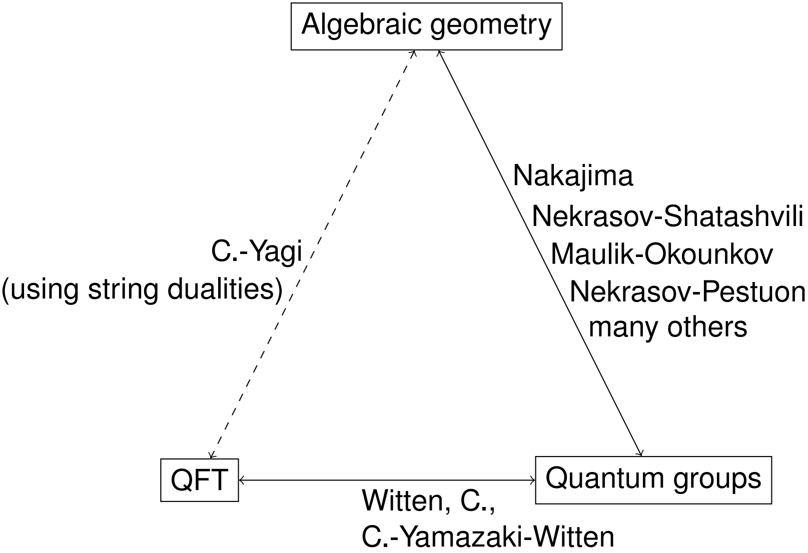
Kevin Costello
Based on joint work with Yamazki and Witten, and with Yagi

String math, 2017


Algebraic geometry
C.-Yagi
(using string dualities)
Nakajima
Nekrasov-Shatashvili
Maulik-Okounkov
Nekrasov-Pestuon
many others
QFT
Quantum groups
Witten, C.,
C.-Yamazaki-Witten

Dimension	Action	Quantum group	Algebraic geometry
3	$\int CS(A)$	$U_q(\mathfrak{g})$	??
4	$\int \mathrm{d}z CS(A)$	Yangian, quantum loop group, elliptic quantum group	Cohomology of finite quiver varieties
5	$\int \mathrm{d}z_1 \mathrm{d}z_2 CS(A)$	Two variable quantum groups, affine Yangians	Cohomology of affine quiver varieties
6	$\int \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 CS(A)$	??	??

In this talk:

- Explain how to see quantum groups directly from 4*d* gauge theory.
- Use this to show from first principles why certain physical systems have Yangian symmetry.

Background

On $\mathbb{R}^2 \times \mathbb{C}$ with coordinates x, y and z, there is a gauge theory with fundamental field

$$A = A_x \mathrm{d}x + A_y \mathrm{d}y + A_{\bar{z}} \mathrm{d}\bar{z} \tag{1}$$

and Chern-Simons action

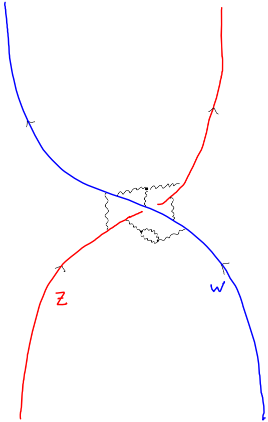
$$\int \mathrm{d}z CS(A). \tag{2}$$

- The theory is topological in the $x - y$ plane and holomorphic in the z-plane.
- Representations of $\mathfrak{g}$ give Wilson line operators in the $x - y$ plane.

R-matrix from Wilson lines

- Put this theory on $\mathbb{R}^2 \times \mathbb{C}$, gauge group $SU(2)$.
- Consider Wilson lines at $z, w \in \mathbb{C}$ for the spin $1/2$ representation of $SU(2)$, which cross in the topological plane.
- The space of states at the end of each Wilson line is $\mathbb{C}^2$.
- Gluon exchange leads to the R-matrix

$$R(z-w) : \mathbb{C}^2_z \otimes \mathbb{C}^2_w \to \mathbb{C}^2_w \otimes \mathbb{C}^2_z.$$

