A geometric recipe for superpotentials

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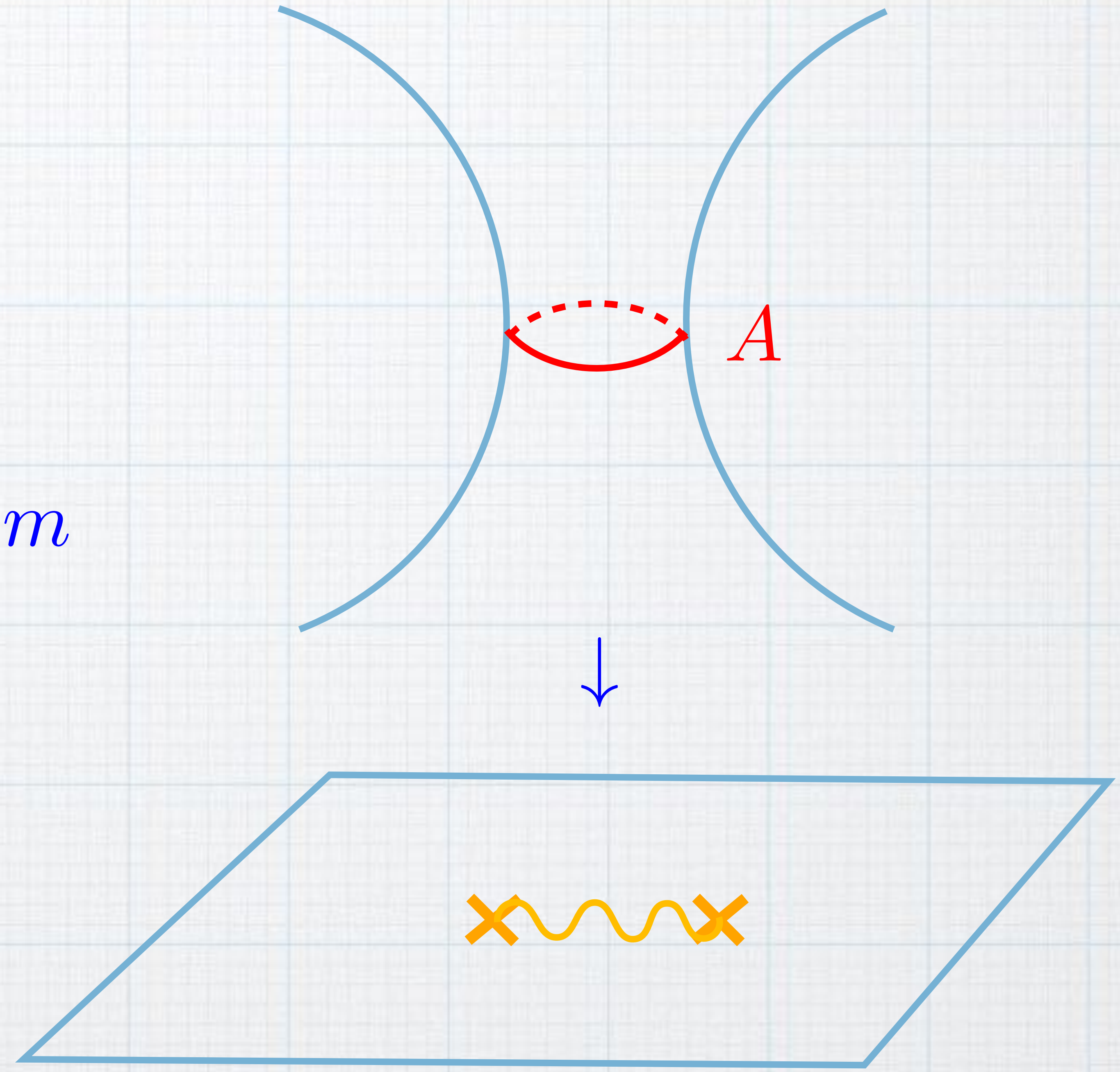
based on [H-Neitzke,'13],[H-Neitzke,'16],
[H-Kidwai,'17],[H-Neitzke,'17]
work in progress [H-Ruetter]

Consider the spectral curve:

$$\Sigma : \; w^2 = z^2 + 2m$$

$$\downarrow$$

$$C = \mathbb{C}$$



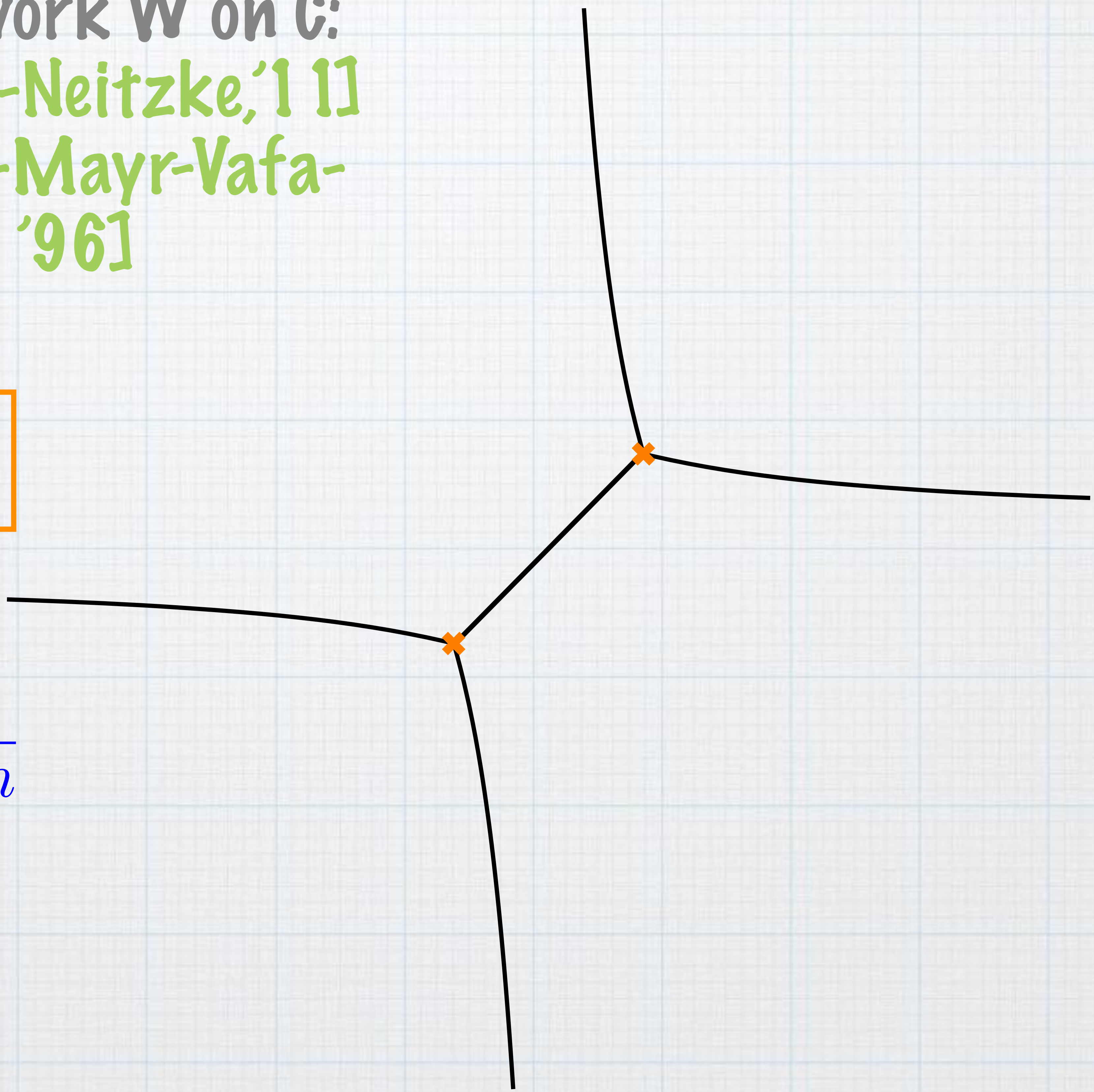
This geometry tells us about the physics of a single N=2 hypermultiplet of mass m

This hypermultiplet is encoded in a spectral network W on C: [Gaiotto-Moore-Neitzke,'11] [Klemm-Lerche-Mayr-Vafa-Warner, '96]

$$e^{-i\vartheta}\lambda(v) \in \mathbb{R}$$

$$\lambda = \sqrt{z^2 + 2m}$$

$$\vartheta = \frac{\pi}{4}$$

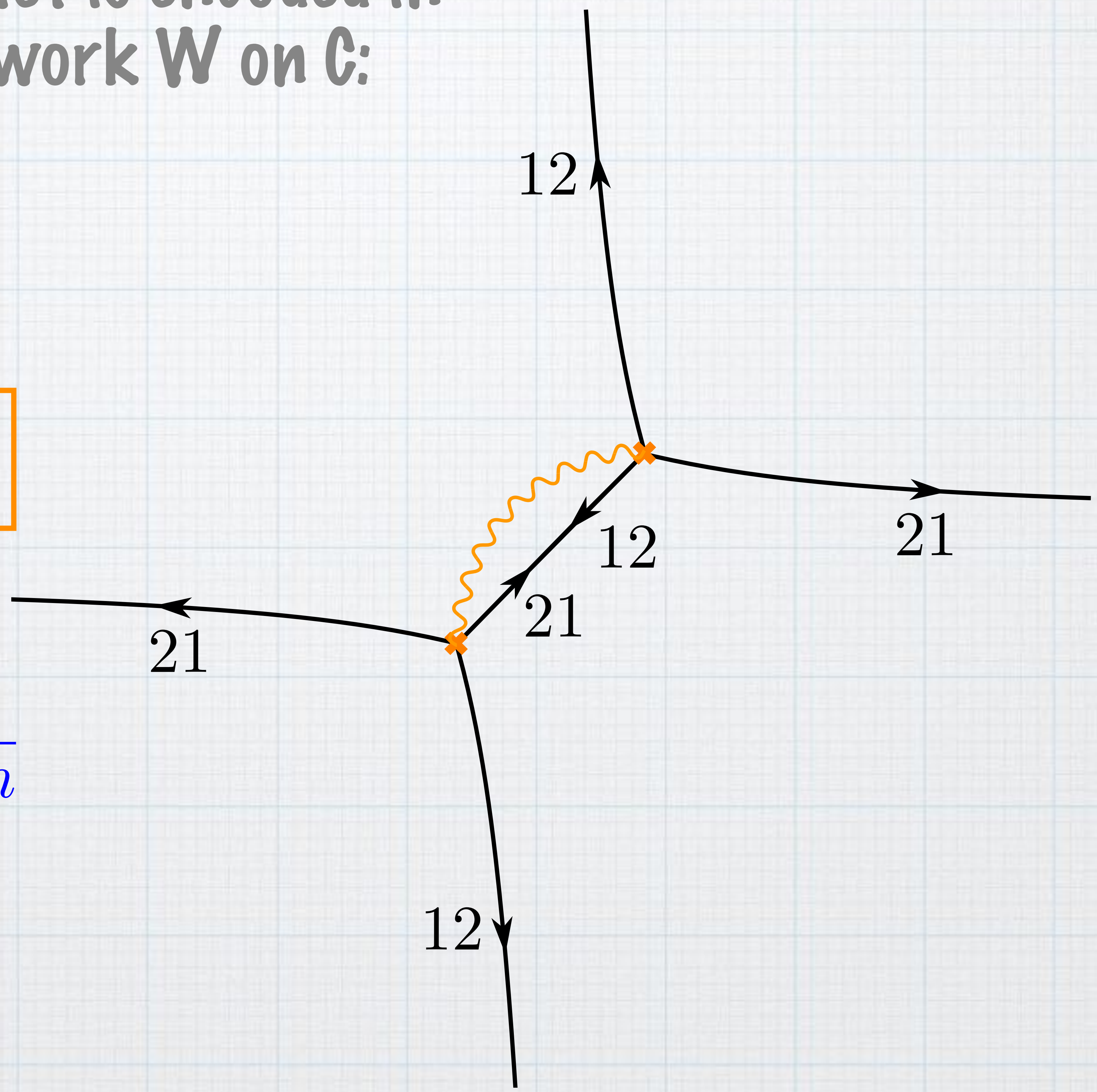


This hypermultiplet is encoded in a spectral network W on C:

$$e^{-i\vartheta}\lambda(v) \lessgtr 0$$

$$\lambda = \sqrt{z^2 + 2m}$$

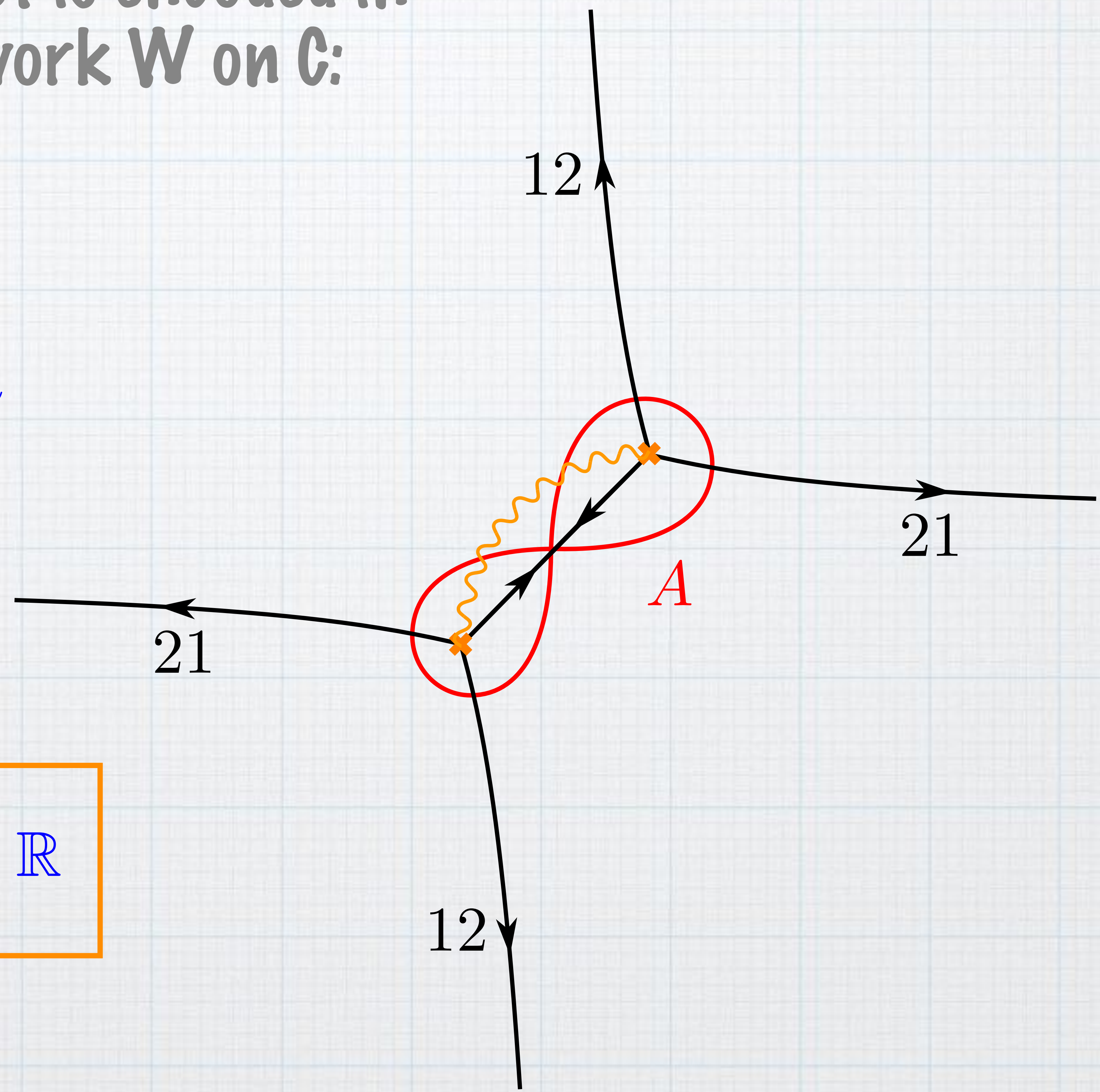
$$\vartheta = \frac{\pi}{4}$$



The hypermultiplet is encoded in a spectral network W on C:

$$\oint_A \lambda = 2\pi i m$$

$$e^{-i\vartheta} \oint_A \lambda \in \mathbb{R}$$



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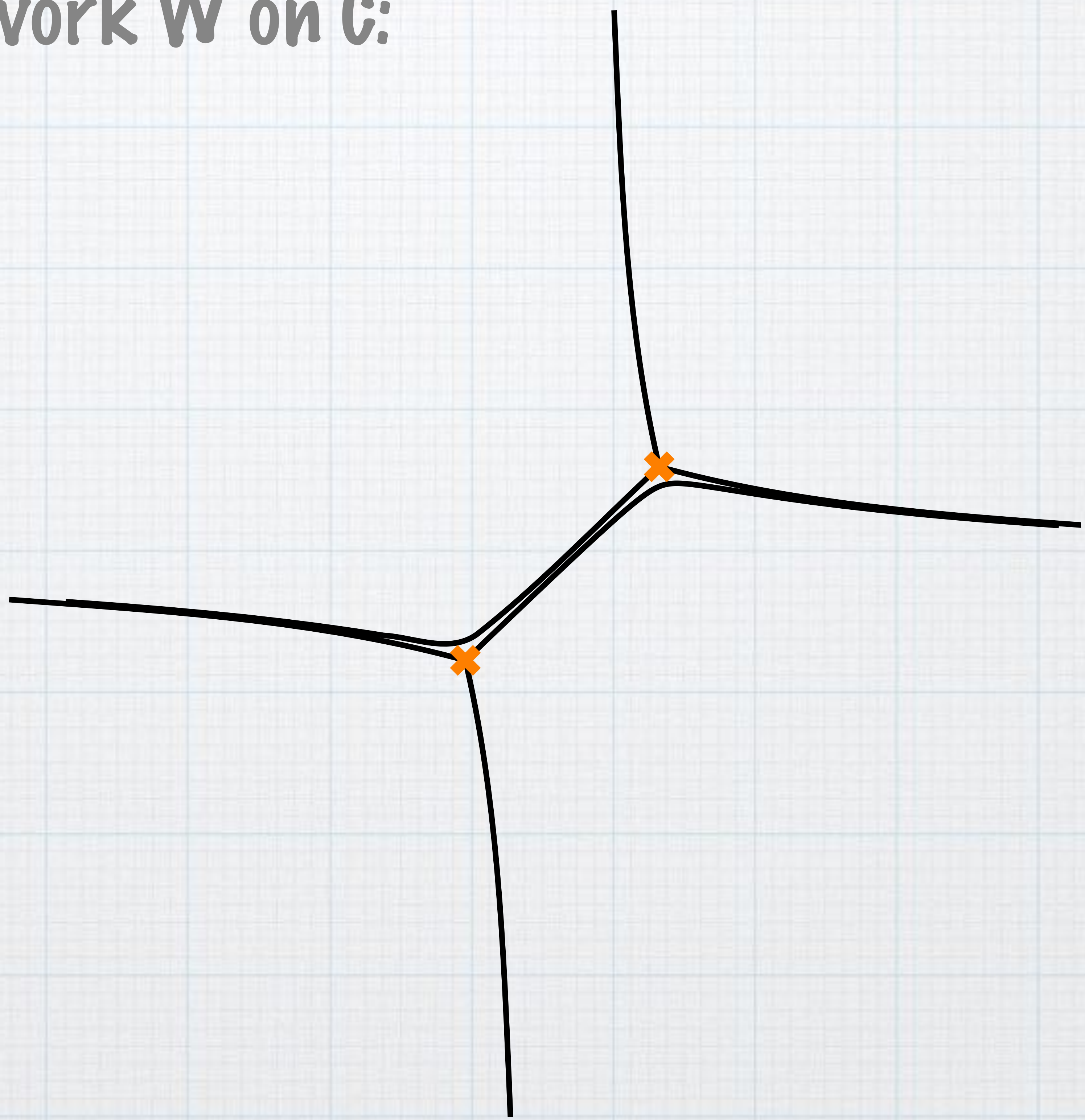
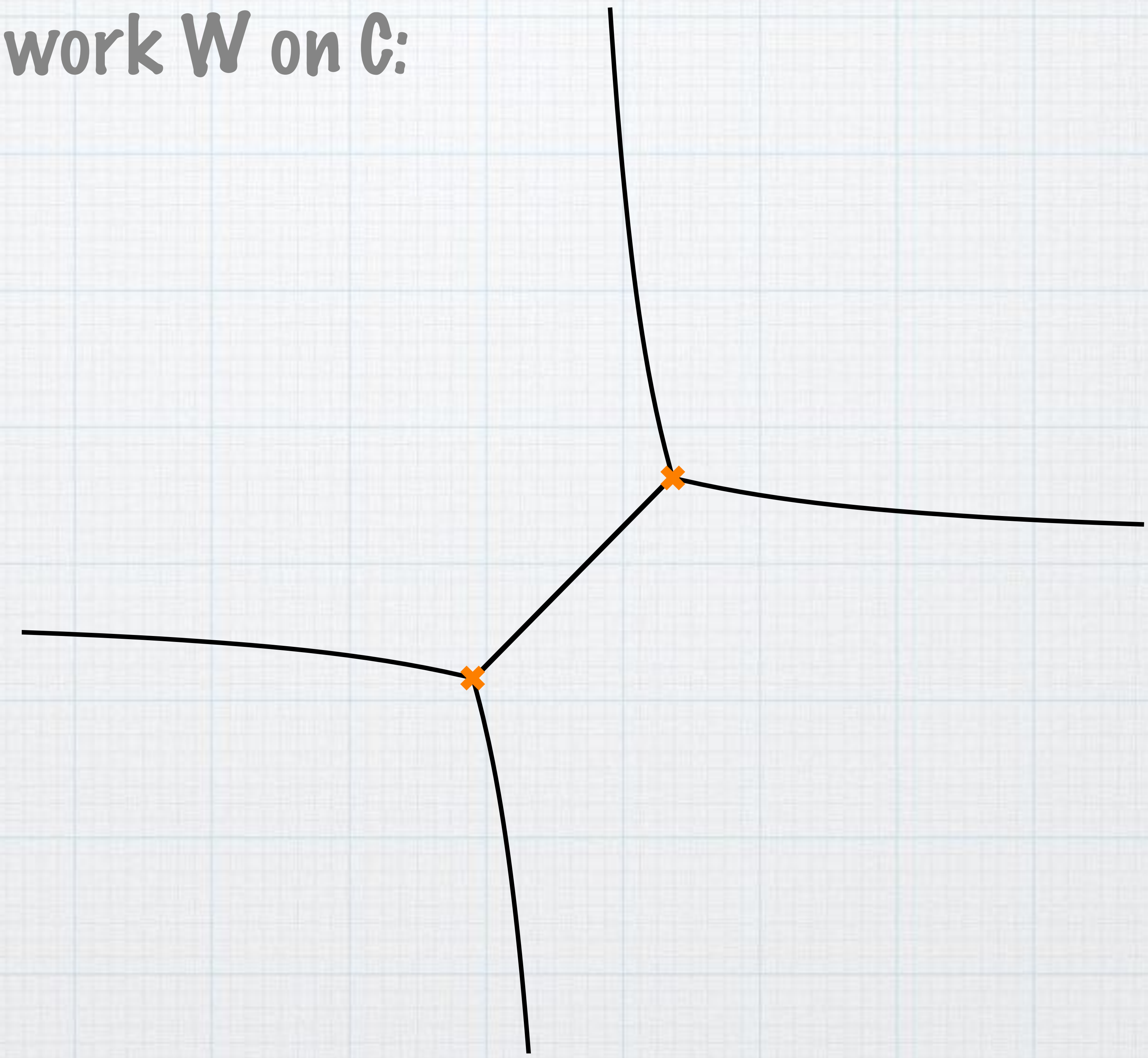


