\mathfrak{M})$ as $Spec\mathcal{Z}(S^{n-1})$

2. Line operators give quasicoherent sheaves on $\mathfrak{M}$

   $\rightsquigarrow$ realize 1-affinization $\mathfrak{M} \longrightarrow \mathfrak{M}^{lin} \longrightarrow \mathfrak{M}^{loc}$ from $(\mathcal{Z}(S^{n-2}), \otimes)$ by Tannakian reconstruction

[7] E_n stack: Francis

Defects

- Surface defects give sheaves of categories over $\mathfrak{M}$.

  $\rightsquigarrow$ detect abelian varieties or topological tori as automorphism groups (e.g. Seiberg-Witten for $4d$ $\mathcal{N} = 2$)

- $\cdots$
- Cobordism Hypothesis[8]: Entire TFT detected by $n-1$-category of boundary conditions $\mathcal{Z}(pt)$

[8] Lurie

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$\mathcal{N} = 4$ SYM

Focus on 4d "geometric Langlands" theory:

$\mathcal{N} = 4$ super-Yang-Mills, following Kapustin-Witten.

G_c compact Lie group $\leftrightarrow$ $G = G_{\mathbb{C}}$ complex reductive

$\widehat{\mathcal{B}}_G$: $\mathcal{N} = 4$ SYM in GL twist with gauge group G_c, $\Psi = \infty$

$\widehat{\mathcal{A}}_{G^\vee}$: $\mathcal{N} = 4$ SYM in GL twist with gauge group $G_c^\vee$, $\Psi = 0$

S-duality: Equivalence $\widehat{\mathcal{A}}_{G^\vee} \simeq \widehat{\mathcal{B}}_G$

Local operators in $\mathcal{N} = 4$

In $\widehat{\mathcal{A}}_{G^\vee} \simeq \widehat{\mathcal{B}}_G$: local operators $\longleftrightarrow$

$$H^*(BG^\vee) \simeq \mathbb{C}[\mathfrak{h}^*]^W \simeq \mathbb{C}[\mathfrak{g}^*]^G$$

(vanishing Poisson bracket)

$\rightsquigarrow$ space of characteristic polynomials

$$\mathfrak{M}^{loc} \simeq \mathfrak{h}^*//W \simeq \mathfrak{g}^*//G$$

coadjoint quotient of $\mathfrak{g}^*$.

Coulomb branch of $\mathcal{N} = 4$ SYM

Local operators in Ω-background

The Ω-background doesn't affect ring of $\mathcal{N}=4$ local operators:

$$\mathbb{C}[\mathfrak{g}^*]^G \xleftarrow{\epsilon\to 0} U_\epsilon\mathfrak{g}^G = Z(U_\epsilon\mathfrak{g})$$

$$\mathbb{C}[\mathfrak{g}^*]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W \xleftarrow{\sim} U_\epsilon\mathfrak{g}^G$$

Harish-Chandra isomorphism: deformation quantization preserves Casimirs / center of enveloping algebra

Line operators in $\mathcal{N} = 4$

Line operators in $\widehat{\mathcal{A}}_{G^\vee}$: derived Satake category, $LG_+^\vee$-equivariant sheaves on affine Grassmannian $Gr^\vee = LG^\vee/LG_+^\vee$

$$\widehat{\mathcal{A}}_{G^\vee}(S^2) = Shv_{LG_+^\vee}(Gr^\vee)$$

Ω-background: add equivariance for loop rotation $\mathbb{C}^\times$

$$\widehat{\mathcal{A}}_{G^\vee}(S^2_\epsilon) = Shv_{LG_+^\vee \rtimes \mathbb{C}^\times}(Gr^\vee)$$

Line operators in $\mathcal{N}=4$

Line operators in $\widehat{\mathcal{B}}_G$:

$$\widehat{\mathcal{B}}_G(S^2) = D^b(\mathfrak{g}^*/G)$$

coherent sheaves on $\mathfrak{g}^*/G$
$\leftrightarrow$ G-equivariant $Sym\,\mathfrak{g}$-modules