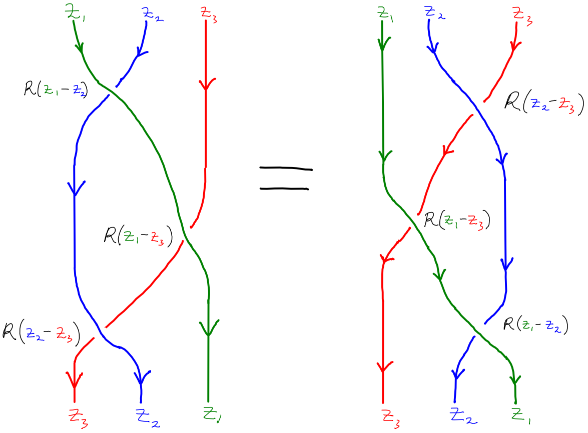


Theorem

This is the R-matrix for zero-field six-vertex model (related to XXX spin chain).

Yang-Baxter equation

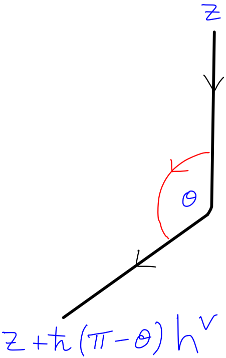
The Yang-Baxter equation follows from topological invariance of the theory on $\mathbb{R}^2$:



Framing anomaly

Classically, Wilson lines must live at a constant value of z.

At the quantum level, if they bend they shift in the z plane:



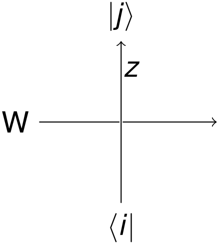
Here $h^\vee$ is the dual Coxeter number.

Where does the Yangian algebra come from?

1. Koszul dual of local operators. (Too abstract!)
2. RTT presentation (recent work with Witten and Yamazaki).

We will show that for any Wilson line in the theory, the Yangian acts on the states at the end of the Wilson line.

Take $\mathfrak{g} = \mathfrak{gl}_n$, consider vertical Wilson line in the fundamental representation at $z \in \mathbb{C}$, arbitrary horizontal Wilson line W.
Place incoming/outgoing states $\langle i|$ and $|j\rangle$ on the fundamental Wilson line.



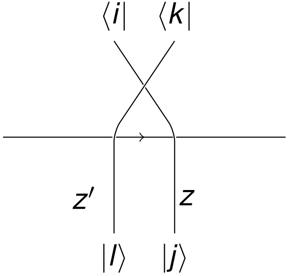
This gives an operator

$$T^i_j(z) : W \to W.$$

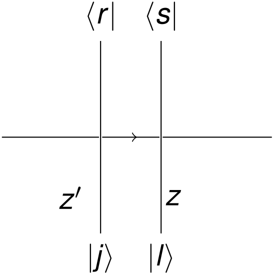
If we expand near $z \sim \infty$, we find operators

$$T^i_j(z) = \delta^i_j + z^{-1} t^i_j[0] + z^{-2} t^i_j[1] + \dots$$

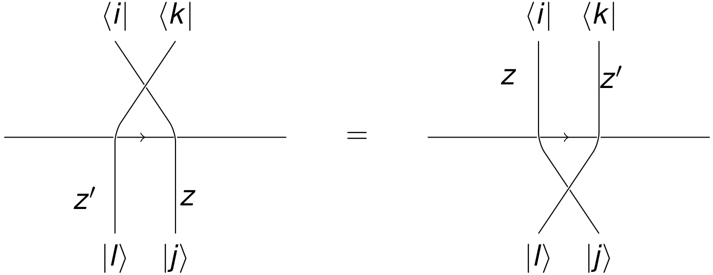
The operators $t^i_j[k]$ will give a representation of the Yangian. Commutation relations are deduced from crossing vertical Wilson lines:



$$= \sum R^{ik}_{rs}(z - z')$$



Topological invariance means location of crossing of vertical lines doesn't matter:



This leads to the relation

$$\sum_{r,s} R^{ik}_{rs}(z-z')T^r_j(z')T^s_l(z) = \sum_{r,s} T^i_r(z)T^k_s(z')R^{rs}_{jl}(z-z').$$

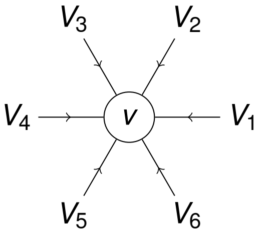
This is the RTT relation for the Yangian for $\mathfrak{gl}_n$.