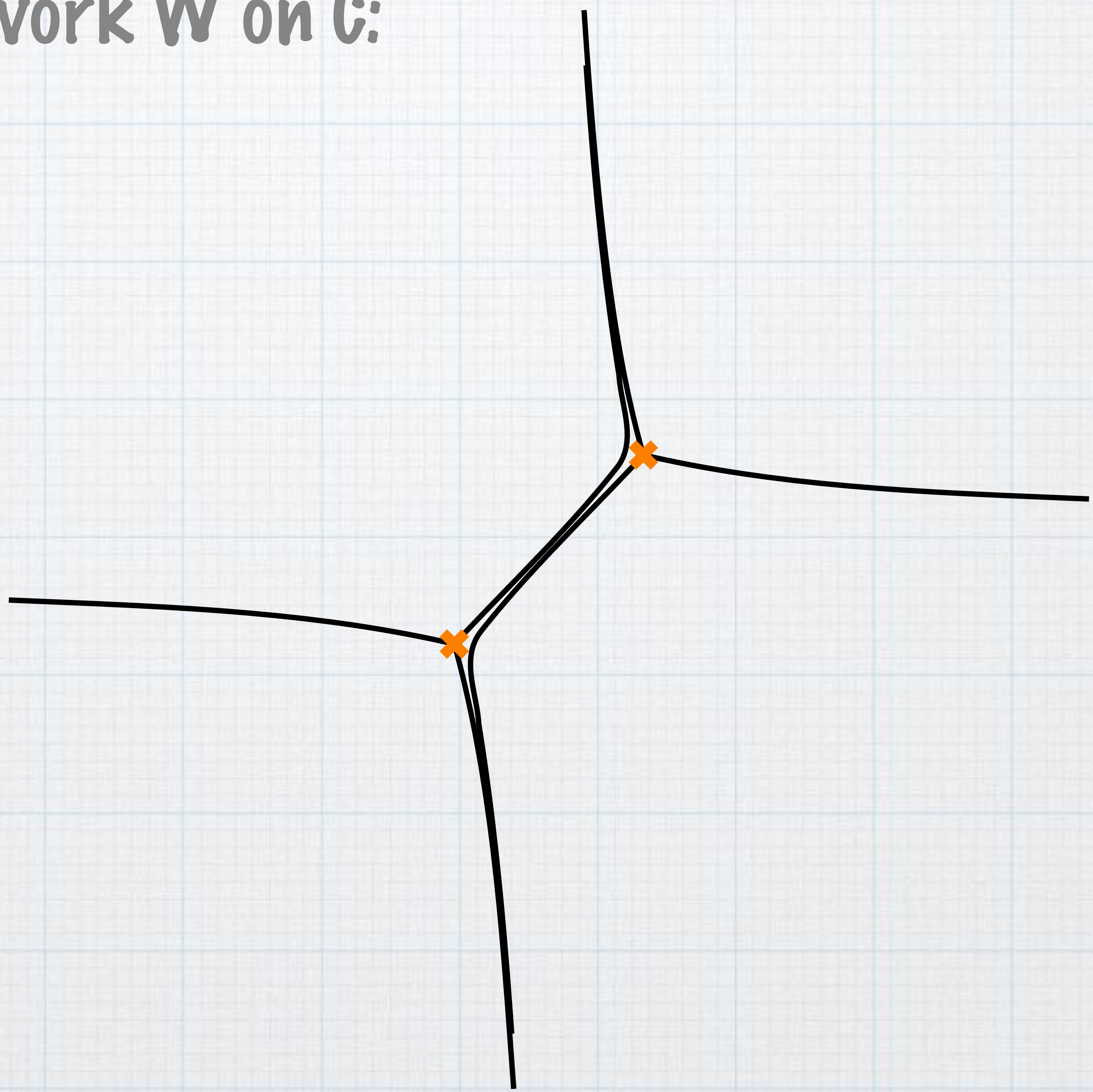
$\vartheta = 0^-$

This hypermultiplet is encoded in a spectral network W on C:



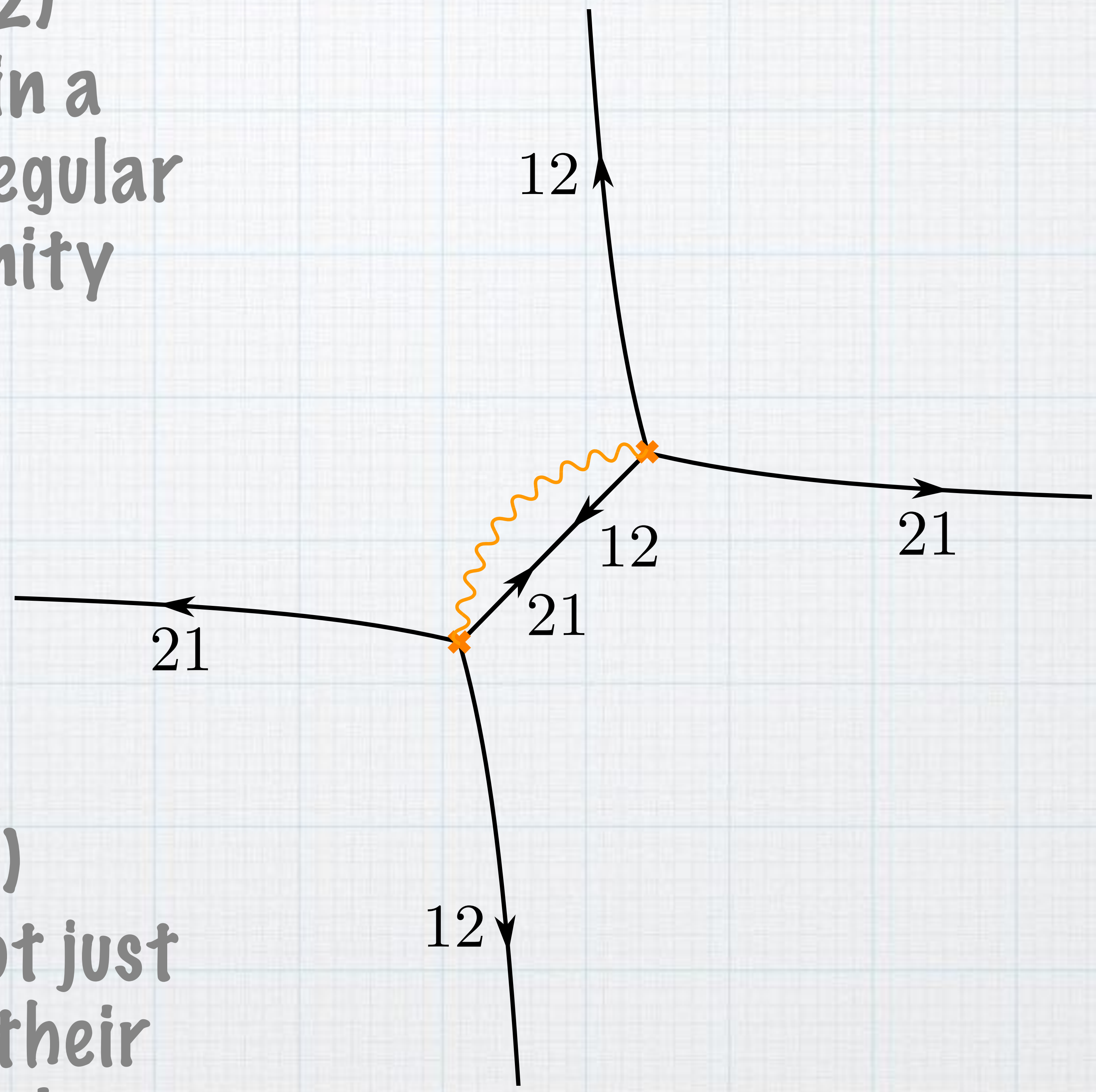
$\vartheta = 0$

This hypermultiplet is encoded in a spectral network W on C:



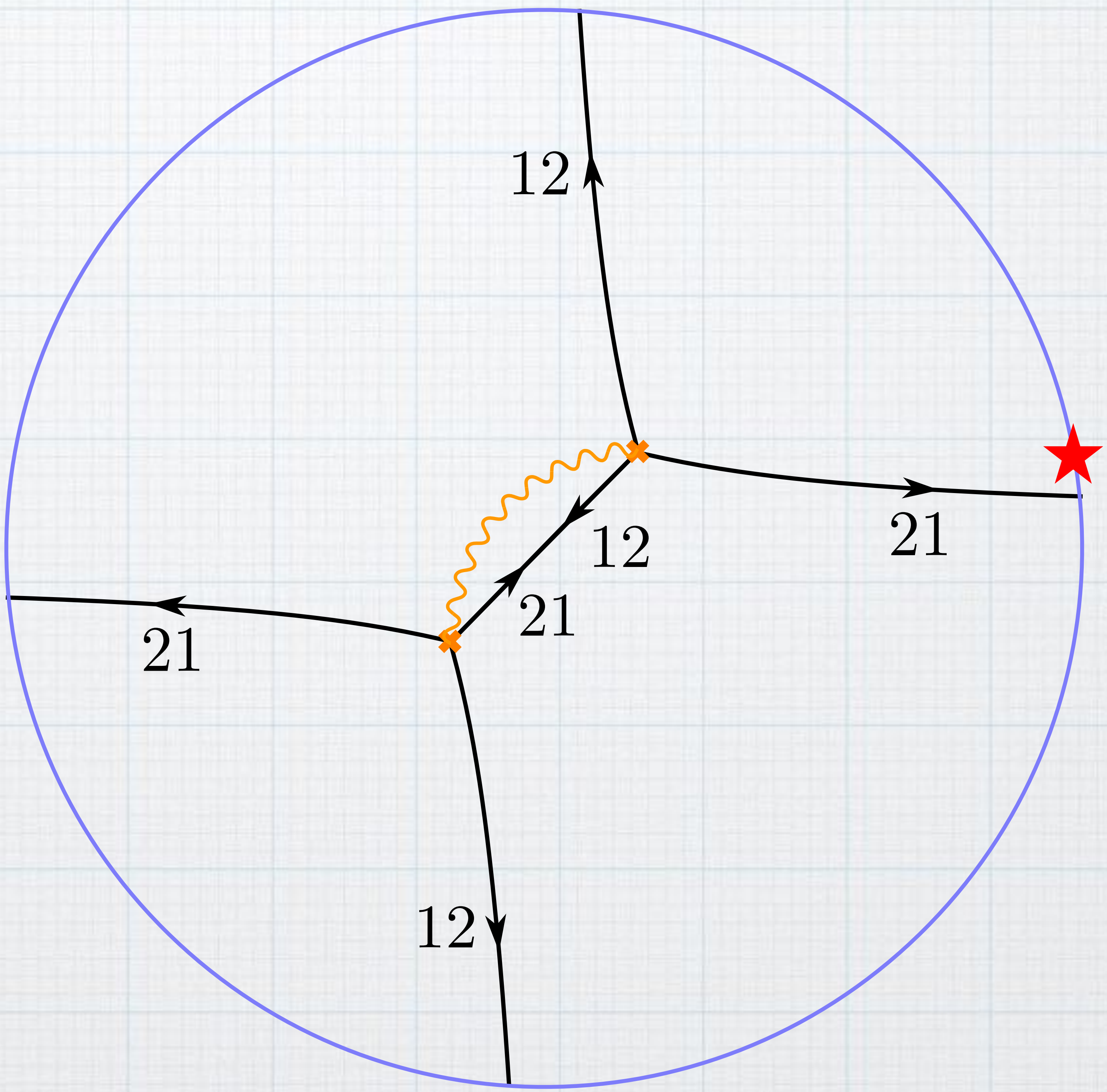
$\vartheta = 0^+$

Consider flat SL(2) connections on C in a bundle E with an irregular singularity at infinity



Such flat SL(2) connections are not just characterised by their monodromy, but also come with Stokes data

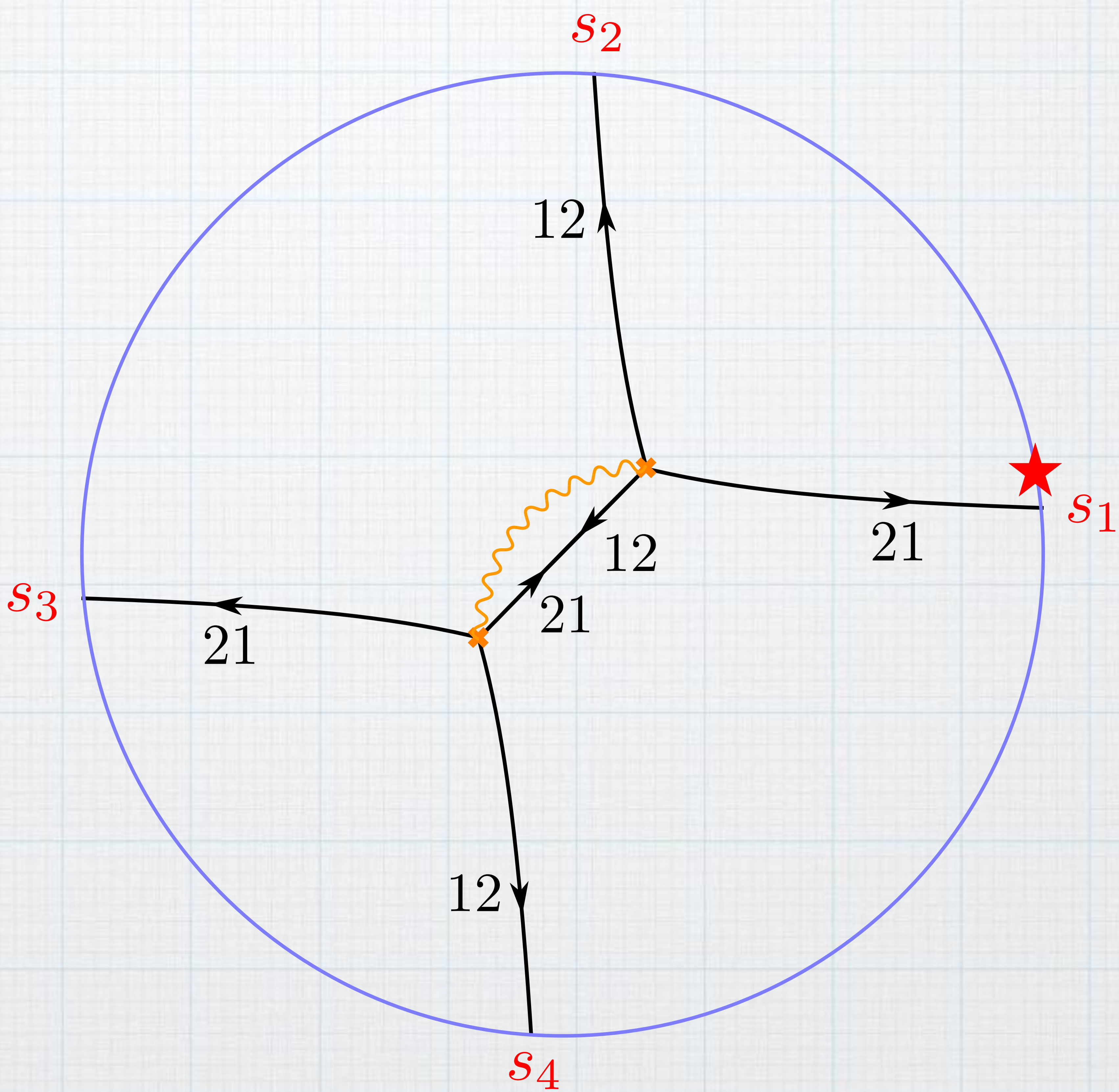
Their moduli space has complex dimension 2 if we fix a trivialisation of E at infinity



Choose sections

$$\nabla s_i(z) = 0$$

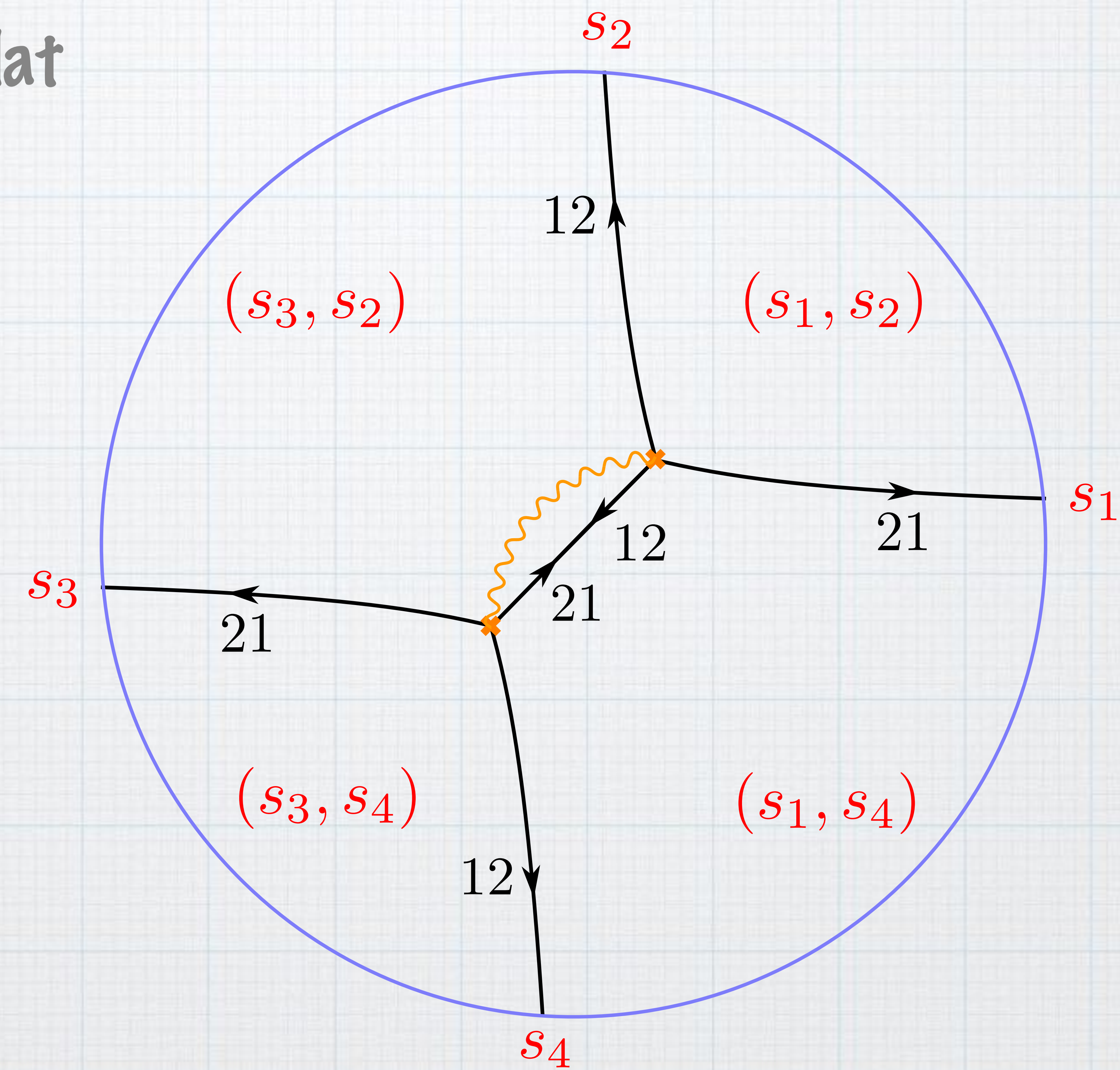
that are asymptotically small



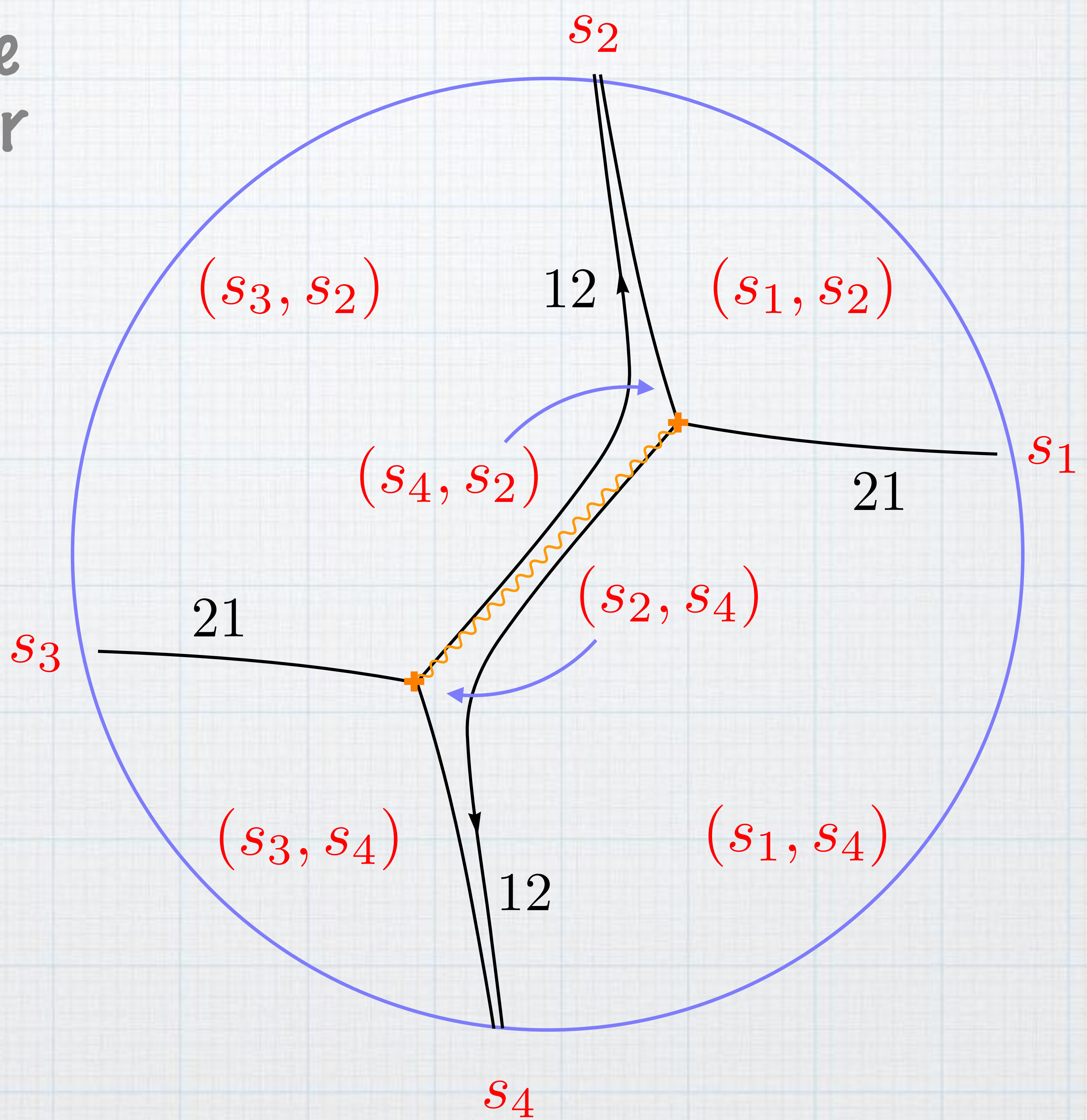
Then we may "abelianize" the flat connection