Turn on Ω-background:

$$\widehat{\mathcal{B}}_G(S^2_\epsilon) = Mod(U_\epsilon\mathfrak{g}/G)$$

quantize to G-equivariant $U\mathfrak{g}$-modules
Harish-Chandra bimodules

S-duality for line operators in $\mathcal{N} = 4$

Line operators in $\widehat{\mathcal{A}}_{G^\vee}$ and $\widehat{\mathcal{B}}_G$ (with or without Ω-background) identified by Derived Geometric Satake Theorem[9]:

$$Shv_{LG_+^\vee \rtimes \mathbb{C}^\times}(Gr^\vee) \simeq Mod(U_\epsilon \mathfrak{g}/G)$$

$\rightsquigarrow$ moduli stack of $\widehat{\mathcal{A}}_{G^\vee} \simeq \widehat{\mathcal{B}}_G$ given by

$$\mathfrak{M}_\epsilon^{lin} \simeq \mathfrak{g}_\epsilon^*/G$$

Stack structure $\leftrightarrow$ residual gauge symmetry along Coulomb branch

[9] Bezrukavnikov-Finkelberg

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The $\mathcal{N} = 4$ mold

Shifted symplectic geometry perspective:

$\mathfrak{g}^*/G$ is odd shifted cotangent bundle of pt/G,

Affinization map

$$\mathfrak{M}^{lin} = \mathfrak{g}^*/G \xrightarrow{\quad\chi\quad} \mathfrak{M}^{loc} = \mathfrak{h}^*//W$$

is a shifted integrable system:

"Mold for integrable systems"[10]

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[10] Ngô

Quantum mold

Quantum local operators $Z(U_\epsilon\mathfrak{g})$
$\rightsquigarrow$ commuting operators on $\mathfrak{M}_\epsilon^{lin}$:
map of mildly noncommutative spaces[11]

$$\mathfrak{M}_\epsilon^{lin} \xrightarrow{\chi_\epsilon} \mathfrak{M}_\epsilon^{loc}$$

i.e., tensor functor from modules over local operators to line operators:

$$Mod(U_\epsilon\mathfrak{g}/G) \xleftarrow{\chi_\epsilon^*} Mod(Z(U_\epsilon\mathfrak{g}))$$

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[11] E_2 stacks

Probing $\mathfrak{M}^{lin}$

Study $\mathfrak{M}^{lin}$ by mapping in and intersecting Poisson varieties[12]
$\leftrightarrow$ studying module categories over line operators[13]

G-phase spaces

$$\begin{array}{ccccc} X \times_{\mathfrak{g}^*/G} Y & \longrightarrow & Y & & \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & \mathfrak{g}^*/G & \xrightarrow{\chi} & \mathfrak{h}^*//W \end{array}$$

All (classical and quantum) G-phase spaces carry commuting Hamiltonians from map to $\mathfrak{M}^{loc}$

- source of (classical and quantum) integrable systems.

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[12] shifted coisotropic maps

[13] E_2-algebras over E_3-category of line operators

Probing $\mathfrak{M}^{lin}$

Physically: use line operators to study 3d boundary conditions –

(e.g., 3d $\mathcal{N} = 4$ theories with suitable global (G or $G^\vee$) symmetry)

and 3d reductions of 4d theory on interval[14]

$\rightsquigarrow$ Poisson moduli spaces and Ω-background quantization.

All resulting theories carry commuting operators from 4d local operators.

[14] BZ-Dimofte-Neitzke

Basic boundary conditions 1

Two basic examples:

- $\mathbb{D}_G$: Quotient map

$$\mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$$

$\leftrightarrow$ Dirichlet boundary condition for $\widehat{\mathcal{B}}_G$

Quantum version: $Mod(U_\epsilon\mathfrak{g}) \longleftarrow Mod(U_\epsilon\mathfrak{g}/G)$.