This explains the Yangian for $\mathfrak{gl}_n$. What about other groups? We need extra relations. These come form *vertices*.

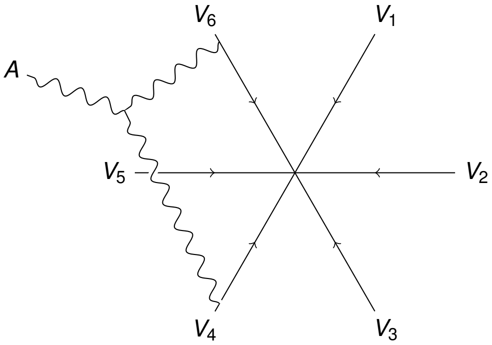
Classically we can make a gauge invariant configuration where k Wilson lines in representations $V_1, \ldots, V_k$ meet with an invariant tensor

$$v \in V_1 \otimes \cdots \otimes V_k$$

at the vertex.



At the quantum level there may be anomalies to quantizing the vertices. These come from Feynman diagrams:



Formula for the anomaly:

$$\partial_z X^a \left(\sum_{1 \leq i < j \leq n} (\theta_{ij} - \pi) f^a_{bc} t_{b;i} t_{c;j} \right) v.$$

This is a linear map $\mathfrak{g} \to V_1 \otimes \cdots \otimes V_n$. X is ghost field.

Anomalies can potentially be cancelled by varying 2 parameters:

1. The angles between the Wilson lines in the topological plane.
2. The position of the Wilson lines in the z-plane.

By the framing anomaly these are equivalent.

With 3 Wilson lines V_1, V_2, V_3 we can cancel the anomalies if there are *exactly two* copies of $\mathfrak{g}$ in $V_1 \otimes V_2 \otimes V_3$.

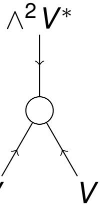
If the quadratic Casimir acts by c_i on V_i then the angles between the three Wilson lines are given by the formula

$$\theta_{12} = \pi - \pi \frac{(c_1 - c_2 - c_3)(c_2 - c_3 - c_1)}{-c_1^2 - c_2^2 - c_3^2 + 2c_1c_2 + 2c_1c_3 + 2c_2c_3}$$
$$\theta_{23} = \pi - \pi \frac{(c_2 - c_3 - c_1)(c_3 - c_1 - c_2)}{-c_1^2 - c_2^2 - c_3^2 + 2c_1c_2 + 2c_1c_3 + 2c_2c_3}$$
$$\theta_{31} = \pi - \pi \frac{(c_3 - c_1 - c_2)(c_1 - c_2 - c_3)}{-c_1^2 - c_2^2 - c_3^2 + 2c_1c_2 + 2c_1c_3 + 2c_2c_3}$$

Examples

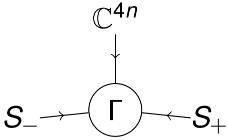
1. $V_1 = V_2 = V$ fundamental representation of $\mathfrak{sl}_n$, $V_3 = \wedge^2 V$.

$$\theta_{12} = \pi \frac{n-1}{n}$$
$$\theta_{23} = \pi \frac{2}{n}$$
$$\theta_{31} = \pi \frac{n-1}{n}$$



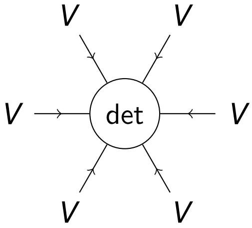
2. $\mathfrak{g} = \mathfrak{so}(4n)$, $V_1 = S_+$, $V_2 = S_-$, $V_3 = \mathbb{C}^{4n}$, invariant tensor $\Gamma \in S_+ \otimes S_- \otimes \mathbb{C}^{4n}$

$$\theta_{12} = \pi \frac{n}{2n-1}$$
$$\theta_{23} = \pi \frac{2n-2}{2n-1}$$
$$\theta_{31} = \pi \frac{n}{2n-1}$$



1. V is the fundamental representation of $\mathfrak{sl}_n$,

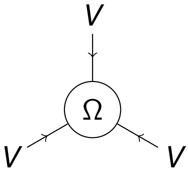
   $\det \in V^{\otimes n}$

   is a vertex with angles $2\pi/n$.