[GMN,'11]

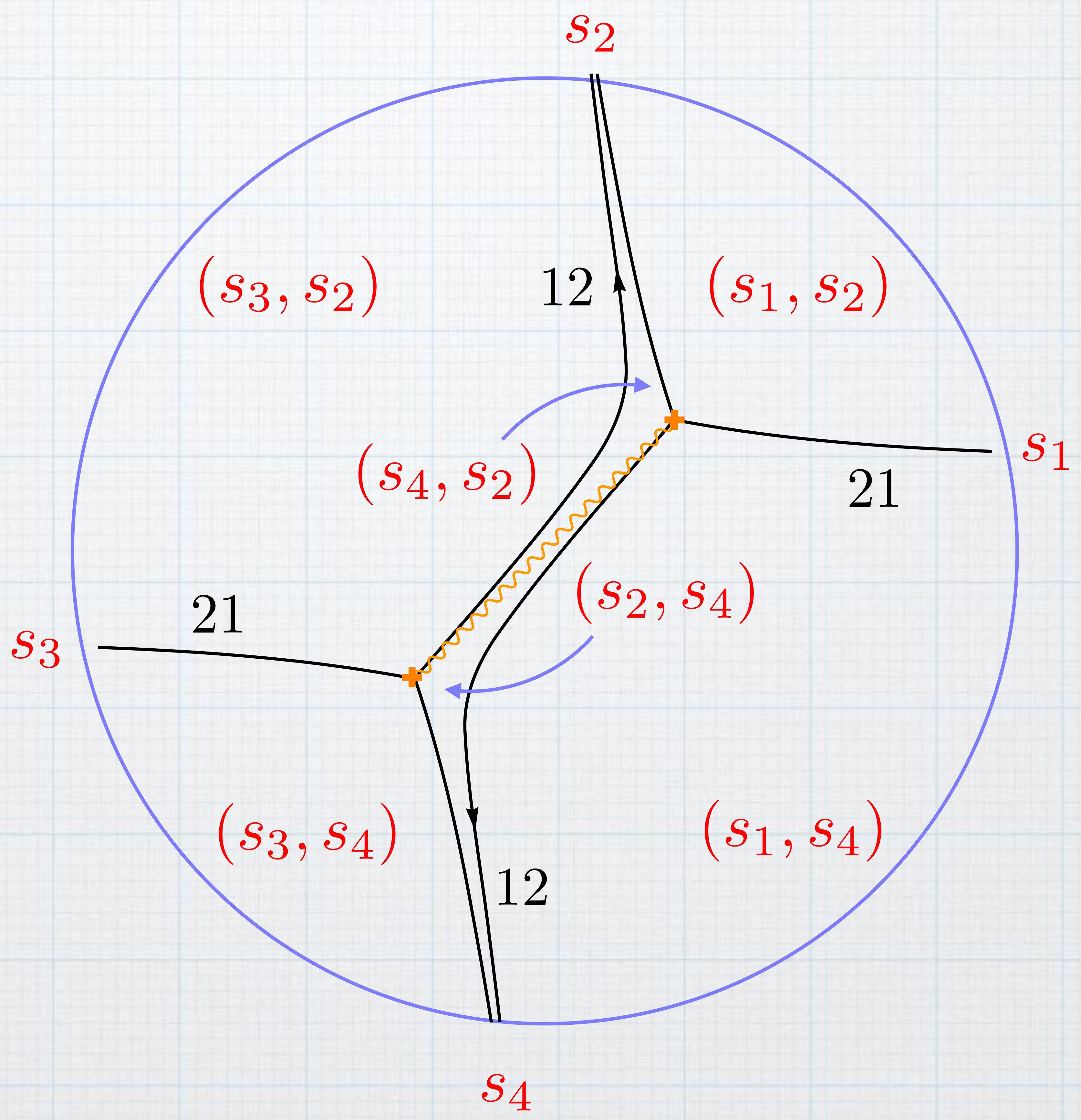
[HN,'13]



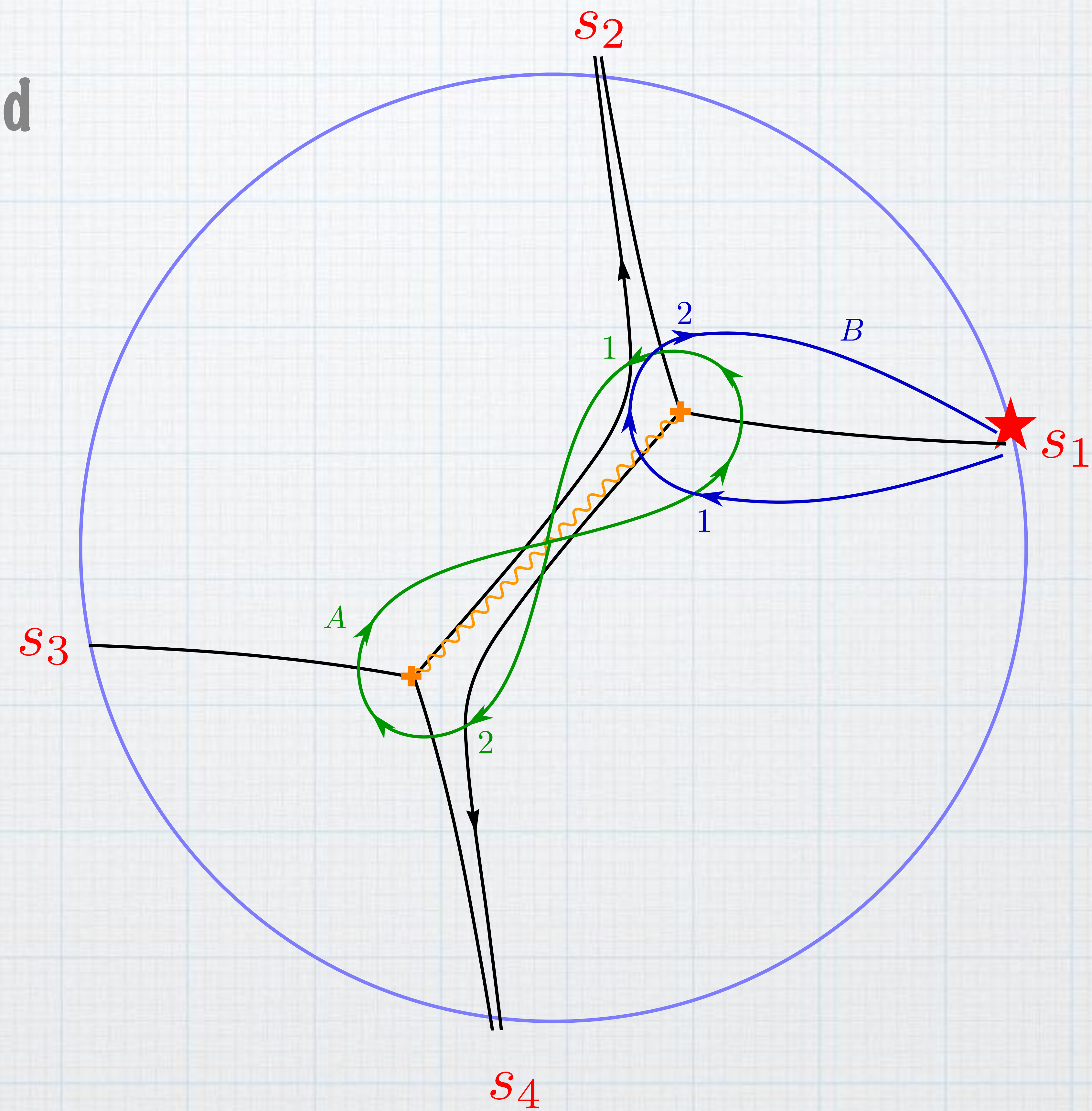
More precisely, we want to do this for either resolution



Finding such a gauge implies that we can lift the flat connection to a GL(1) flat connection on the spectral cover



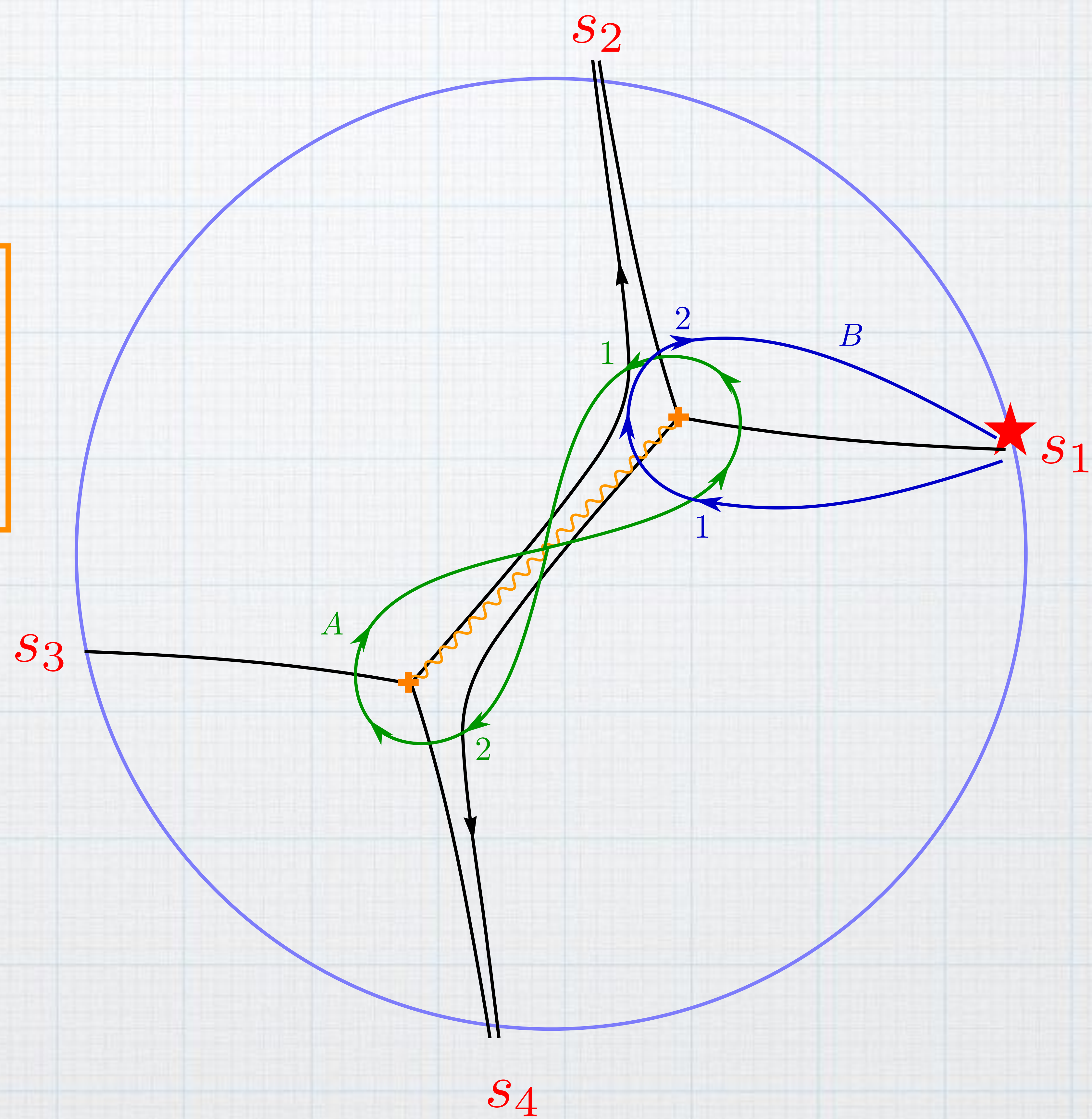
Introduce an A and B-cycle on the spectral cover



Then we find the invariants:

$$X^{-} = \mathrm{Hol}_A \nabla^{\mathrm{ab}}$$

$$Y^{-} = \mathrm{Hol}_B \nabla^{\mathrm{ab}}$$

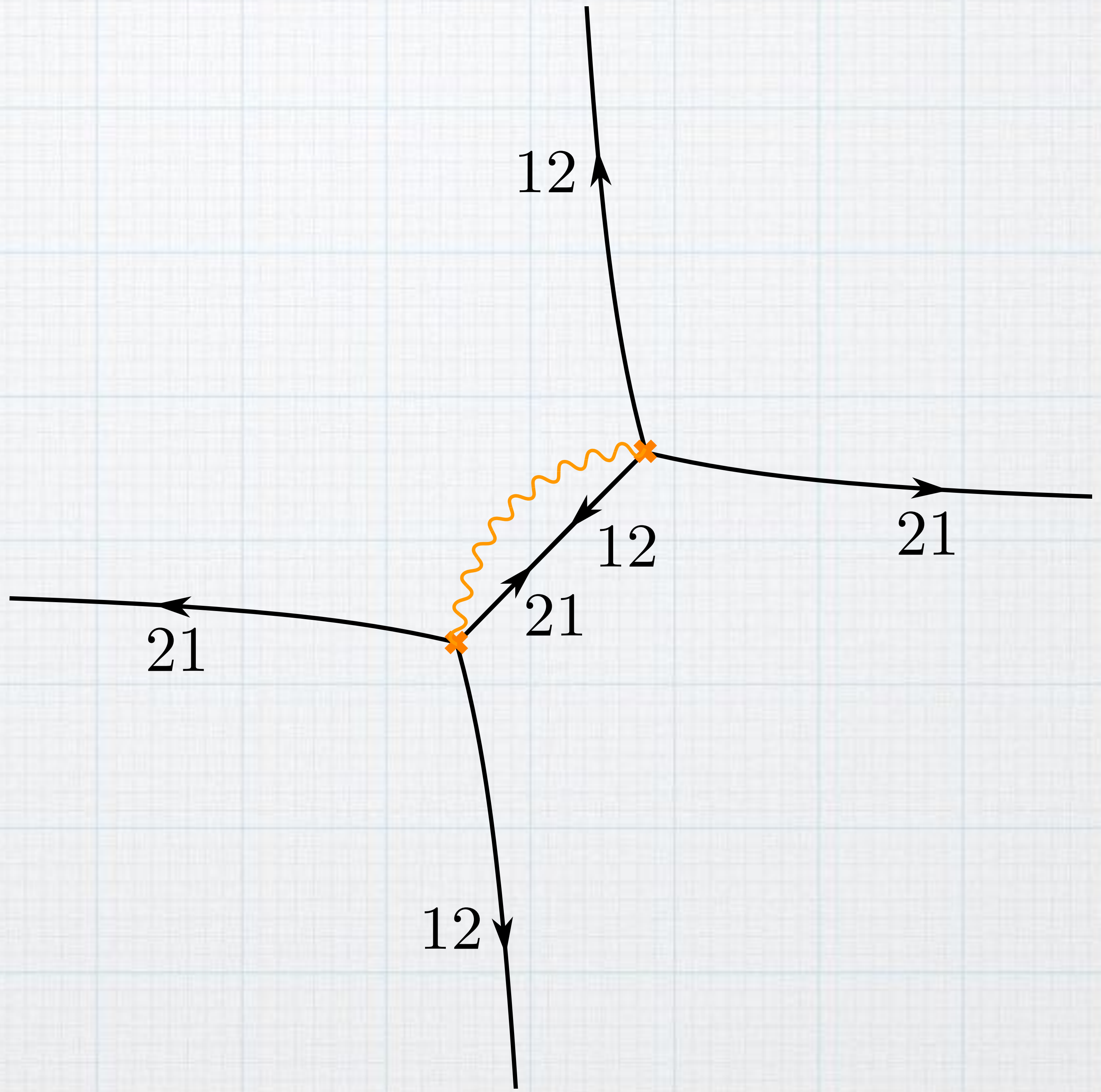


We call

$$X = X^{-} = X^{+}$$

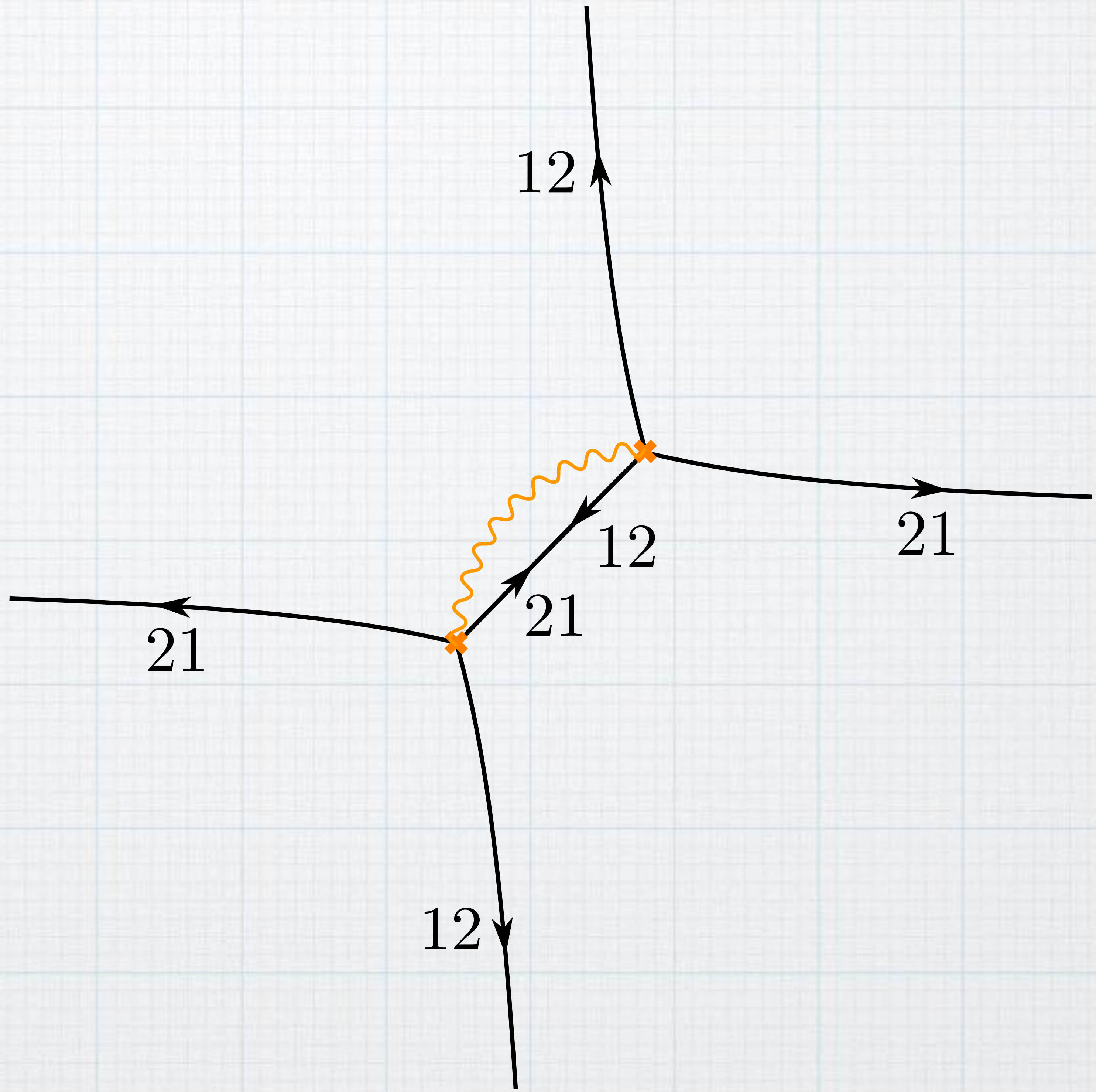
$$Y = \sqrt{Y^{-}Y^{+}}$$

the **spectral coordinates** with respect to the network with saddle



$$x = \log X$$
$$y = \log Y$$

are Darboux coordinates on the moduli space of irregular flat connections



There is a distinguished 1-dimensional complex Lagrangian subspace of flat SL(2) connections, locally given by

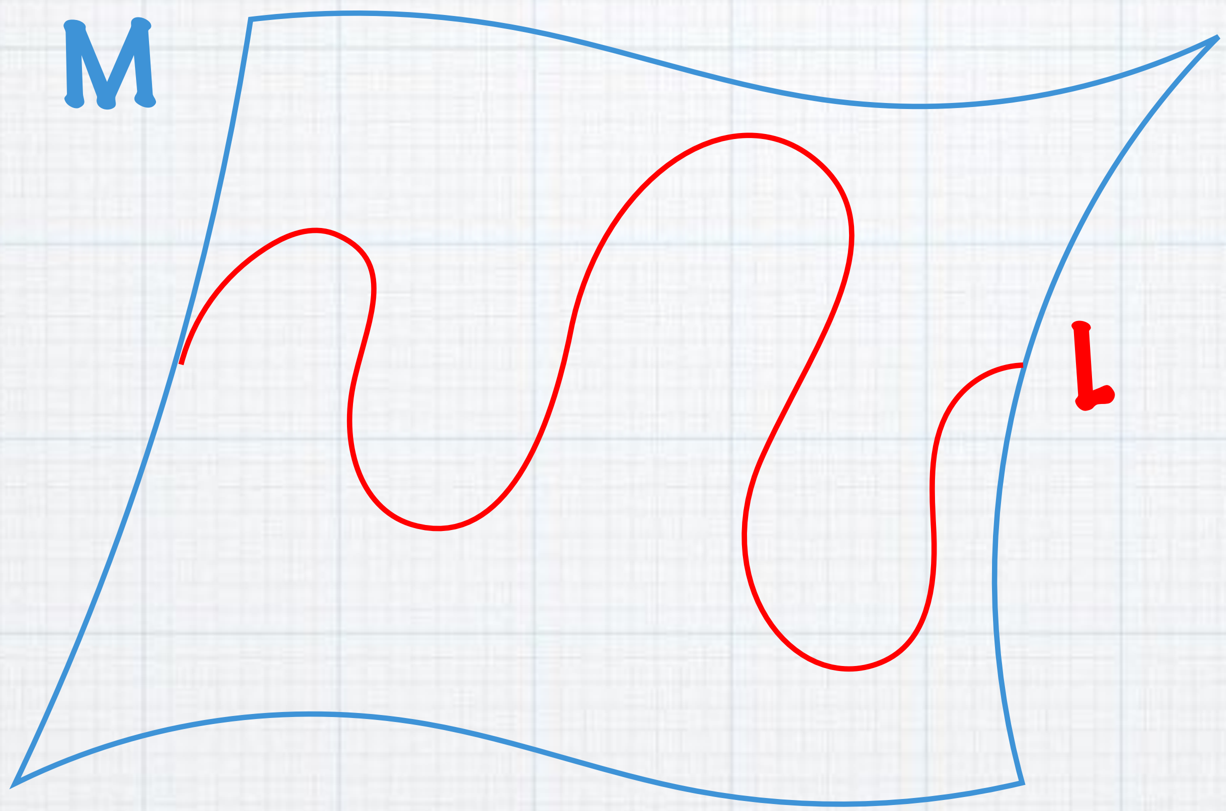
$$\left(-\epsilon^2\partial_z^2 + z^2 + 2m\right) y(z) = 0$$

This is the so-called space of opers

What are the spectral coordinates X and Y restricted to this subspace?