Ungauging G

Given map $X \longrightarrow \mathfrak{g}^*/G$ (e.g. boundary condition),
pair with $\mathbb{D}_G = \mathfrak{g}^*$:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\mu} & \mathfrak{g}^* & & \\ \downarrow{\scriptstyle G} & & \downarrow & & \\ X & \longrightarrow & \mathfrak{g}^*/G & \xrightarrow{\chi} & \mathfrak{h}^*//W \end{array}$$

$\leftrightarrow$ $\widetilde{X}$ holomorphic Hamiltonian G-space

$\rightsquigarrow$ commuting Hamiltonians on any Hamiltonian reduction of $\widetilde{X}$,
e.g. $T^*(\Gamma\backslash G/K)$

Ungauging G

Quantum version of pairing with $\mathbb{D}_G$:

A noncommutative space $X_\epsilon \longrightarrow \mathfrak{M}_\epsilon^{lin}$
(module category for line operators $Mod(U_\epsilon\mathfrak{g}/G)$)

$\leftrightarrow$ a quantum Hamiltonian G-space $\widetilde{X}_\epsilon$:
a [de Rham] categorical representation[15] of G:

e.g., Quantization of T^*M for holomorphic G-space M

Local operators $Z(U\mathfrak{g})$
$\rightsquigarrow$ G-invariant commuting Hamiltonians

e.g., Harish-Chandra system of commuting operators on $\Gamma\backslash G/K$

[15] module category for $\mathcal{D}$-modules on G

Basic boundary conditions 2

- $\mathbb{N}_{G^\vee}$: Kostant section

$$\mathfrak{g}^*/G \xrightarrow{\ \chi\ } \mathfrak{h}^*//W, \qquad \mathfrak{h}^*//W \xrightarrow{\ Kos\ } \mathfrak{g}^*/G$$

$\leftrightarrow$ Neumann boundary condition for $\widehat{\mathcal{A}}_{G^\vee}$,
regular Nahm pole boundary condition for $\widehat{\mathcal{B}}_G$

lands in regular locus $\mathfrak{g}^{*,reg}$

Quantum version: Whittaker reduction

$$Mod(U_\epsilon\mathfrak{g}/G) \xleftarrow{\ \chi_\epsilon^*\ } Mod(Z(U_\epsilon\mathfrak{g})), \qquad Mod(U_\epsilon\mathfrak{g}/G) \xrightarrow{\ Whit\ } Mod(Z(U_\epsilon\mathfrak{g}))$$

Gauging $G^\vee$

Given map $X \longrightarrow \mathfrak{g}^*/G$, can instead pair with $\mathbb{N}_{G^\vee}$
$\leftrightarrow$ gauge 3d $G^\vee$-symmetry[16]

intersect with Kostant section
(quantum: take Whittaker reduction)

$$\begin{array}{ccccc} \overline{X} & \xrightarrow{\mu} & \mathfrak{h}^*//W & & \\ \downarrow & & \downarrow {\scriptstyle Kos} & & \\ X & \longrightarrow & \mathfrak{g}^*/G & \xrightarrow{\chi} & \mathfrak{h}^*//W \end{array}$$

$\rightsquigarrow$ integrable system on Coulomb branch of 3d gauge theories

Note: only probes *regular* part of $\mathfrak{g}^*/G$:
carries Hamiltonian action of *regular centralizers* in G

[16] cf. Dimofte talk

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“Integrating” integrable systems

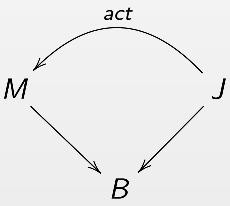
Typical integrable system $M \longrightarrow B$:

- Restrict to generic part $B^{reg} \subset B$

$\rightsquigarrow$ family of abelian groups (tori)

- “Nice” singular fibers contain dense abelian groups

$\rightsquigarrow$ can integrate system:



family of abelian groups with $Lie(J) \simeq T^*B$
realizing Hamiltonian flows

Regular Locus

Regular element $x \in \mathfrak{g}^*$:

Centralizer $Z_G(x)$ of dimension $rk(\mathfrak{g})$ ($\Rightarrow$ abelian)

- $\mathfrak{g}^{*,reg} \subset \mathfrak{g}^*$ open, codim 3 complement

$$\mathfrak{sl}_2^{*,reg} = \mathfrak{sl}_2^* \setminus \{0\}$$

- Kostant section lands in $\mathfrak{g}^{*,reg}$

Regular Centralizers

- Consider centralizers of elements of Kostant section:

$$\begin{array}{ccc} \{Z_G(Kos(\lambda))\} & \longrightarrow & \{\lambda\} \\ \downarrow & & \downarrow \\ J & \longrightarrow & \mathfrak{h}^*//W \end{array}$$

family of abelian groups over $\mathfrak{h}^*//W$
group scheme of regular centralizers

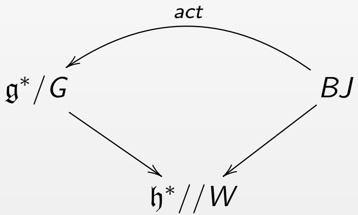
- $\mathfrak{g}^{*,reg}$ is G-orbit of Kostant section $\rightsquigarrow$

$$\mathfrak{g}^{*,reg}/G \simeq BJ \longrightarrow \mathfrak{h}^*//W$$

family of shifted abelian groups, with Kostant section as unit

A Fundamental Lemma

Ngô Lemma: The action of BJ on itself extends to an action on all of $\mathfrak{g}^*/G$.

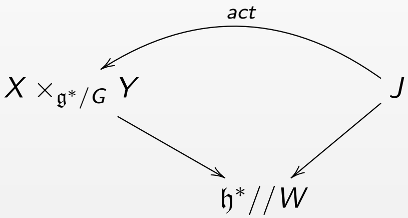


i.e., can integrate our shifted integrable system!

Proof: Hartogs...

Integrating G-integrable systems

Corollary:[17] Canonical integration of Hamiltonian flows on any G-phase space to action of J



Examples:

- Any Hamiltonian reduction $\Gamma_{O_1}\backslash\backslash T^*G//_{O_2}K$

- Moduli space of any reduction of $\mathcal{N}=4$ on an interval (for example 3d $G^\vee$-gauge theories)

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[17] BZ-Gunningham

Integrating Hitchin system

Ngô's application: to topology of spaces of Higgs bundles

$$Higgs_G(\Sigma) \sim Map(\Sigma, \mathfrak{g}^*/G)$$

Crucial: regular centralizers disconnected!
"smart integration"
$\rightsquigarrow$
Produce abelian groups acting on cohomology of Hitchin fibers
$\rightsquigarrow$
(...)
$\rightsquigarrow$
proof of Fundamental Lemma

Main Construction

BZ - Gunningham: Geometric construction and quantization of Ngô action:

- Identify commutative[18] tensor category $\mathcal{K}_\epsilon$, the Kostant category,

$\epsilon = 0$: sheaves on groupscheme J of regular centralizers,
$\epsilon \neq 0$: sheaves on $\mathfrak{h}^*//W_{aff} \sim H^\vee//W$

- Construct central action of $\mathcal{K}_\epsilon$ on $\mathfrak{g}^*_\epsilon/G$
(equivalently, on all categorical representations of G)
deforming action of J

[18] Symmetric monoidal / E_∞

Hidden Symmetry

Construction gives hidden symmetry of line operators in $\mathcal{N} = 4$

$\rightsquigarrow$

• universal integration of Hamiltonian system on all quantum G-phase spaces

• large commutative action on quantized moduli space of any reduction of $\mathcal{N} = 4$ on interval

Translations

Role of $\mathfrak{h}^*//W_{aff} \sim H^\vee//W$:

> Quantum G-phase spaces live over $\mathfrak{h}^*//W_{aff}$ (not $\mathfrak{h}^*//W$).