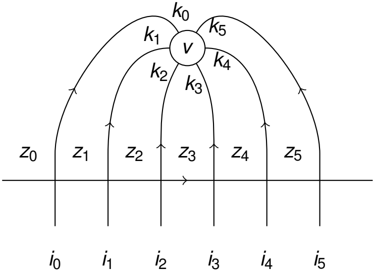
2. V is the fundamental representation of G_2, invariant 3-form

   $\Omega \in V^{\otimes 3}$

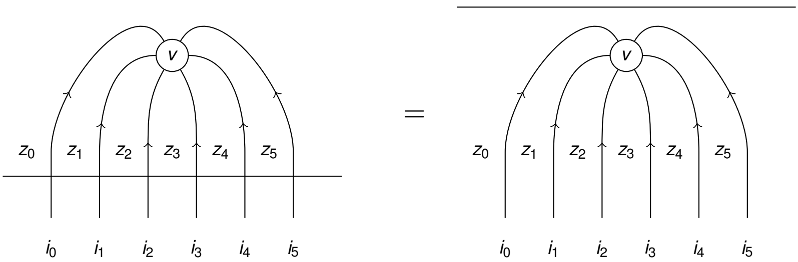
   defines a vertex with angles $2\pi/3$.

Vertices add relations to the RTT algebra. We can cross a vertex with a horizontal line:



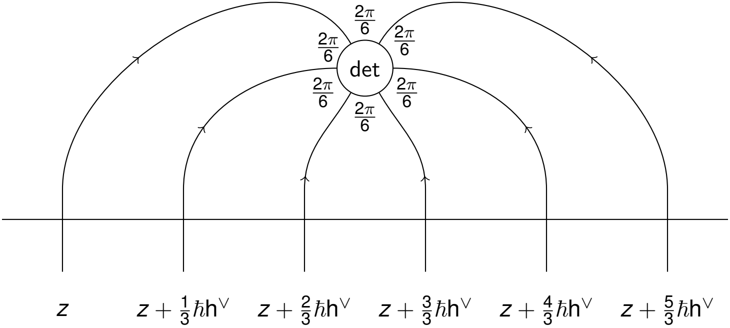
$$= \quad \sum_{k_0,\dots,k_5} v_{k_0,\dots,k_5} T_{i_0}^{k_0}(z_0)\dots T_{i_5}^{k_5}(z_5)$$

Vertices add relations to the RTT algebra. We can cross a vertex with a horizontal line:



$$\sum_{k_0,\ldots,k_5} v_{k_0,\ldots,k_5} T_{i_0}^{k_0}(z_0)\ldots T_{i_5}^{k_5}(z_5) = v_{i_0,\ldots,i_5}$$

Examples : quantum determinant. V fundamental representation of $\mathfrak{sl}_n$, vertex coming from $\det$ ($n = 6$).



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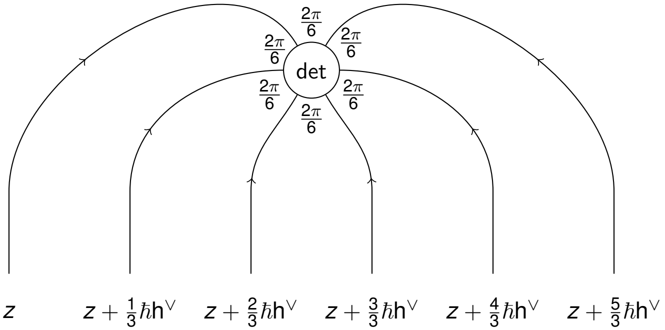


Examples : quantum determinant. V fundamental representation of $\mathfrak{sl}_n$, vertex coming from $\det$ ($n = 6$).