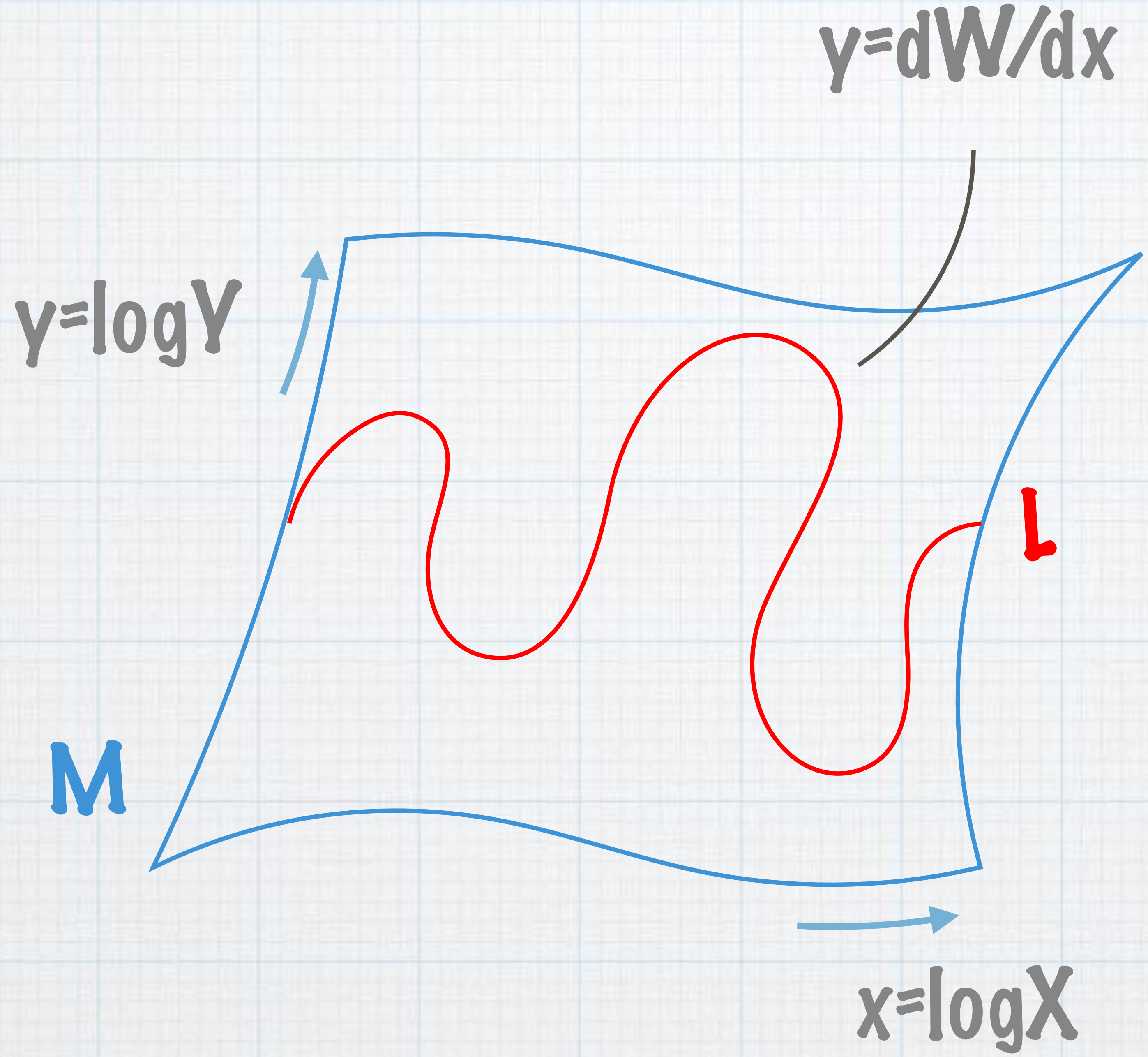
$$X = e^{\frac{2\pi i m}{\epsilon}}$$

$$Y = \sqrt{\frac{\Gamma(\frac{1}{2} - \frac{m}{\epsilon})}{\Gamma(\frac{1}{2} + \frac{m}{\epsilon})}}$$



M = moduli space of irregular flat connections

L = space of harmonic oscillator opers



M = moduli space of irregular flat connections

L = space of harmonic oscillator opers

The generating function of the subspace of opers in the spectral coordinates reads

$$y = \frac{\partial W^{\mathrm{oper}}(x)}{\partial x}$$

and we find

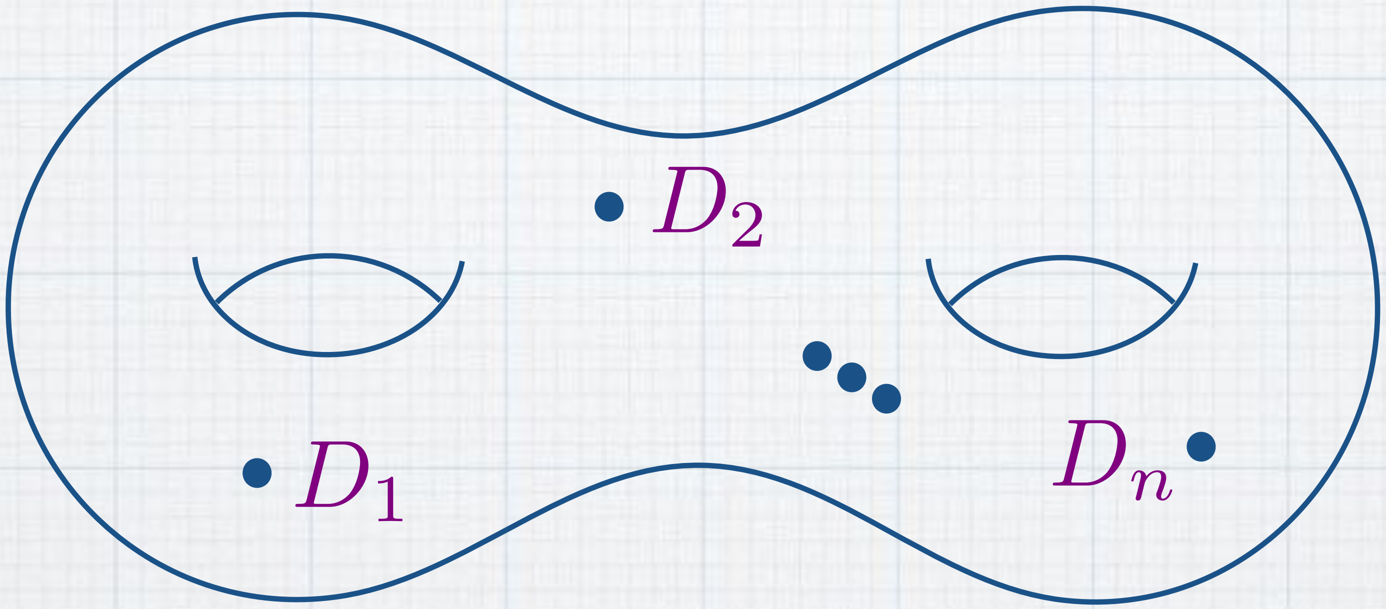
$$W^{\mathrm{oper}}(m) = \frac{1}{2}\int_{\frac{1}{2}}^{m} dm' \log \frac{\Gamma(\frac{1}{2} - \frac{m'}{\epsilon})}{\Gamma(\frac{1}{2} + \frac{m'}{\epsilon})}$$

This is a baby example of a proposal by Nekrasov, Rosly and Shatashvili saying that:

The **effective twisted superpotential** for a theory of class S is equal to the generating function of opers on C in a special choice of Darboux coordinates

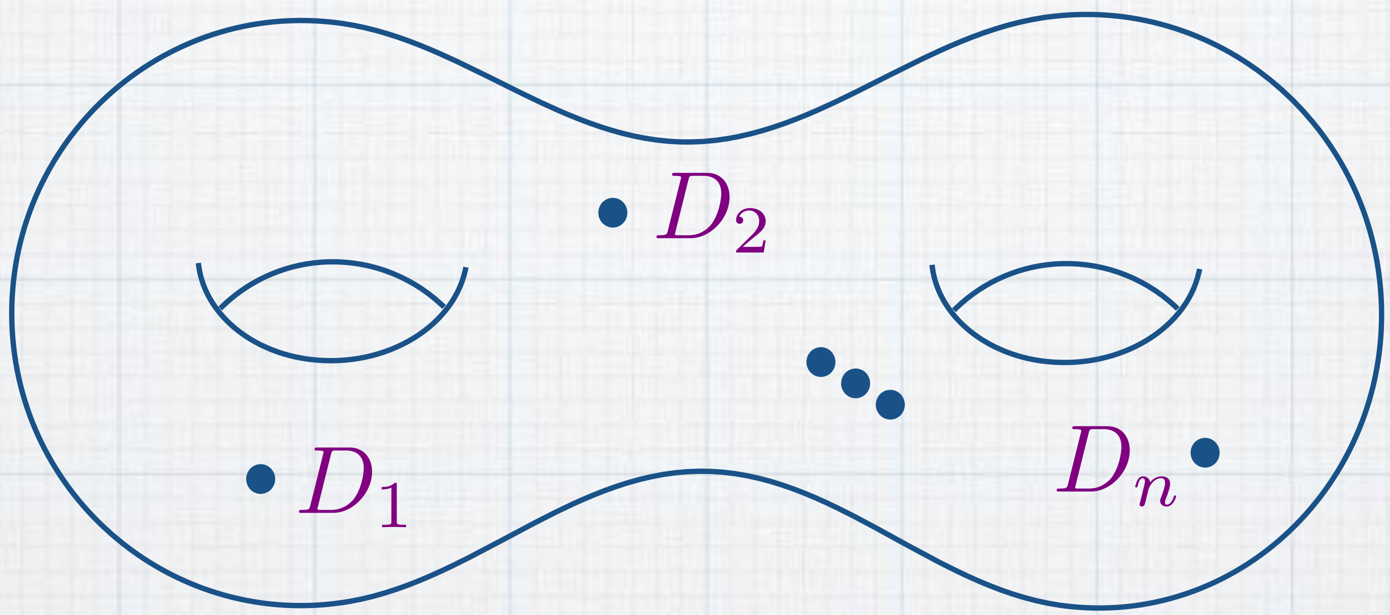
(where they defined these coordinates for K=2 on a surface C with regular punctures)

Compactification of the six-dimensional (2,0) theory X[su(K)] on a Riemann surface C



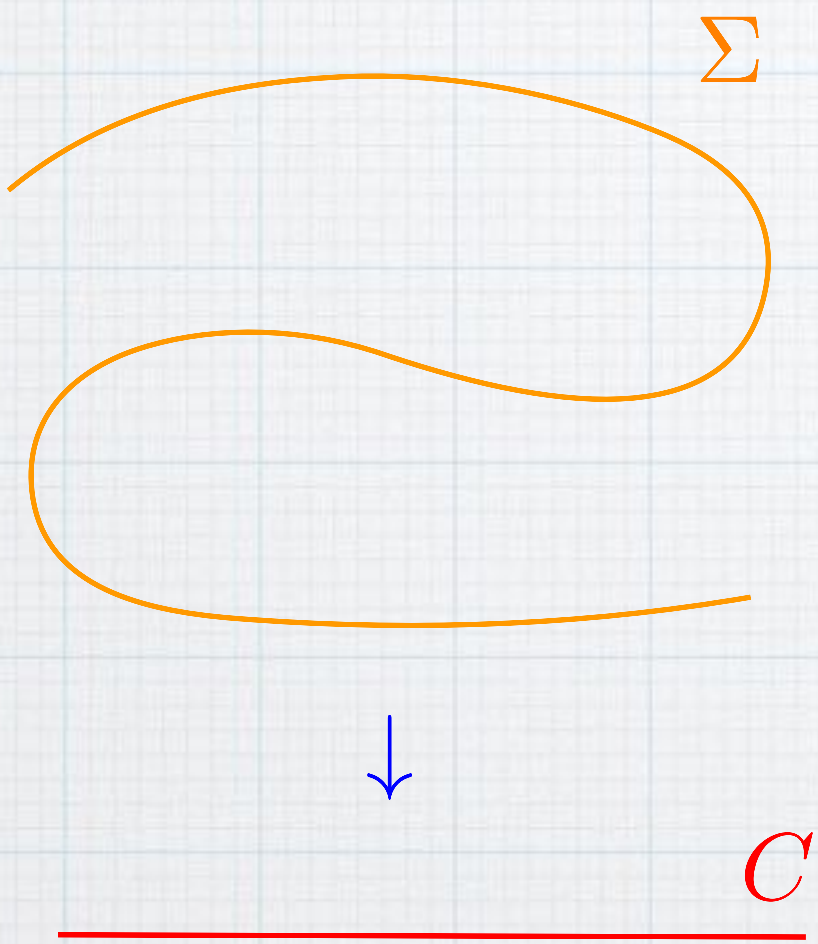
defines a four-dimensional N=2 theory T=S[K,C,D] of class S [Gaiotto, GMN,'11]

Microscopic properties of T encoded in the surface C together with some defect data D at the punctures.



For instance, UV gauge couplings are identified with the complex structure parameters of C

Each spectral curve



$$\Sigma \subset T^*C$$

$$\Sigma : w^K + \varphi_2 w^{K-2} + \ldots + \varphi_K$$

$$\downarrow$$

$$C$$

characterises a Coulomb vacuum of the theory T

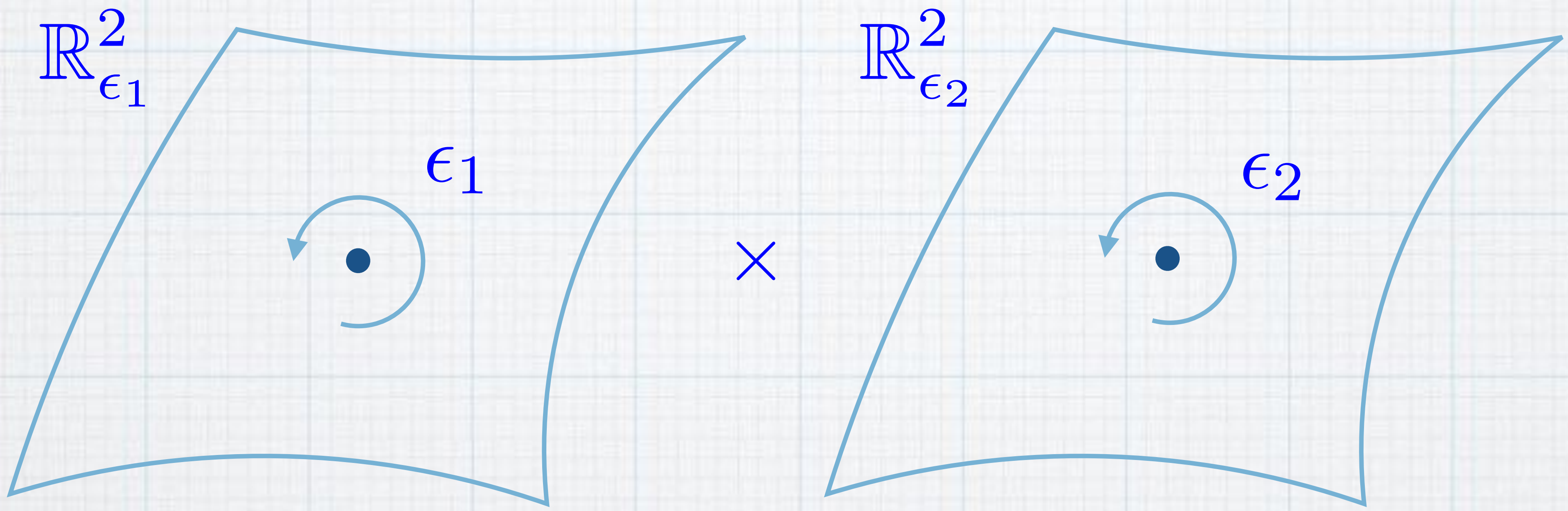
The low energy of T is encoded in the prepotential F

$$a_D = \frac{\partial \mathcal{F}}{\partial a}$$

the periods are defined by

$$a = \int_A \lambda \qquad a_D = \int_B \lambda$$

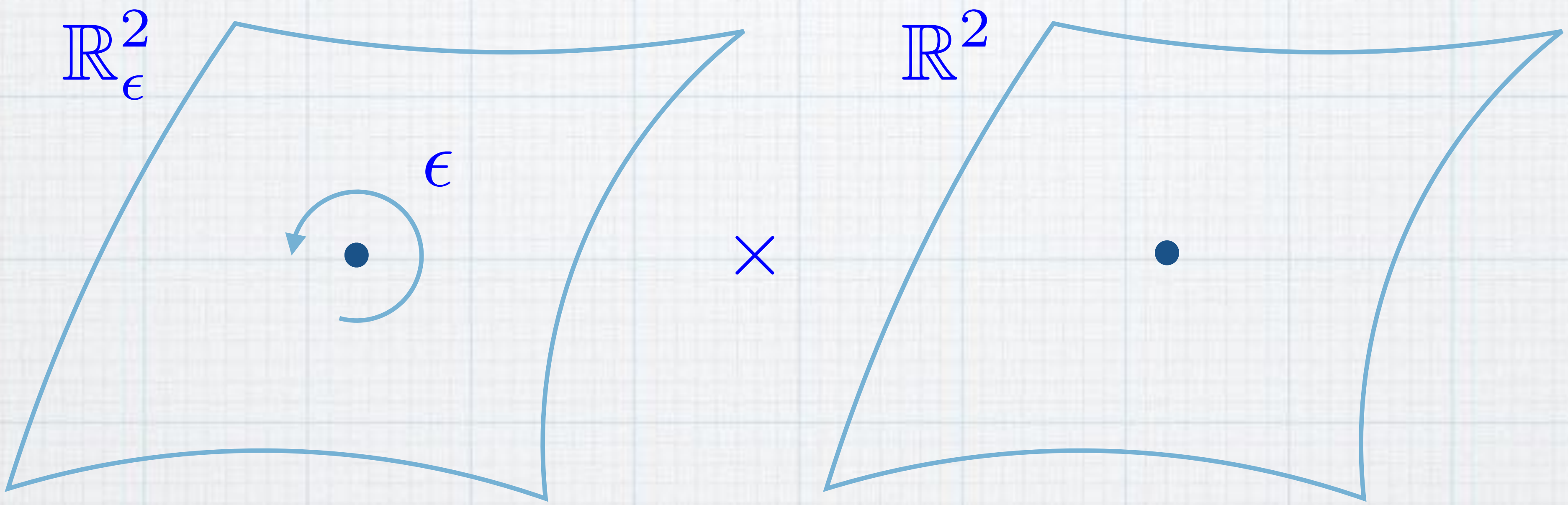
The prepotential F may be computed from first principles by considering a deformation of T



Equivariant localisation leads to the Nekrasov partition function

$$Z^{\text{Nek}}(a, \epsilon_1, \epsilon_2) = \exp \mathcal{F}(a, \epsilon_1, \epsilon_2)$$

Instead, if we only turn on only one deformation parameter



the resulting theory preserves N=(2,2) super-Poincare symmetry

In the infra-red limit T has an effective two-dimensional description in terms of abelian twisted chiral multiplets, coupled to an effective twisted superpotential [Nekrasov-Shatashvili,'09]

$$\widetilde{W}^{\text{eff}} = \lim_{\epsilon_2 \to 0} \epsilon_2 \, \mathcal{F}(\epsilon_1 = \epsilon, \epsilon_2)$$

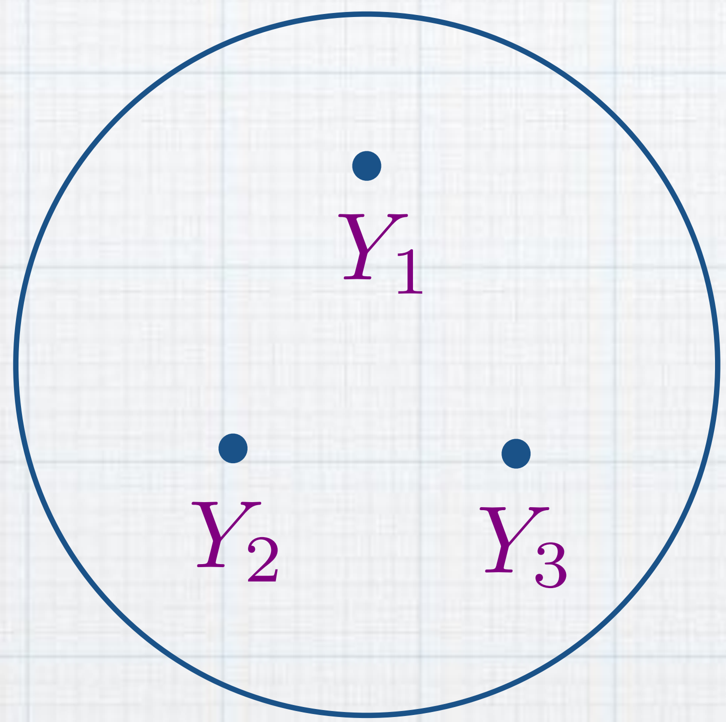
which can be computed if T has a Lagrangian prescription

$$\widetilde{W}^{\text{eff}} = \widetilde{W}^{\text{cl}} \log q + \widetilde{W}^{1-\text{loop}} + \widetilde{W}^{\text{inst}}(q)$$