Example: On $H = \mathbb{C}^\times$,

eigensystem $\{z\frac{d}{dz}f = \lambda f\} \leftrightarrow$ connection $d - \lambda\frac{dz}{z}$
depends (up to gauge) only on monodromy

$$exp(2\pi i\lambda) \in \mathbb{C}^\times = H^\vee,$$

i.e. on

$$[\lambda] \in \mathbb{C}/\mathbb{Z} = \mathfrak{h}^*/\Lambda.$$

Translations

- $U\mathfrak{g}$-modules at fixed central character $\lambda \in Spec(Z_\mathfrak{g}) = \mathfrak{h}^*//W$ depend on λ only up to translation: translation functors

- Eigensystems[19] of higher Casimirs

$$Z(U\mathfrak{g}) \to \mathcal{D}(\Gamma \backslash G/K)$$

depend on λ only up to translation: shift operators

Categorified Harish-Chandra System:
We build spectral decomposition of $Mod(U\mathfrak{g})$, $Mod(\mathcal{D}(\Gamma \backslash G/K))$ etc. over $\mathfrak{h}^*//W_{aff}$:

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[19] Harish-Chandra

The Construction

Geometric source of hidden symmetry:

- Use S-dual description[20] of $\mathfrak{M}^{lin}$ as derived Satake category:

$$\widehat{\mathcal{A}}_{G^\vee}(S^2) = (Shv_{LG^\vee_+}(Gr^\vee), *)$$

- can tensor sheaves on any space by locally constant sheaves!

- Description[21] of $H_*^{G\times\mathbb{C}^\times}(Gr)$
$\rightsquigarrow$ local systems are identified with sheaves on $\mathfrak{g}^{*,reg} = BJ$
$\rightsquigarrow$ BJ acts on $\mathfrak{g}^*/G$

Same works $U(1)$-equivariantly, i.e., for $\epsilon \neq 0$.

[20] Bezrukavnikov-Finkelberg
[21] Bezrukavnikov-Finkelberg-Mirkovic

The Kostant category, abstractly

- Modules for equivariant homology of affine Grassmannian[22]

$$\mathcal{K}_\epsilon := Mod(H_*^{G\times\mathbb{C}^\times}(Gr))$$

– quantized phase space of Toda lattice for G

- Whittaker Hecke category for $G^\vee$,
$\mathcal{D}_\epsilon(N_\psi \backslash G^\vee /_\psi N) = End(\mathbb{N}_{G^\vee})$

$\leftrightarrow$ Ω-deformed line operators in $G^\vee$ 3d $\mathcal{N} = 4$ SYM[23]

[22] Bezrukavnikov-Finkelberg-Mirkovic
[23] Teleman

The Kostant category, concretely

(Take $\epsilon \neq 0$.)

- Modules for affine nil-Hecke algebra[24]:
  explicit combinatorial description: divided-difference operators on $\mathfrak{h}^*$

- Sheaves on coarse quotient[25]

$$\mathfrak{h}^{\vee,*}//W_{aff} \sim H^{\vee}//W$$

$W_{aff} \simeq \Lambda \rtimes W$ affine Weyl group

[24] Kostant-Kumar
[25] Lonergan, Ginzburg

Other aspects

Other structures captured by $\widehat{\mathcal{A}}_{G^\vee} \simeq \widehat{\mathcal{B}}_G$ decompose over $\mathcal{K}$:

- Langlands parameters for categorical representations of G in

  $$\mathfrak{h}^*//W_{aff} \sim H^\vee//W \sim G^\vee//G^\vee$$

  conjugacy classes in dual group
  [local geometric Langlands program]

- Decompose class $\mathcal{D}$-modules $\mathcal{D}(G/G)$ over $H^\vee//W$:
  spectral decomposition into Lusztig character sheaves with fixed central character

- Homology of character varieties of surfaces $Loc_G(\Sigma)$ decomposes into "Kostant eigenhomology" parametrized by $H^\vee//W$ – categorifies description of point counts[26]

[26] Hausel-Letellier-Villegas

The End

Thank You!