This gives the relation (using $\mathrm{h}^\vee = n$ for $\mathfrak{sl}_n$)

$$\sum_{k_r} \mathrm{Alt}(k_0, \ldots, k_{n-1}) T_{i_0}^{k_0}(z) T_{i_1}^{k_1}(z+2\hbar) \cdots T_{i_{n-1}}^{k_{n-1}}(z+2(n-1)\hbar) = \mathrm{Alt}(i_0, \ldots, i_{n-1}), \quad (3)$$

which is the *quantum determinant*. This presents the Yangian for $\mathfrak{sl}_n$.

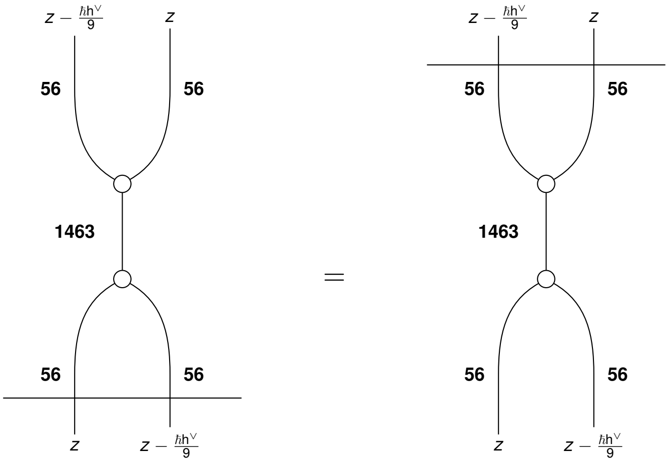
Other groups

For all simple groups except $E8$ this method produces a presentation of the Yangian (and the quantum loop group, elliptic quantum group) directly from field theory. Strategy:

1. Present the group as the automorphisms of its smallest dimensional representation preserving a list of invariant tensors.
2. Lift these invariant tensors to vertices between Wilson lines.
3. Add the corresponding relations to the RTT relation.

- $\mathfrak{so}_n$, $\mathfrak{sp}_{2n}$: quadratic relation from invariant tensor in $V^{\otimes 2}$.
- $G2$, $F4$: quadratic plus cubic relation in lowest-dimesional representation (**7** and **26**)
- $E6$: only a cubic relation from the invariant tensor in $\mathbf{27}^{\otimes 3}$.
- $E7$: quadratic plus quartic relations from invariant tensors in powers of **56**.

The quartic relation for $E7$:



Why is this useful? We can embed certain physical systems in the 4*d* gauge theory. They will then acquire Yangian (or quantum loop group / elliptic quantum group) symmetry, explaining many results in the physics literature.

1. Two-dimensional integrable field theories can be engineered by considering surface operators instead of line operator. This explains why they have quantum group symmetry, and gives many new examples.
2. Twisted supersymmetric gauge theories can be realized by embedding the 4*d* gauge theory in string theory and applying string dualities. Explains quantum group symmetry here (e.g. Nekrasov-Shatashvili, Maulik-Okounkov, ...).
3. In this talk I will explain why the Yangian appears in the algebra of monopole operators in 3d $N = 4$ gauge theories. This is a result of Bullimore, Dimofte, Gaiotto and Braverman, Finkelberg, Kamnitzer, Kodera, Nakajima, Webster, Weekes.

Embedding the 4d gauge theory in string theory

- Consider a $D5$ brane on $x_0, \ldots, x_5$ in $\mathbb{R}^{10}$
- Introduce Ramond-Ramond 2 -form with

$$\mathrm{d}C_2 = dx_3\,(dx_4 dx_5 - dx_6 dx_7)$$

Theorem (C., J. Yagi)

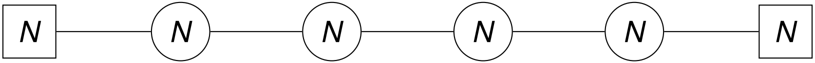
After performing a twist the theory on the $D5$ brane is the 4d Chern-Simons theory related to the Yangian.

Topological directions are x_0, x_1, holomorphic plane is x_2, x_3. This set up is T-dual to M-theory on

$$S^1 \times \mathbb{R}^2 \times \underline{\mathbb{R}^2} \times \left(\underline{S^1} \times \underline{\mathbb{C}_q} \times \mathbb{C}_{-q} \right) \tag{4}$$

with $\underline{M5}$ brane on underlined directions.