For theories of class S the effective twisted superpotential is the Yang-Yang function of the quantised Hitchin system
[Nekrasov-Shatashvili,'09]

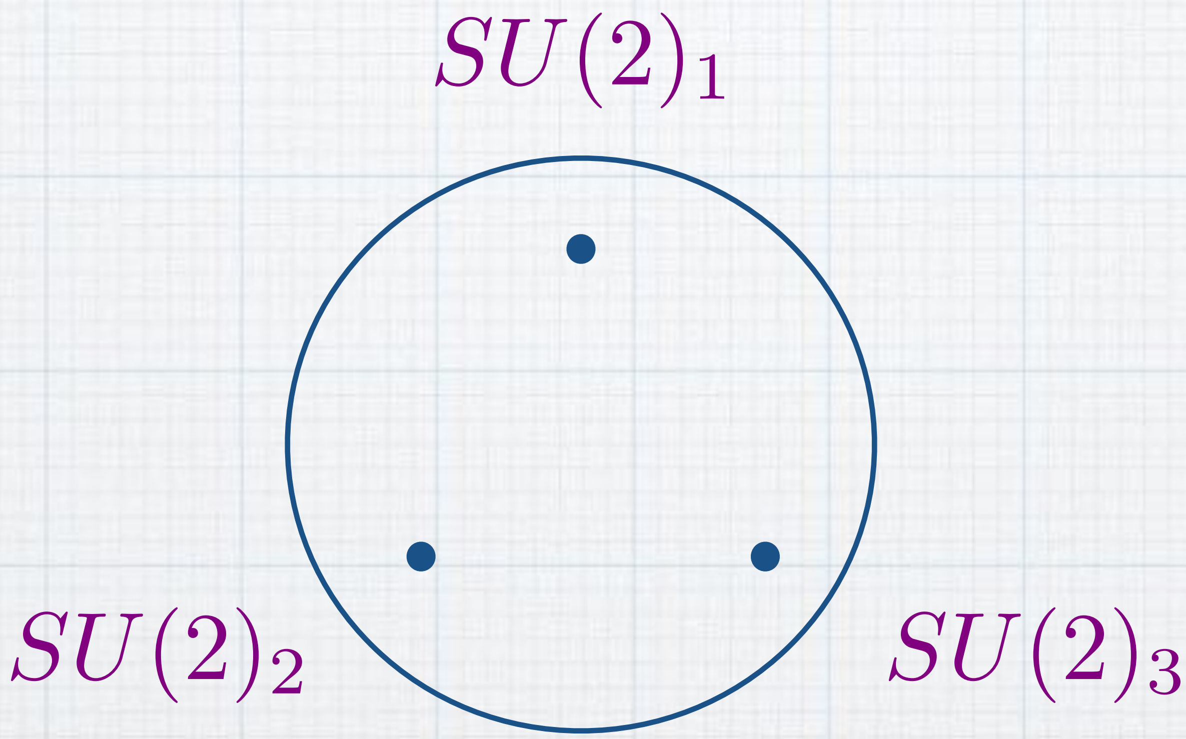
The opers are the eigenvalues of this quantum integrable system

The building blocks of superconformal N=2 theories of class S[K,C,D] are the three-punctured spheres



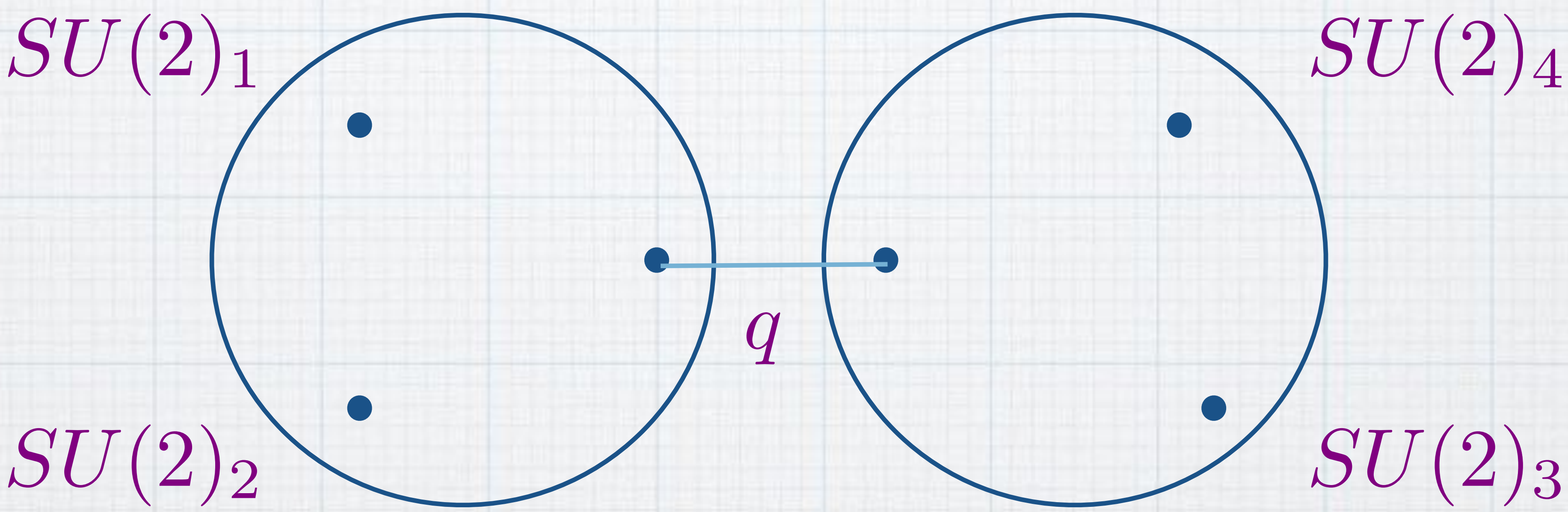
where the Y are Young diagrams with K boxes

For K=2 there is a single building block



corresponding to a half-hypermultiplet in the trifundamental representation

Gluing two of these building blocks



leads to the superconformal SU(2) theory coupled to four hypermultiplets

This is the main example for NRS:

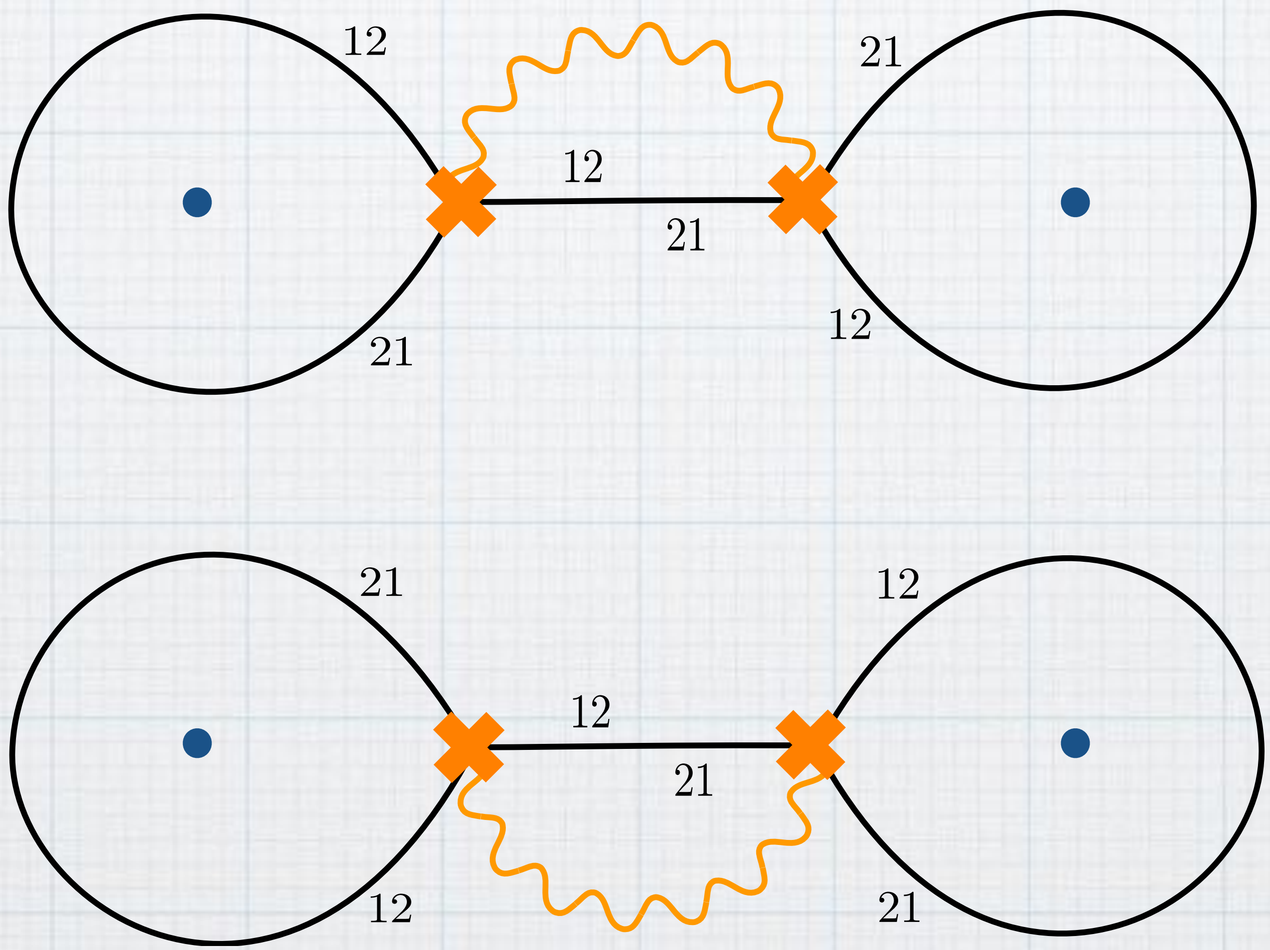
They introduced Darboux coordinates on the moduli space of flat SL(2) connections with regular singularities on C

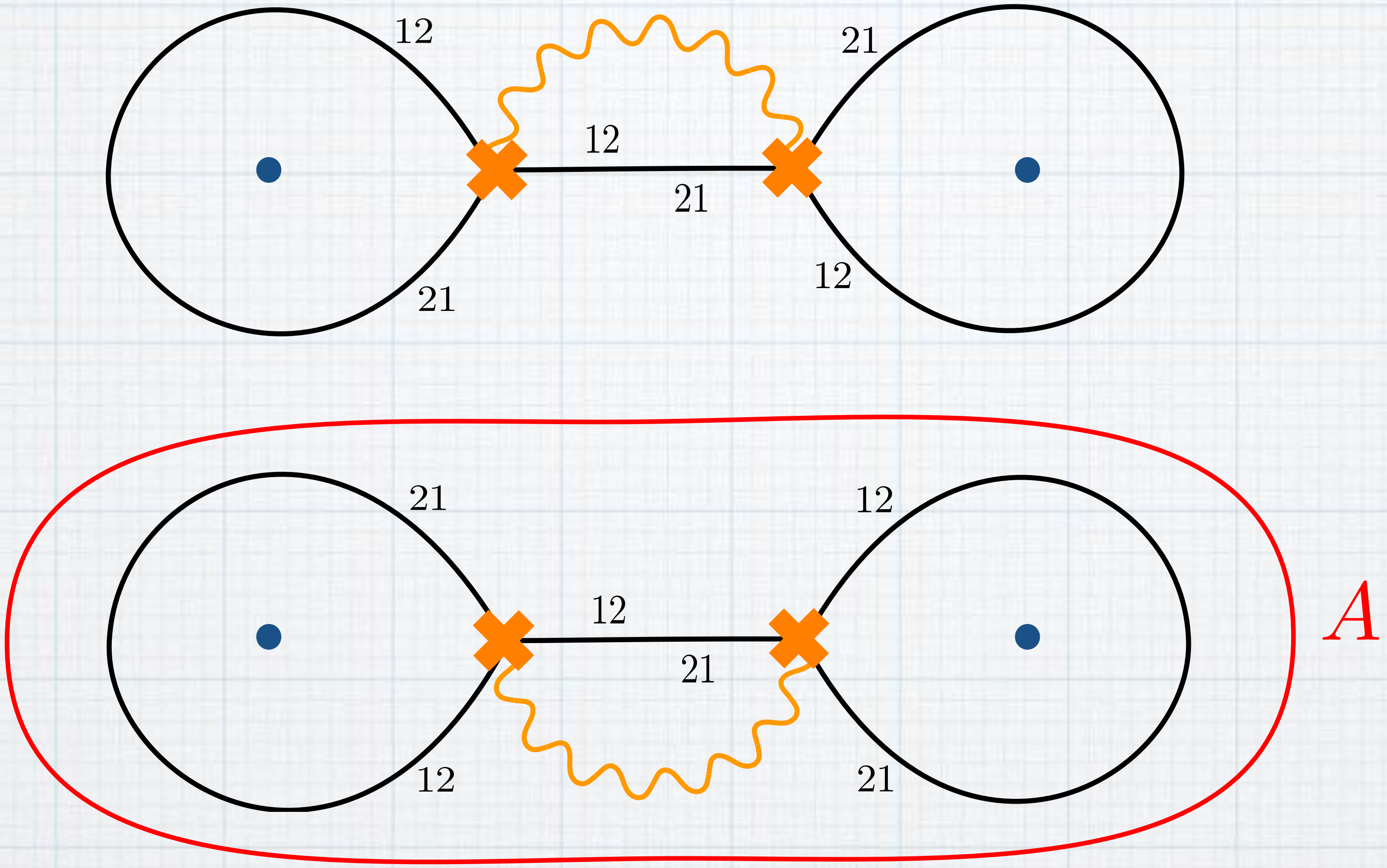
And noticed that the brane of opers is characterised by the Fuchsian differential equation

$$y''(z) = \left(\frac{\delta_0}{z^2} + \frac{\delta}{(z-q)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{z(z-1)} + \frac{H}{z(z-q)(z-1)} \right) y(z)$$

also known as Heun's oper

The NRS Darboux coordinates may be found as well by abelianizing with respect to a so-called Fenchel-Nielsen network [H-Kidwai,'17][H-Neitzke,'13]

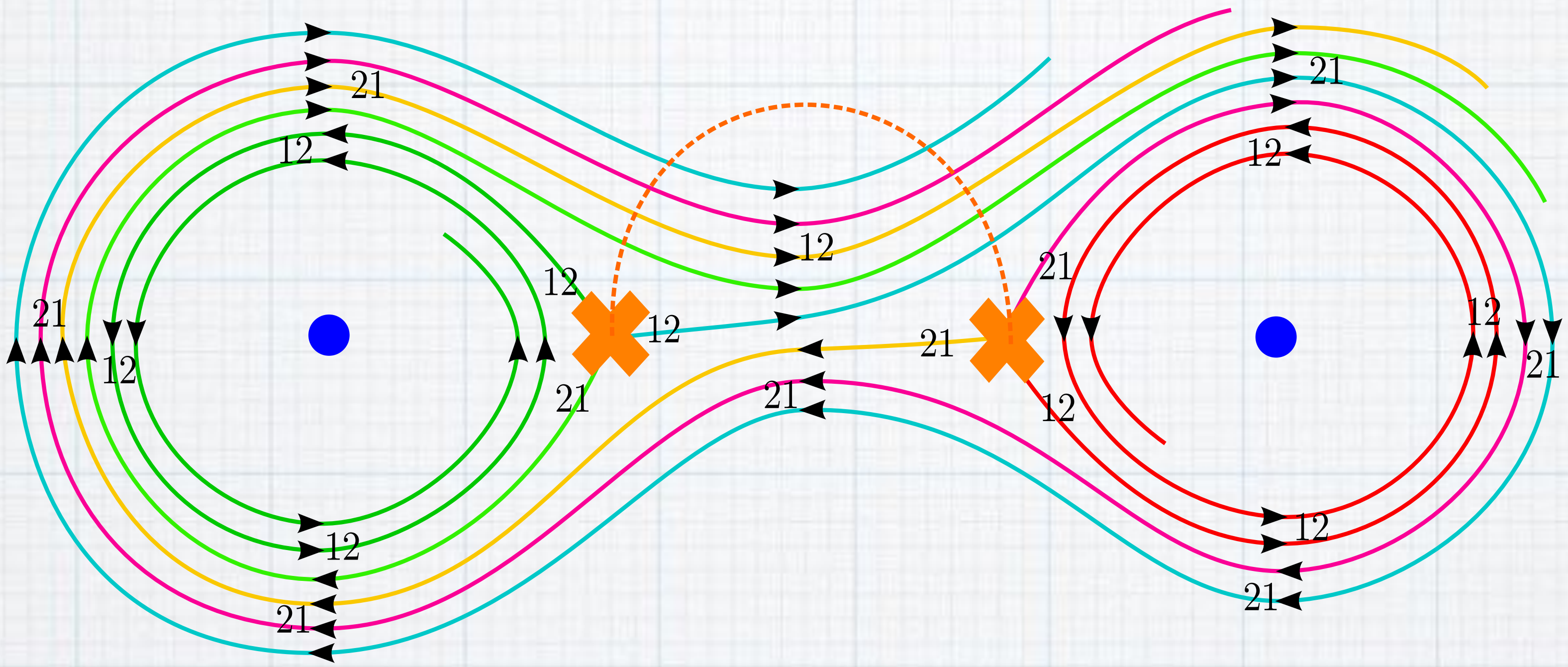




defined by a Strebel differential

$$e^{-i\vartheta} \oint_A \lambda \in \mathbb{R}$$

Changing the phase slightly



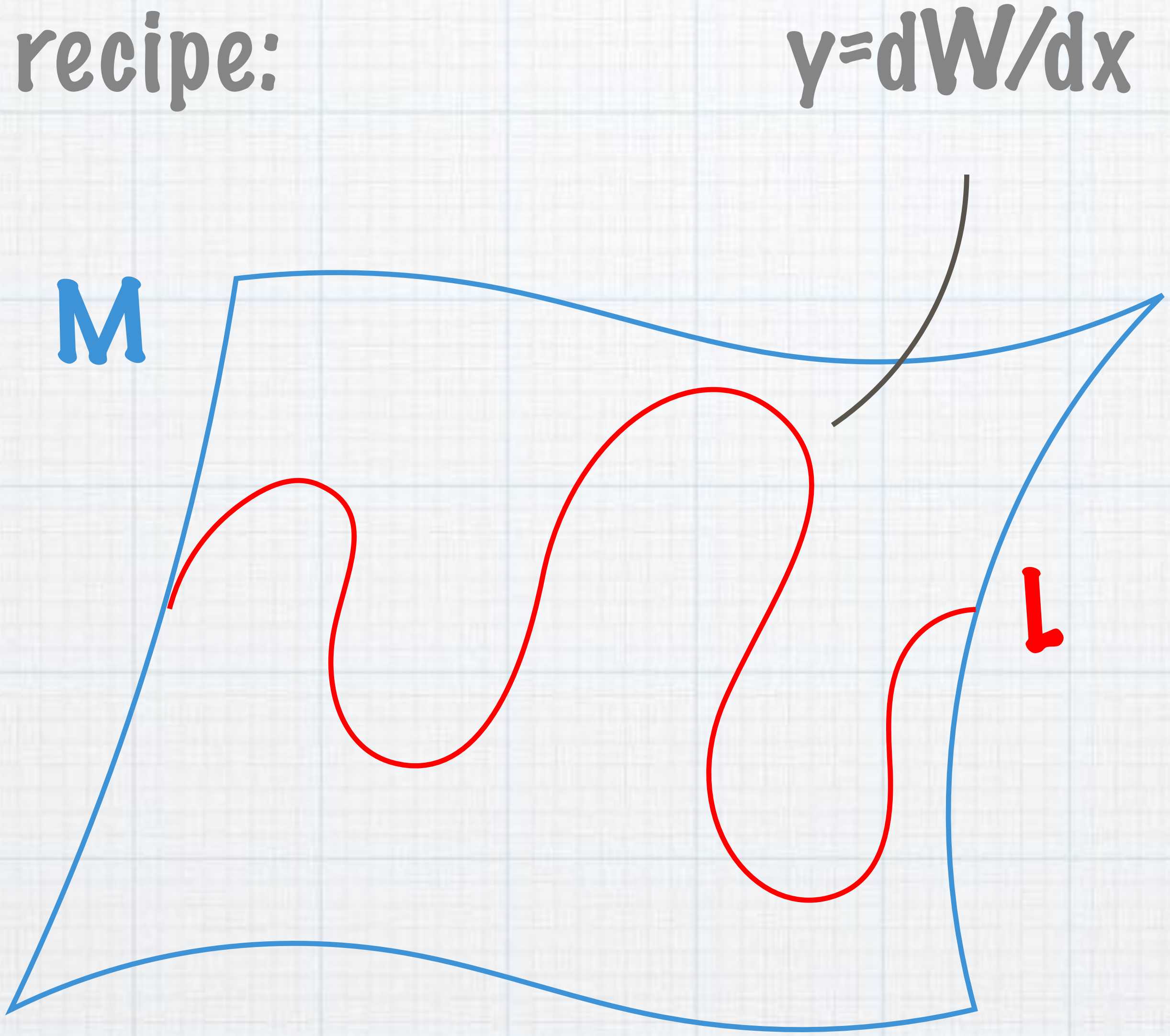
there are two resolutions of the network

The NRS coordinates are obtained as the spectral coordinates:

$$X = X^- = X^+$$

$$Y = \sqrt{Y^- Y^+}$$

Geometric recipe:



L = space of Heun's opers

M=moduli space of flat connections on 4-punctured sphere

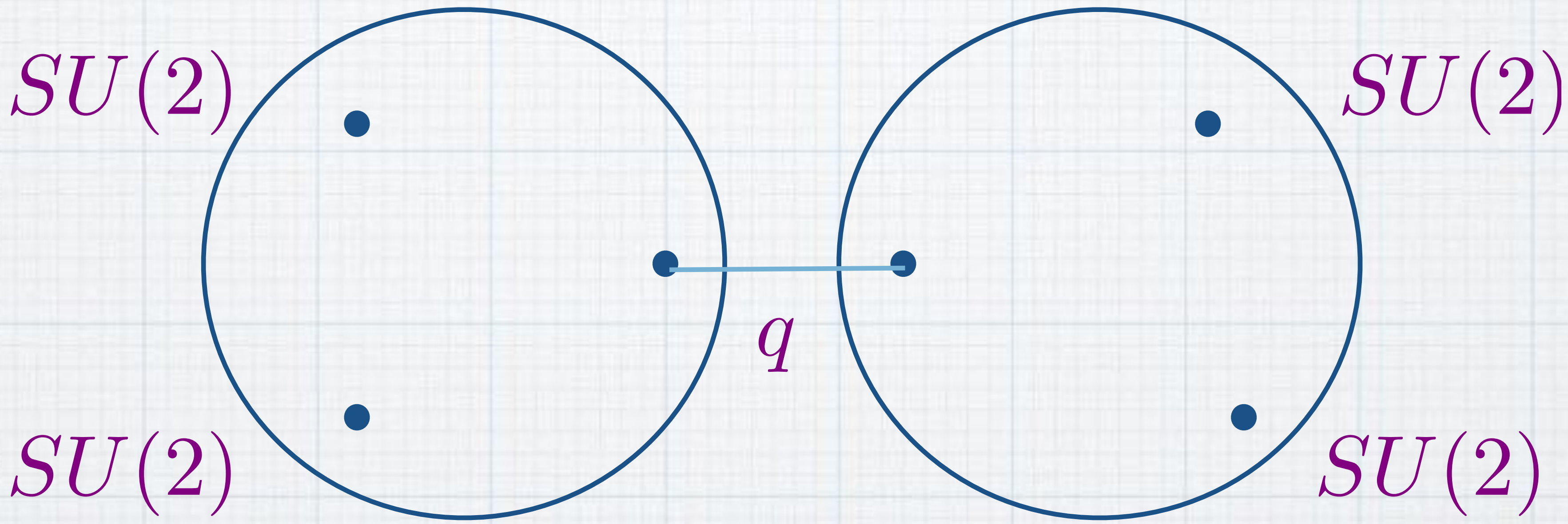
Geometric recipe:

1) Compute the monodromy representation for any flat connection on the 4-punctured sphere in terms of the NRS Darboux coordinates

2) Compute the monodromy representation for the Heun differential equation in an expansion of q

3) Extract the generating function of Heun opers by comparing the two results

The monodromy representation of the Heun differential equation may be computed by perturbing in q



In the limit q->0 the Heun opers are described as a hypergeometric oper on either three-punctured sphere

$$y''(z) = \left(\frac{\delta_0}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta}{z(z-1)} \right)$$

As a result we find [H-Kidwai,'17]:

$$W^{\text{oper}} = \widetilde{W}^{\text{cl}} \log q + \widetilde{W}^{1-\text{loop}} + \widetilde{W}^{\text{inst}}(q)$$

with

$$\widetilde{W}^{\text{cl}} = -\frac{a^2}{\epsilon}$$

$$\widetilde{W}^{1-\text{loop}} = -\frac{1}{2}\Upsilon\left(-\frac{a}{\epsilon}\right) - \frac{1}{2}\Upsilon\left(\frac{a}{\epsilon}\right) + \frac{1}{2}\sum_{i=1}^{4}\Upsilon\left(\frac{\epsilon + a + m_l}{2\epsilon}\right) + \frac{1}{2}\sum_{i=1}^{4}\Upsilon\left(\frac{\epsilon - a + m_l}{2\epsilon}\right)$$

$$\widetilde{W}^{\text{inst}} = \left(\frac{a^2}{4\epsilon^2} + \frac{\prod_{l=1}^{4} m_l}{4(a+\epsilon)(a-\epsilon)}\right) q + \mathcal{O}(q^2)$$

This computation may be generalised to any theory of class S, by computing the generating function of opers in terms of the spectral coordinates defined by a (higher rank) Fenchel-Nielsen network

What is spectral network on C?

[Gaiotto-Moore-Neitzke,'12]

First fix a tuple of k-differentials over C

$$u = (\varphi_2, \dots, \varphi_K)$$

This defines a ramified spectral covering of degree K over C

$$\Sigma : \quad \lambda^K + \varphi_2 \lambda^{K-2} + \ldots + \varphi_K = 0 \quad \subset T^*C$$

What is spectral network on C?

Now fix a phase

$$\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$$

The trajectories of the spectral network are then defined as paths on C obeying

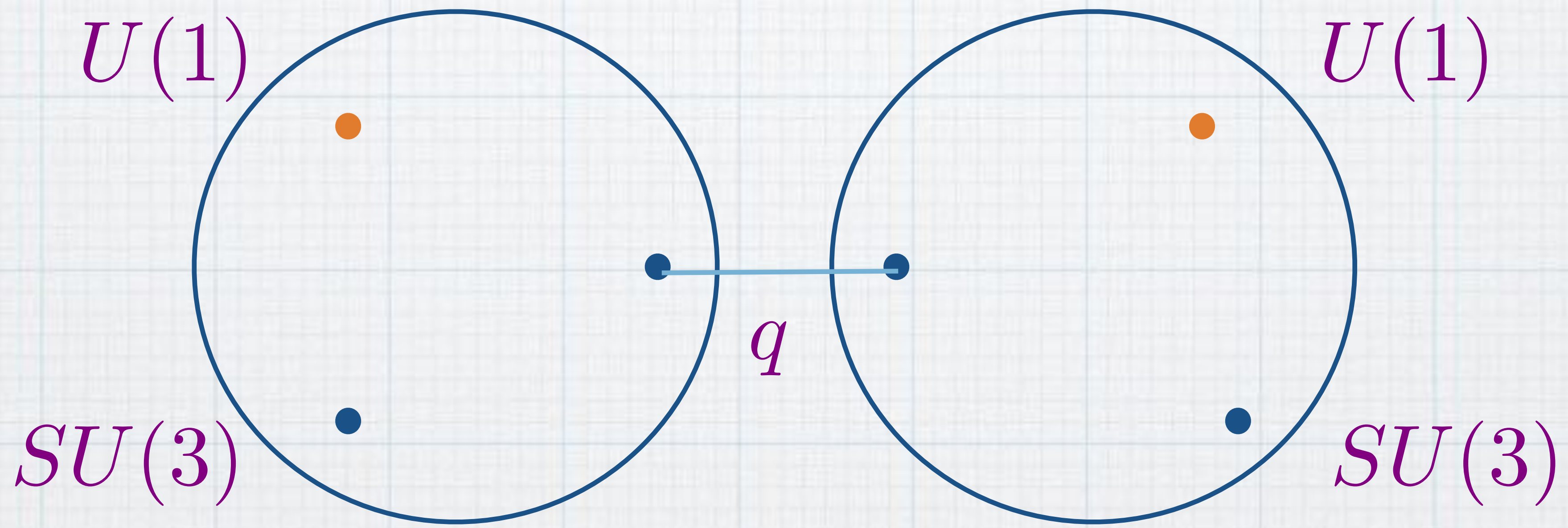
$$e^{-i\vartheta}(\lambda_i - \lambda_j)(v) \in \mathbb{R}_+$$

A (higher rank) Fenchel-Nielsen network is defined by a generalised Strebel differential

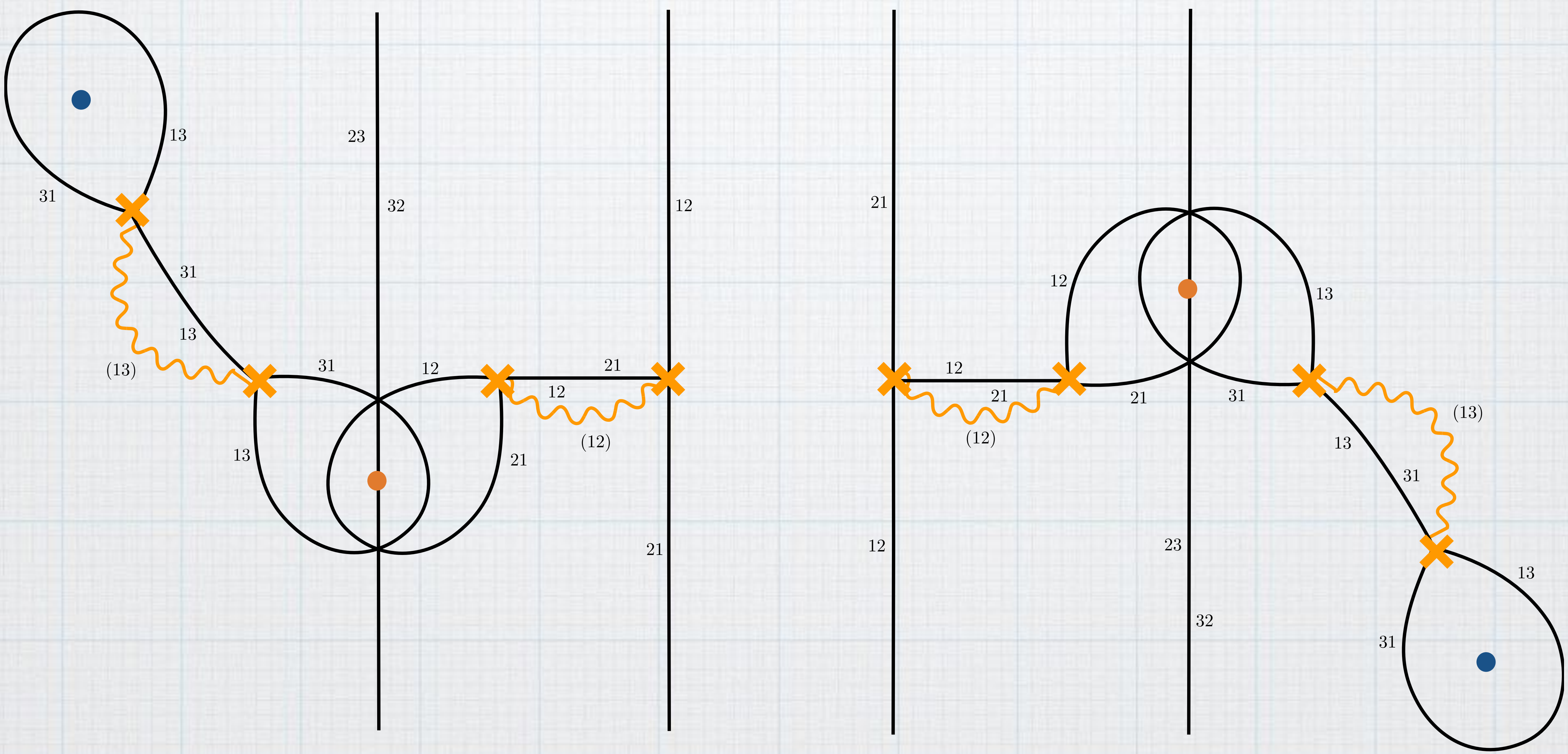
$$e^{-i\vartheta} \oint_A \lambda \in \mathbb{R}$$

for a choice of A-cycles on the spectral curve

Another example is the superconformal SU(3) theory coupled to six hypermultiplets



Generalizations of the NRS Darboux coordinates are found as spectral coordinates for the higher Fenchel-Nielsen network [H-Kidwai,'17]



The space of opers is parametrised by generalised Heun's opers [H-Kidwai,'17]

These are found from generic Fuchsian opers of degree 3 on the four-punctured sphere by restrictions at the minimal punctures:

$$(z-z_*)y'''(z)+p_1(z)y''(z)+p_2(z)y'(z)+p_3(z)y(z)=0$$

analytic

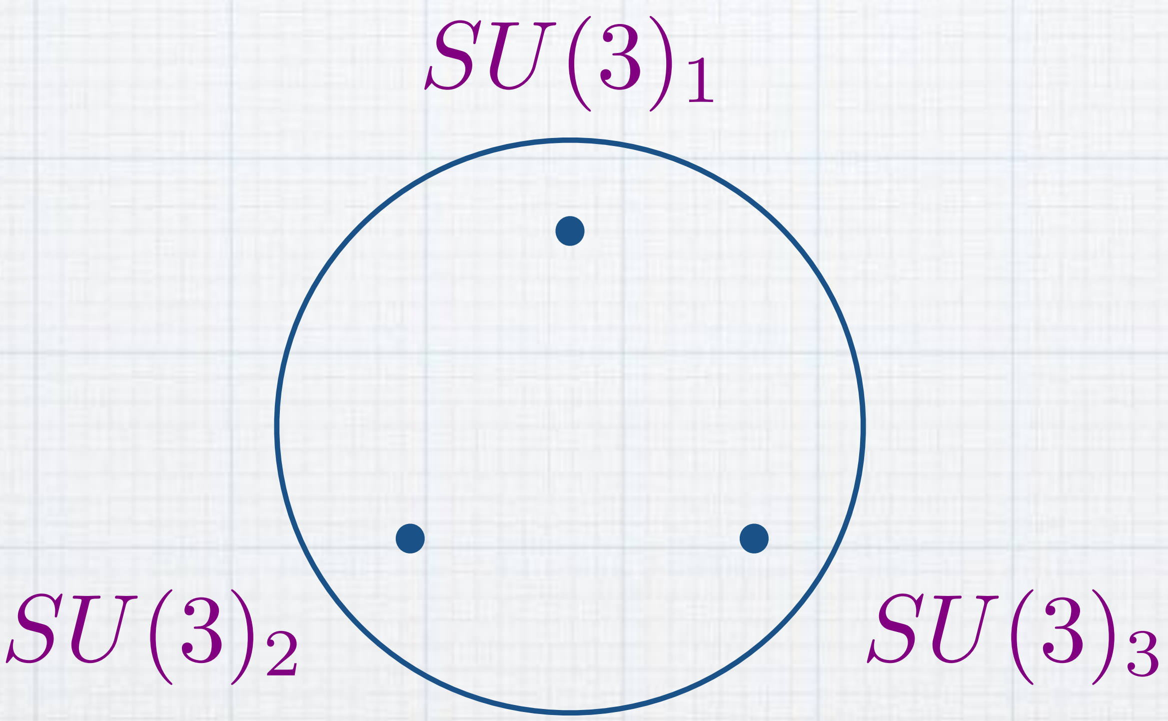
This enforces that two of solutions at the minimal puncture are holomorphic at that puncture, and thus that the monodromy around the minimal puncture is diagonalizable

Following our geometric recipe we compute the generating function of opers by perturbing in the complex structure parameter q

This again yields the known NS effective twisted superpotential

Interestingly, the superconformal SU(2) and SU(3) computations give the superpotential in a perturbation theory in q, but exact in the Omega-deformation parameter

Not all theories of class S have such a simple Lagrangian prescription. Perhaps the simplest example is

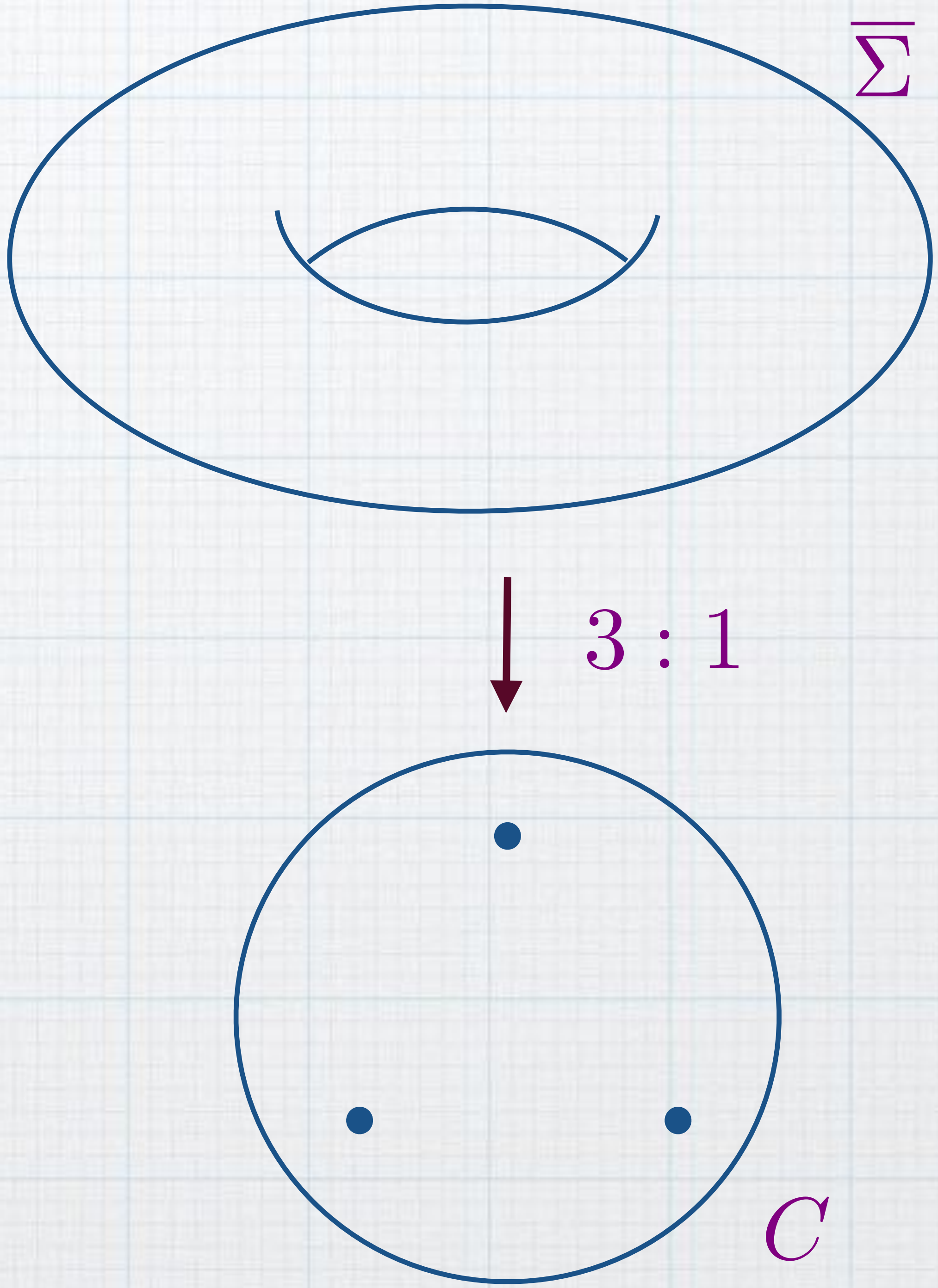


which corresponds to the intrinsically strongly coupled E6 Minahan-Nemeschansky theory

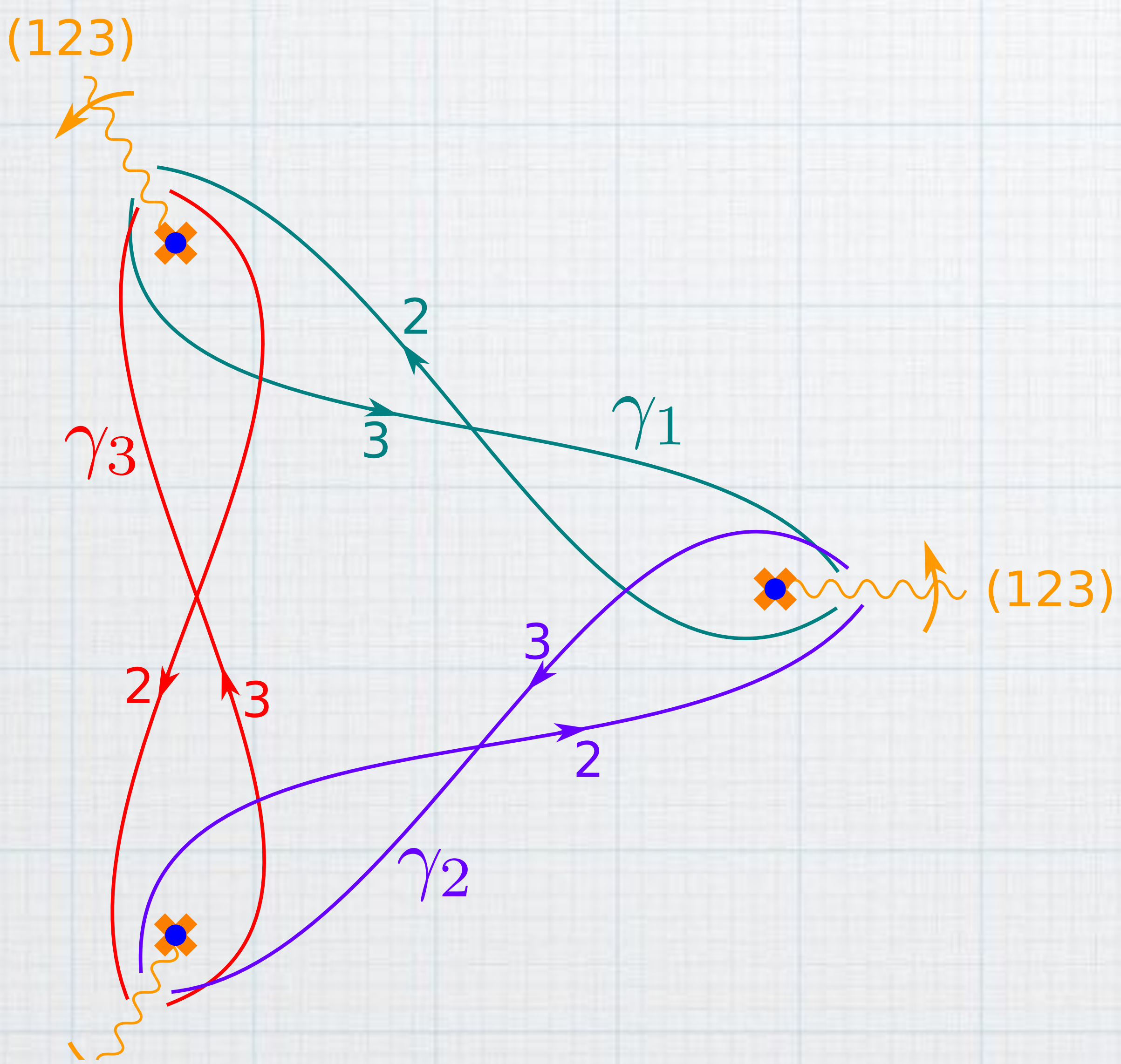
The Coulomb vacua in the massless limit are characterised by:

$$\varphi_2 = 0$$

$$\varphi_3 = \frac{u}{(z^3-1)^2}(dz)^3$$



Higher rank Fenchel-Nielsen networks are labeled by a choice of A-cycle on the spectral curve
[H-Neitzke,'16]

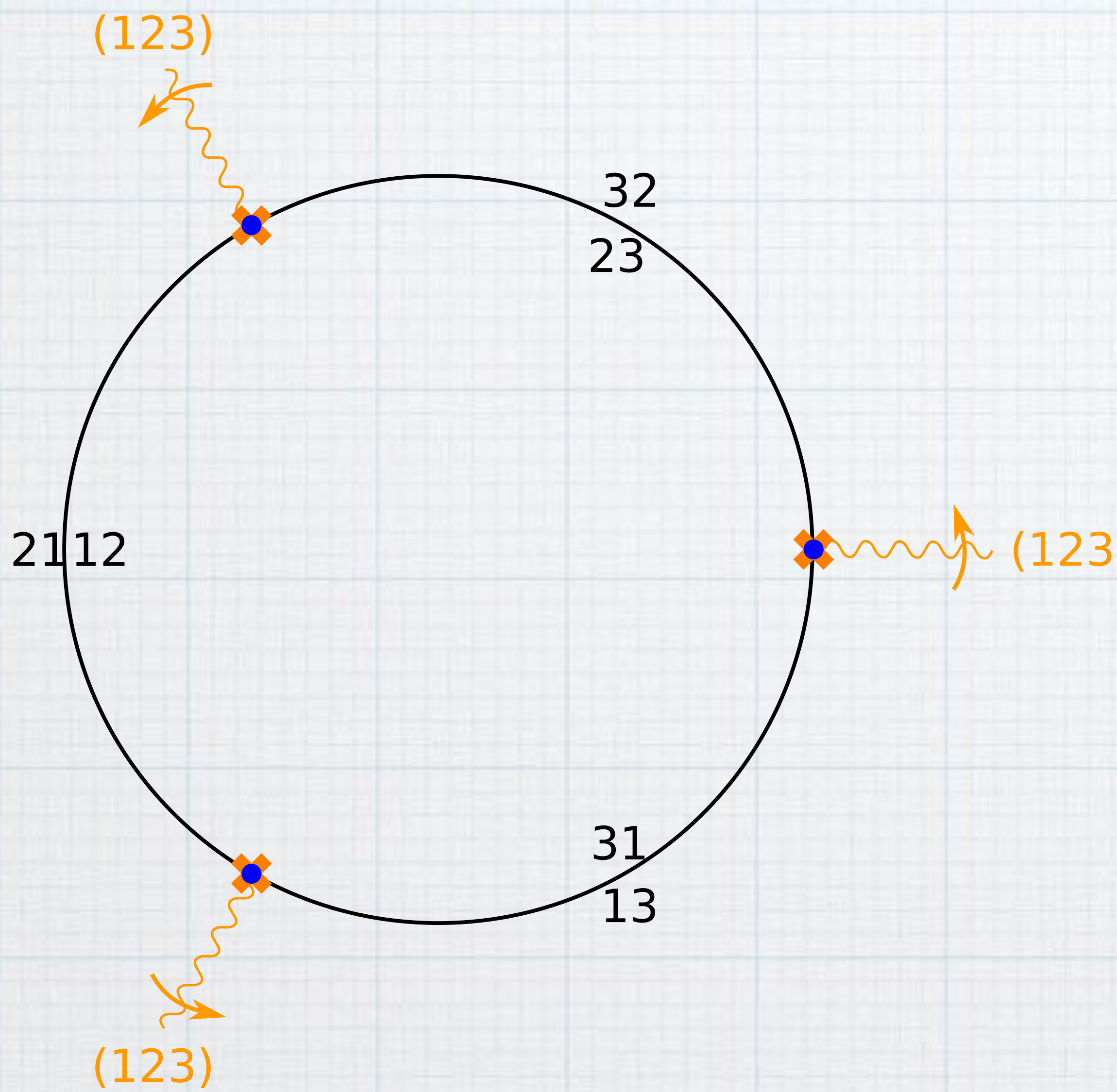


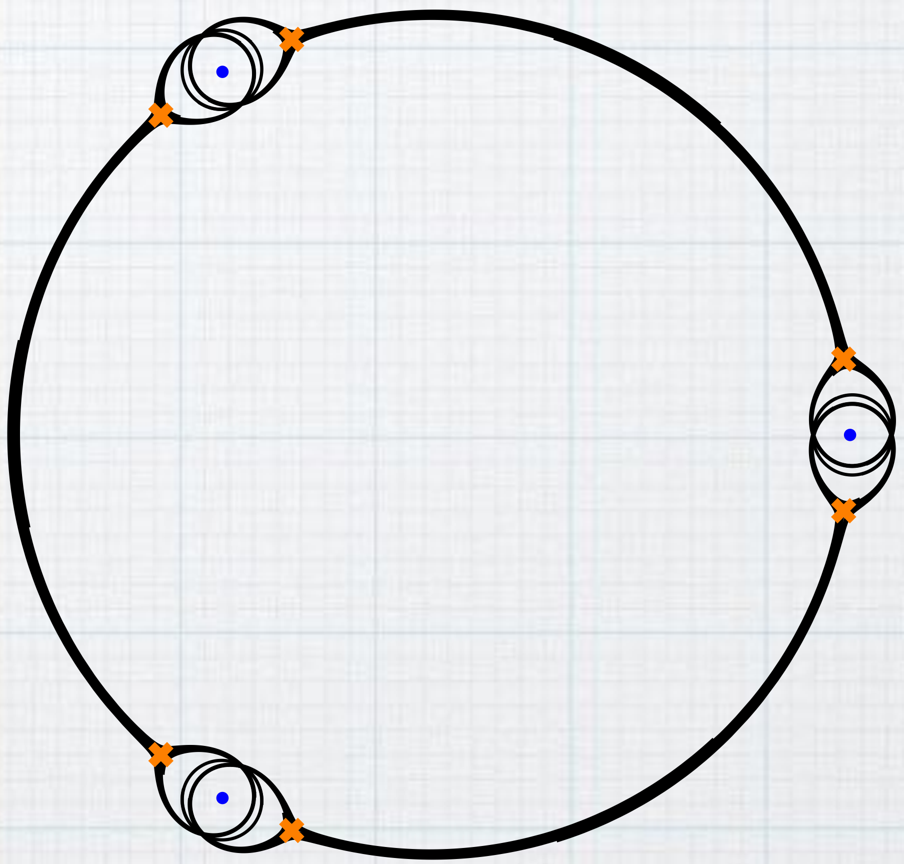
Found at critical phases for which:

$$A = p\gamma_1 + q\gamma_2$$

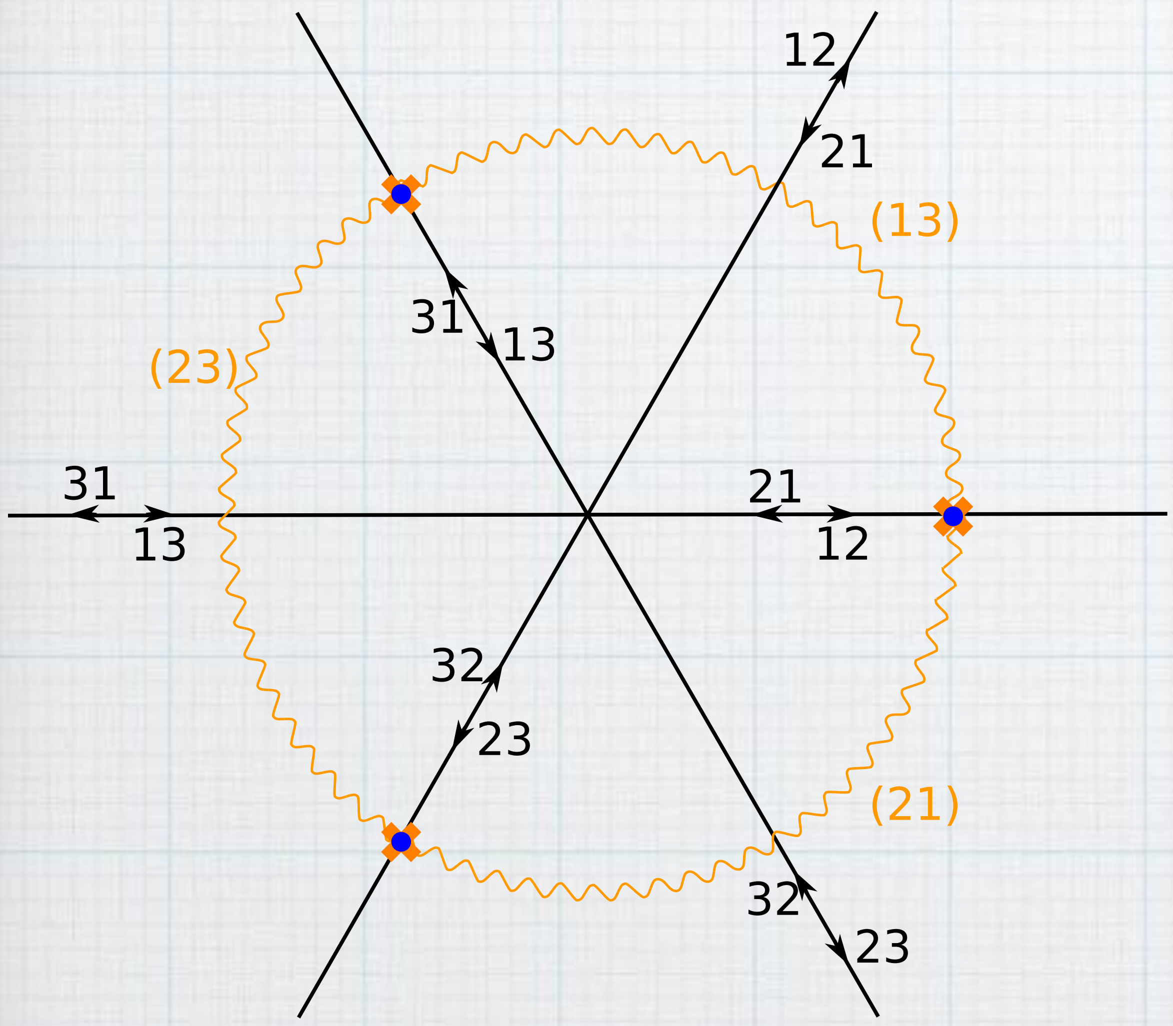
$$e^{-i\vartheta_A} \oint_A \lambda \in \mathbb{R}$$

Simplest network appears when (p,q)=(1,0):

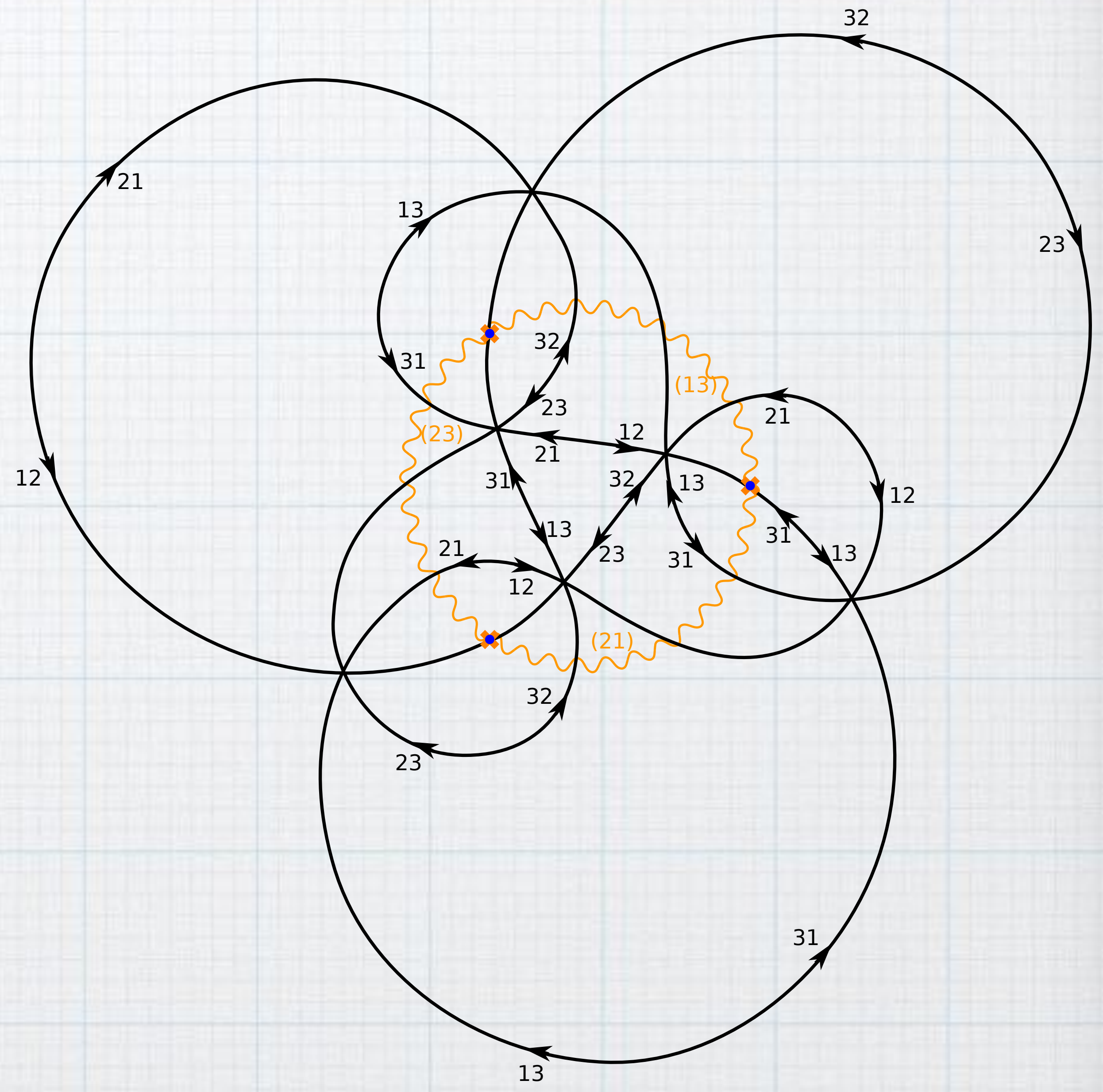




For (p,q)=(1,2):



For (p,q)=(1,3):



For the simplest Fenchel-Nielsen network, we find the monodromy representation [H-Neitzke,'17]:

$$M_0 = \begin{pmatrix} -\frac{Z^2}{4X} & \frac{1}{X} & -\frac{Z\left(1+\frac{Z^2}{4X}\right)Y}{2X} \\ \frac{1}{2}Z\left(1+\frac{Z^2}{4X}\right) & -\frac{Z}{X} & \left(1+\frac{Z^2}{4X}\right)^2 Y \\ \frac{X}{Y} & 0 & \frac{Z}{2} \end{pmatrix}$$

$$M_1 = \begin{pmatrix} -\frac{Z}{2X} & \left(1+\frac{Z^2}{4X}\right)^2 Y & -\frac{Z\left(1+\frac{Z^2}{4X}\right)Y}{2X} \\ 0 & \frac{Z}{2} & 1 \\ \frac{1}{Y} & \frac{Z}{2}\left(1+\frac{Z^2}{4X}\right) & -\frac{Z^2}{4X} \end{pmatrix}$$

$$Z = -1 + X + \sqrt{1 - 14X + X^2}$$

The brane of opers is parametrised by general Fuchsian differential equations of degree 3 on the three-punctured sphere.

Unlike for (generalized) hypergeometric differential equations, their monodromies cannot be computed analytically in terms of known functions

Yet, we can compute asymptotic expansion in the Omega-deformation parameter using the WKB properties of the spectral coordinates

$$X_\gamma(\nabla^{\text{oper}}) = \exp\left(\sum_{n=-1}^{\infty} \epsilon^n \oint^{\gamma} S_n(z)dz\right)$$

where

$$\Psi(z) = \exp\left(\sum_{n=-1}^{\infty} \epsilon^n \int^{z} S_n(z)dz\right)$$

is an asymptotic solution to

$$\nabla^{\text{oper}}\Psi(z) = 0$$

The exact result for X may be found by Borel resummation with respect to the spectral network

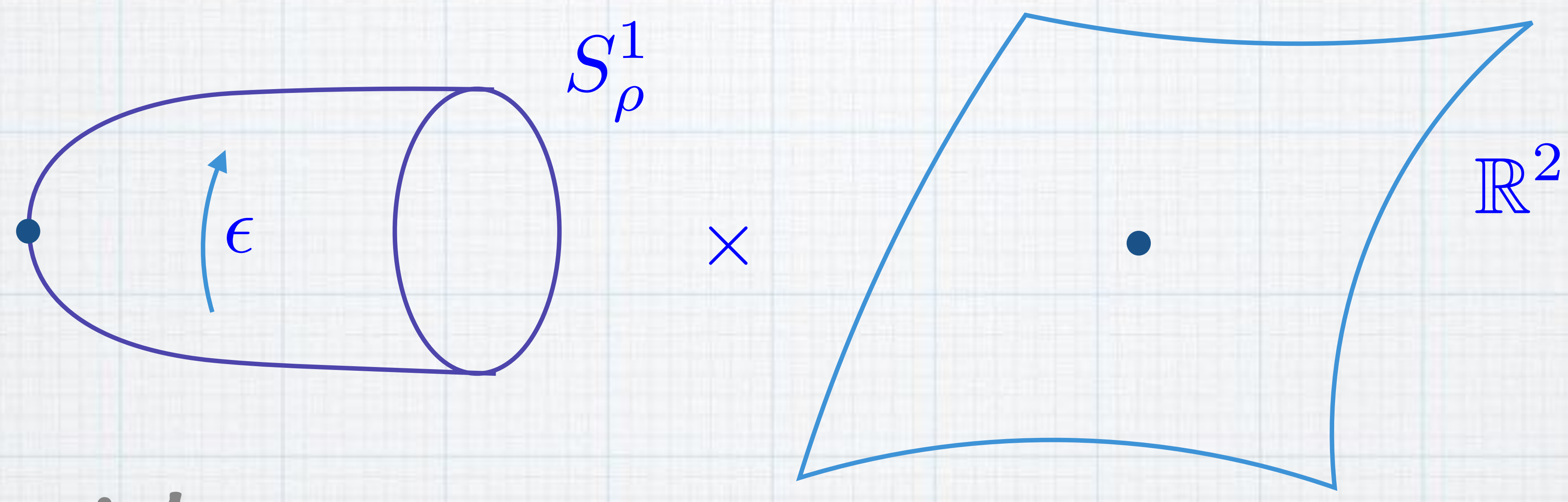
The asymptotic result for the generating function of opers is not very sensitive on the chosen spectral network

This relates the NRS proposal to other proposals computing the effective twisted superpotentials using quantum periods

Can we compare our E6 result??

How do we motivate the NRS correspondence?

Let's consider theory T in the background:



cigar metric

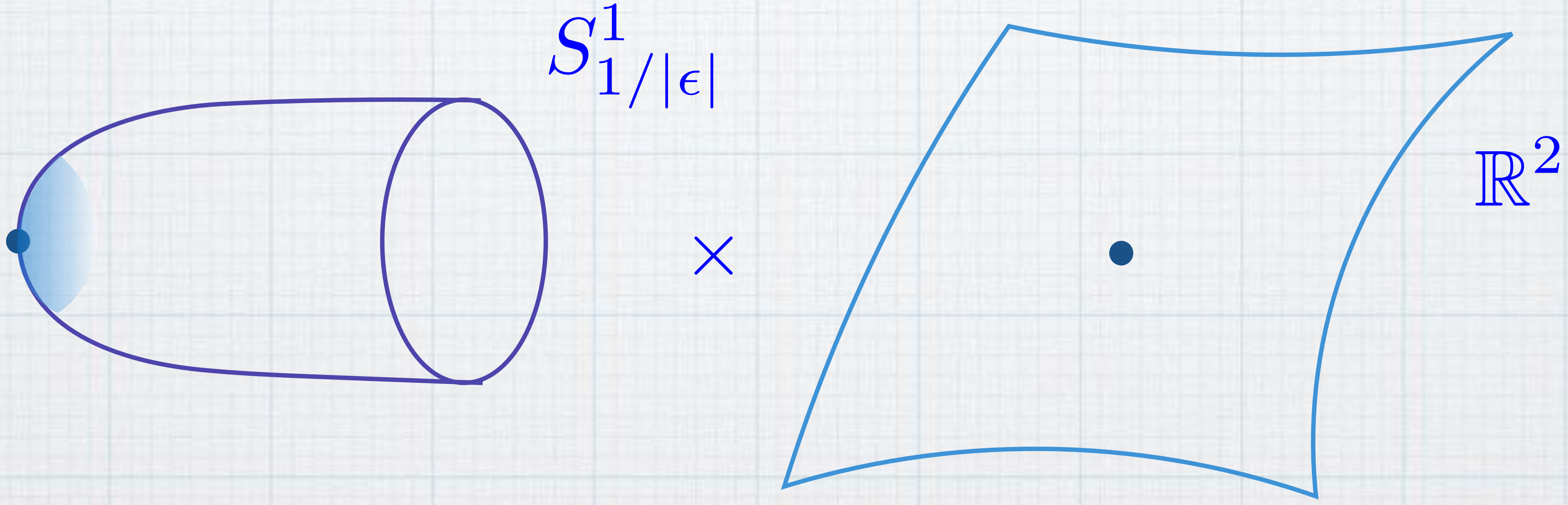
$$ds^2 = dr^2 + f(r)d\varphi^2$$

$$f(r) \sim r^2 \qquad r \to 0$$
$$f(r) \to \rho^2 \qquad r \to \infty$$

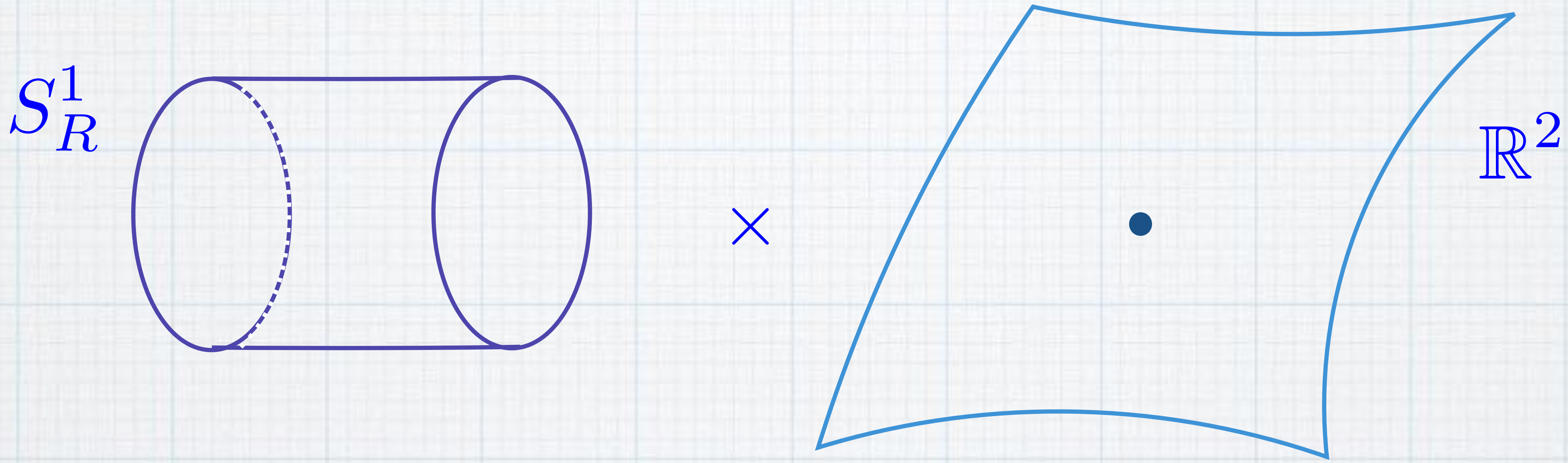
this background still preserves 2d N=(2,2)