1. $F1$ ending on m $D5$'s gives Wilson line in the fundamental representation V.
2. N $D3$ wrapping x_0, x_4, x_5, x_8 passing through m $D5$ gives a Wilson line in $(\mathrm{Sym}^* V)^{\otimes N}$.
3. S-duality: takes us to N $D3$'s passing through m $NS5$'s, and so linear quiver gauge theory.



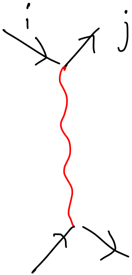
Presence of RR-form in S-dual setting means the linear quiver gauge theory is in an Ω-background.

Conclusion

1. There is a Wilson line whose operators are the monopole operators in the linear quiver gauge theory.
2. Therefore there is a homomorphism from the Yangian to this algebra.
3. Recover the result of Bullimore, Dimofte, Gaiotto and Braverman, Finkelberg, Kamnitzer, Kodera, Nakajima, Webster, Weekes.

For type D and E quivers there is a similar construction. (Rational and elliptic cases should work too, but these are more subtle).

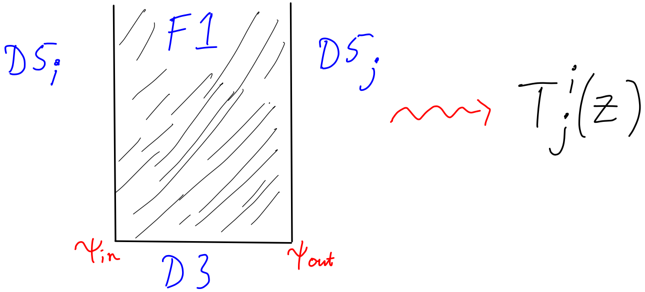
Yangian action comes from RTT presentation. Gauge theory: leading term in the action comes from a fundamental Wilson line (top) exchanging a gluon with a general Wilson line (bottom).



$$\Longrightarrow T^{i}_{j}(z)$$

String theory: fundamental Wilson line on $D5$ comes from $F1$. Incoming/outgoing states come from boundary conditions on the $F1$. Gluon also comes form $F1$.

Conclude that to leading order the action of the Yangian on the $3d$ gauge theory is given by



Apply S-duality: recover the leading term for Yangian action as written down by BDG and BFKKNWW (by monopole operators).

Decoupling

Subtle point: how do we decouple the *NS*5 (or *D*5) brane dynamics to recover precisely the 3*d* $N = 4$ theory?
Recall:

- *D*5 branes on $x_0, \dots, x_5$ in $\mathbb{R}^{10}$.
- *D*3 branes on x_0, x_4, x_5, x_8.
- RR 2-form $\mathrm{d}C_2 = dx_3\,(dx_4 dx_5 - dx_6 dx_7)$.

On the *D*3 brane, impose Dirichlet boundary conditions at $x_8 = \pm\infty$.

On the *D*5 brane, if $z = x_2 + ix_3$ is the holomorphic direction in the effective 4*d* theory, as $z \to \infty$ ask that all fields vanish.

Compactifying the *D*5 brane theory on *z*-plane to x_0, x_1, x_4, x_5 directions gives trivial theory after twisting (infinitely massive).
Compactifying *D*3 on x_8 line gives 3*d* $N = 4$ theory.

Therefore, compactifying *entire D*3-*D*5 *system* gives 3*d* $N = 4$ theory in an Ω background.

Other examples of this process

1. Nekrasov-Shatashvili for finite quivers: consider $D2$-$NS5$ system, apply T and S-duality to get an $F1$-$D5$ system and relate to integrable spin systems.
2. $3d$ $N = 4$ for affine quivers: C., `1705:02500`. Related to a line defect in $5d$ Chern-Simons type theory, and the algebra $U_\hbar(\mathfrak{gl}_n[z_1, z_2])$.
3. Maulik-Okounkov and Etingof-Schiffman result that the affine Yangian acts on cohomology of instanton moduli spaces. This comes from considering surface defect in $5d$ Chern-Simons theory in C., `1610.04144`. (Only proved from this point of view at a special value of the parameters).

In the $5d$ case the RTT presentation has not yet been derived from the point of view of field theory.