IR physics governed by the NS effective twisted superpotential

On the other hand, the Omega deformation can be undone away from the tip of the cigar in favour of a field redefinition [Nekrasov-Witten,'10]



Let's consider theory T in the background:



Compactifying T on the circle is described in the IR by a 3d sigma model into the Hitchin moduli space

$$F_A + R^2[\varphi, \bar{\varphi}] = 0$$

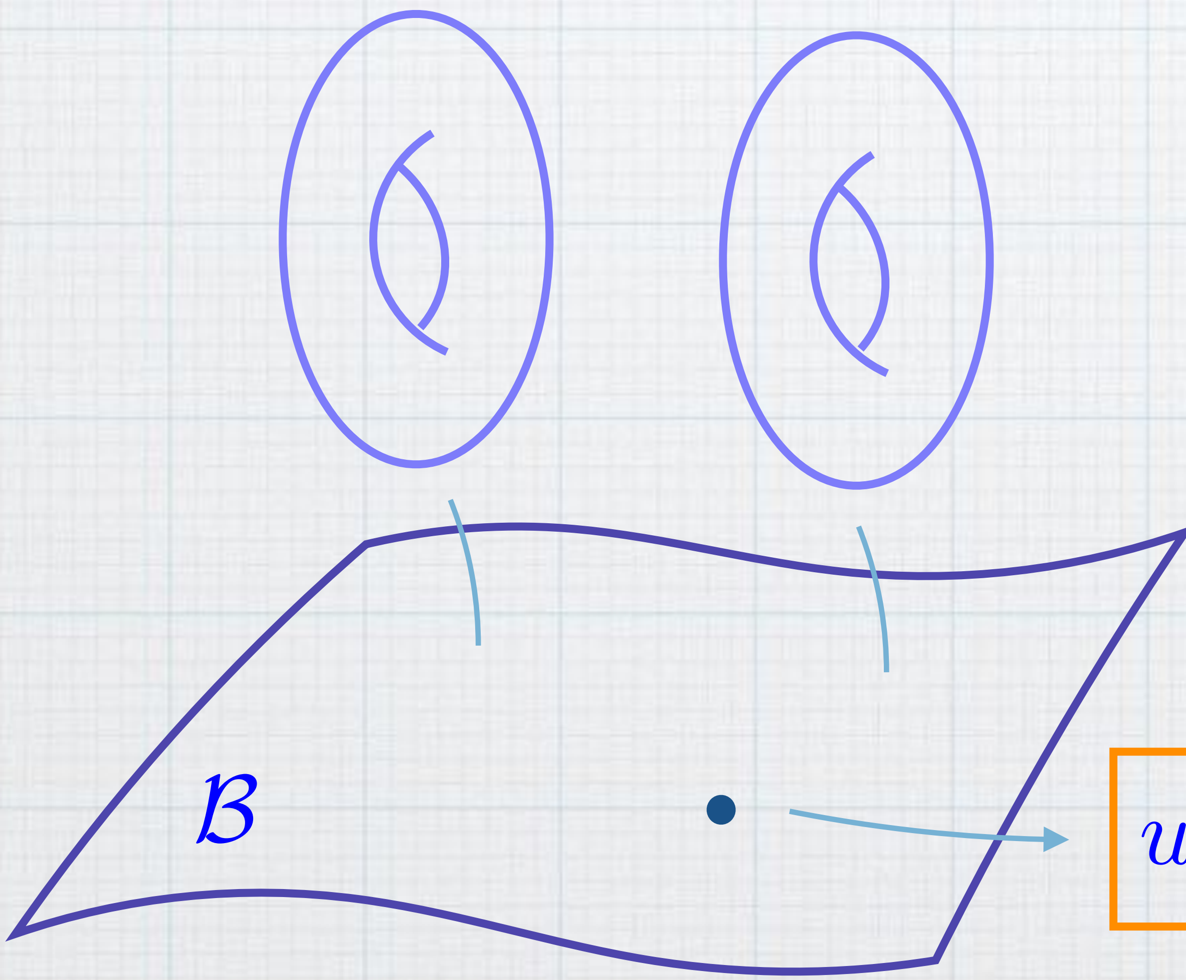
$$\bar{\partial}_A \varphi = 0$$

$$\partial_A \bar{\varphi} = 0$$

The Hitchin moduli space is hyperkahler with parameter

$$\zeta \in \mathbb{P}^1$$

- $\zeta \in \{0, \infty\}$ moduli space of Higgs bundles



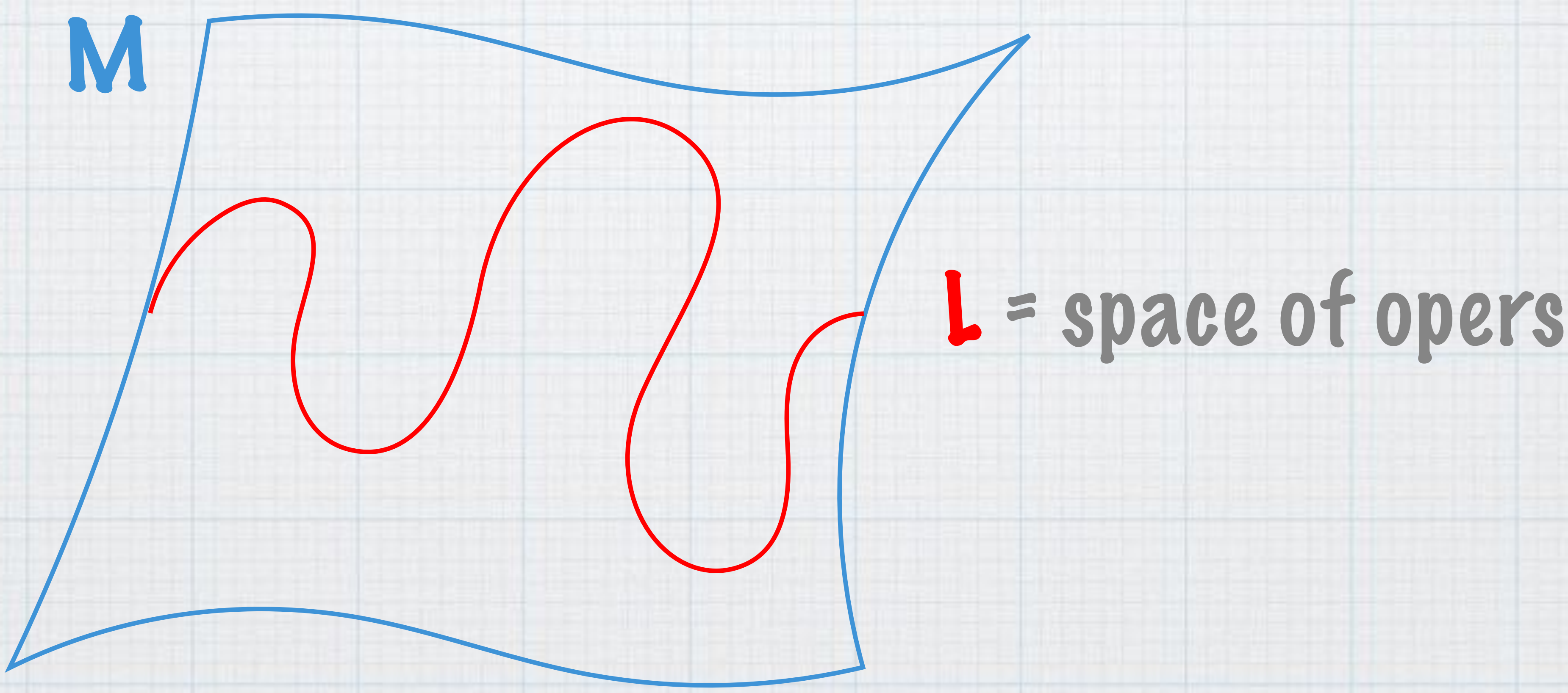
structure of an integrable system

$$u = (\varphi_2, \ldots, \varphi_K)$$

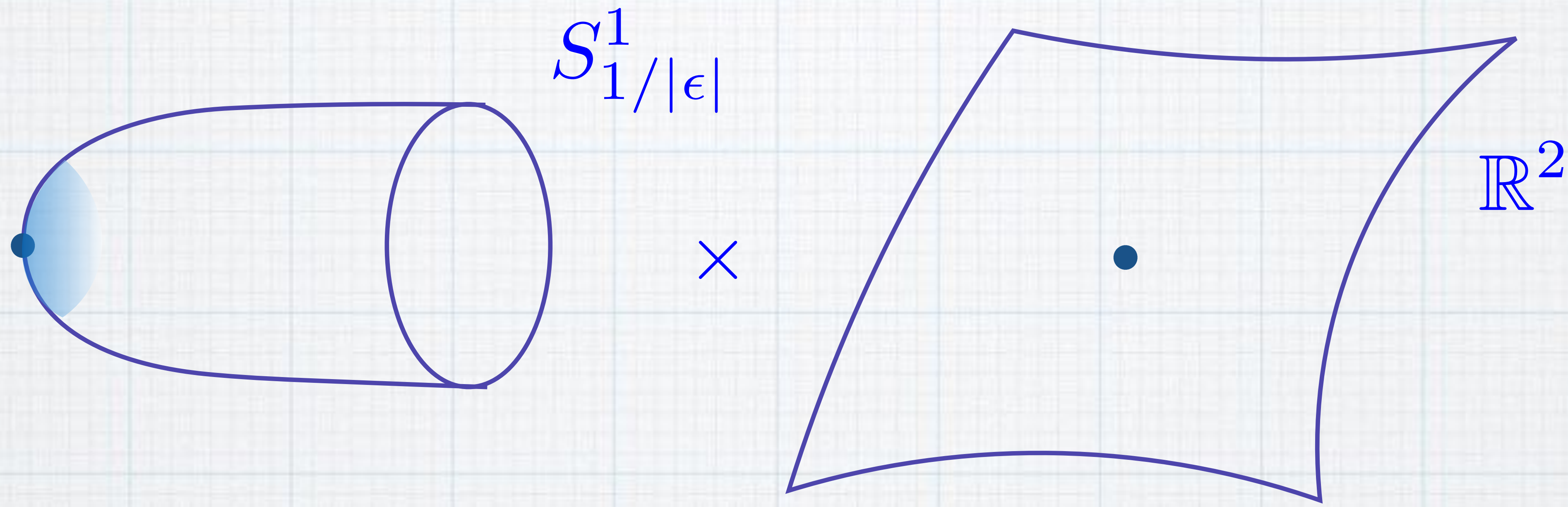
- $\zeta \in \mathbb{C}^*$ moduli space M of flat SL(K,C) connections

$$\nabla = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi}$$

distinguished complex Lagrangian



Let's go back to the background:

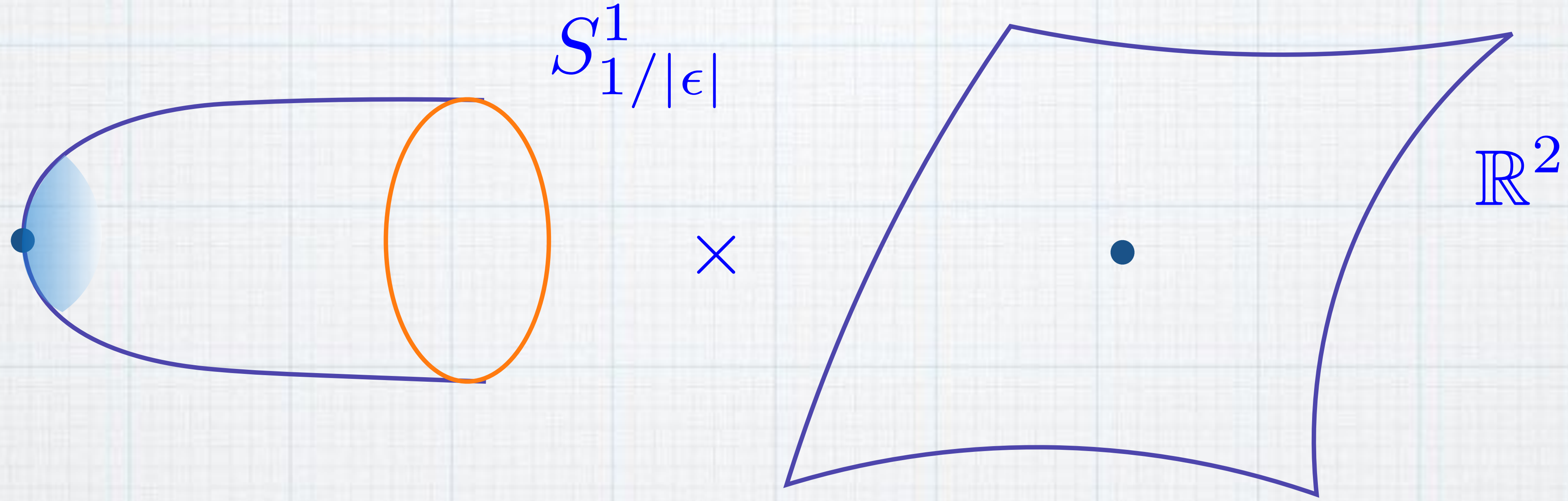


IR description as a 3d sigma model into the Hitchin moduli space with complex structure

$$\zeta = \frac{\epsilon}{|\epsilon|}$$

with an additional brane wrapping the space of opers [Nekrasov-Witten,'10]

Also boundary condition at infinity of the cigar



In 4d this boundary condition preserves a 3d N=2 subalgebra, labelled by a phase

$$\zeta = e^{i\vartheta}$$

General IR boundary conditions are specified by

$$(u, \vartheta)$$

together with a choice of A and B-cycles on the corresponding spectral curve

this data defines a set of spectral coordinates

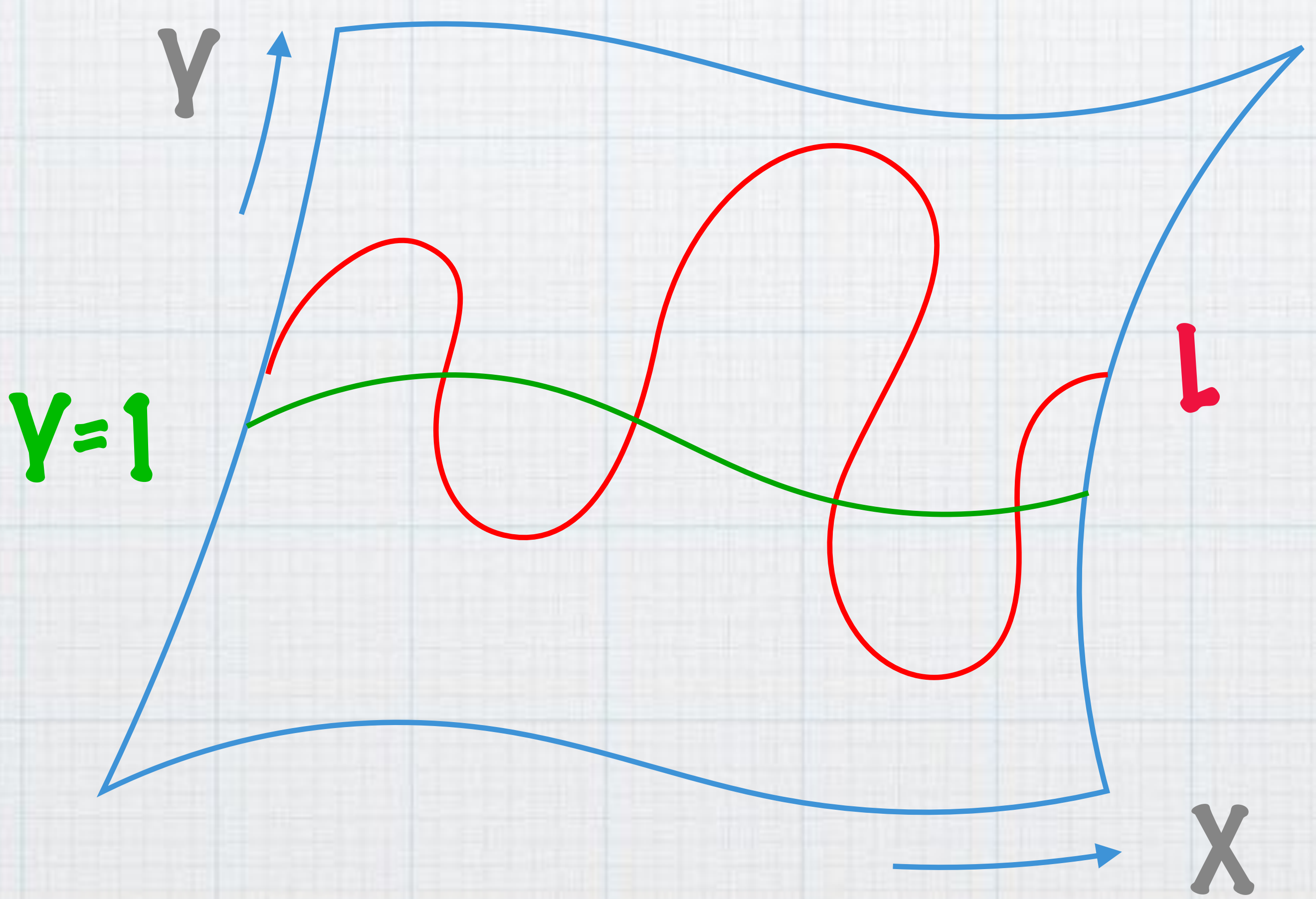
$$(X^i, Y_i)$$

on the moduli space of flat SL(K) connections

Reducing to zero modes in the r-direction, we find a 2d sigma model with superpotential

[Kapustin-Saulina]

$$W^{\text{oper}}(X^i)$$



The NS computation corresponds to a special type of boundary condition

It may be obtained through a UV-IR duality wall:

UV theory

IR theory

$$\mathrm{Re}(e^{-i\vartheta} Z) = c$$
$$\mathrm{Im}(e^{-i\vartheta} Z) = t\,\mathrm{Im}(e^{-i\vartheta} Z^0)$$

The IR theory is characterised by the light particles with

$$\mathrm{Im}(e^{-i\vartheta} Z) = 0$$

This is precisely the Strebel condition! An IR boundary condition is then given by setting Y=1

In particular, in a weakly coupled gauge theory the light particles are the W-bosons found in the Fenchel-Nielsen network determined by

$$\mathrm{Im}(e^{-i\vartheta} \oint_A \lambda) = 0$$

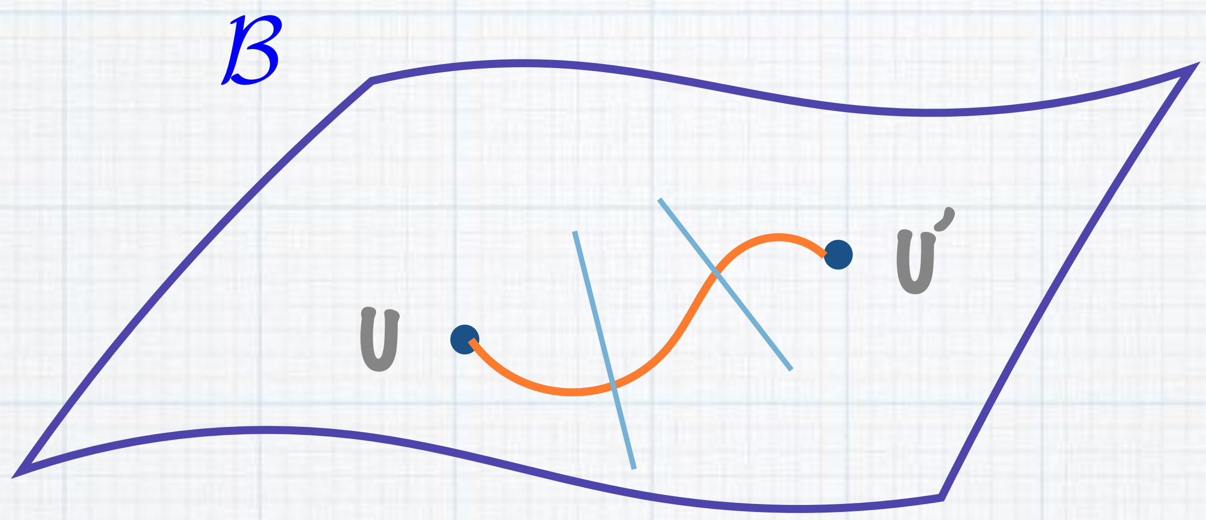
The IR boundary condition is a standard Neumann boundary condition

Using this UV-IR duality wall we have built a set of UV boundary conditions corresponding to (higher) Fenchel-Nielsen networks

For theories of class S of a Lagrangian type the resulting generating function of opers is equal to the effective twisted superpotential

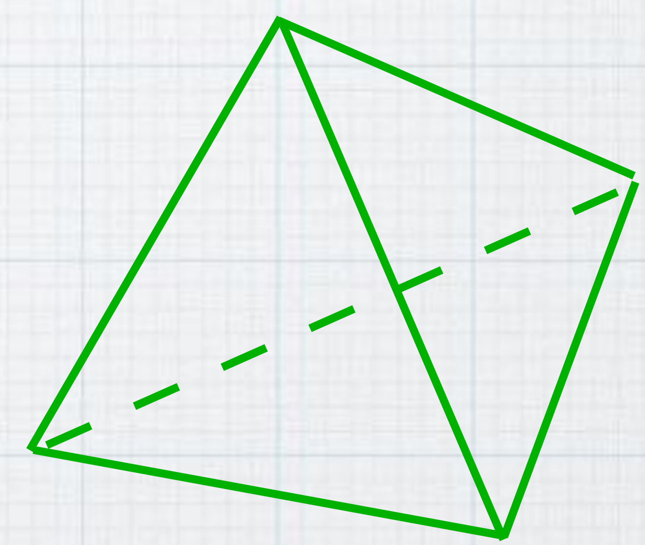
For other IR boundary conditions (corresponding to generic spectral networks) the generating function of opers also has a physical interpretation

Walking along a path from u to any other u′



the spectral network undergoes flips

this corresponds to coupling the 3d boundary to an abelian 3d theory
[Dimofte-Gaiotto-Veen,'13]



the generating function of opers is the effective twisted superpotential of the **total** theory