Bosonization on a lattice in higher dimensions

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Outline

- Review of the Jordan-Wigner transformation
- The toric code
- Bosonization in two dimensions
- Bosonization in general dimensions

Fermion-boson correspondence

Fermion-boson correspondence in 1+1d has many avatars:

- Free massless Dirac fermion $\leftrightarrow$ free massless boson
- Massive Thirring $\leftrightarrow$ sine-Gordon
- Free Majorana fermion $\leftrightarrow$ quantum Ising chain/2d Ising model
- Jordan-Wigner transformation for general fermionic systems on a 1d lattice

I will focus on the latter and will explain how it can be generalized to higher dimensions

Spin chains

The Hilbert space of a spin chain is $V = \otimes_{j=1}^{N} V_j$, where $V_j \simeq \mathbb{C}^2$.

The algebra of observables is $\otimes_{j=1}^{N} End(V_j) \simeq End(V)$.

The Hamiltonian is

$$H^B = \sum_{j=1}^{N} H_j^B,$$

where H_j^B is an observable which has finite range ($[H_j^B, O_k] = 0$ for $|j-k| \gg 0$ for any observable O_k localized on site k).

We will denote by X_j, Y_j, Z_j the standard Pauli matrices acting on site j only. We will assume that $S = \prod_j Z_j$ commutes with H^B. Then the spin chain has a $\mathbb{Z}_2$ symmetry.

Fermionic chains

The Hilbert space of a fermionic chain is $W = \widehat{\otimes} W_i$, where $W \simeq \mathbb{C}^{1|1}$.

The algebra of observables is $\widehat{\otimes}_{j=1}^{N} End(W_j) \simeq \mathrm{Cl}(2N)$, where $\mathrm{Cl}(2N)$ is the complex Clifford algebra with $2N$ generators. This is a $\mathbb{Z}_2$-graded algebra.

The Hamiltonian is

$$H^F = \sum_{j=1}^{N} H_j^F,$$

where each H_j^F is an even (bosonic) observable which has a finite range.

We will denote by $c_j, c_j^\dagger$ the fermionic creation-annihilation operators acting on site j.

Jordan-Wigner transformation

Since $\mathrm{Cl}(2N)$ is isomorphic to a matrix algebra of size $2^N \times 2^N$, the algebras of observables of the two models are abstractly isomorphic.

The Jordan-Wigner transformation is a special isomorphism which maps even local observables of the fermionic chain to local observables of the spin chain commuting with S:

$$c_j \mapsto \frac{1}{2}(X_j + iY_j) \prod_{k=1}^{j-1} Z_k, \quad c_j^\dagger \mapsto \frac{1}{2}(X_j - iY_j) \prod_{k=1}^{j-1} Z_k$$

In particular, this implies:

$$c_j^\dagger c_j \mapsto \frac{1}{2}\left(1 - Z_j\right), \quad (-1)^{c_j^\dagger c_j} \mapsto Z_j.$$

The inverse transformation is also easily written.

The toric code

The toric code (Kitaev) is a soluble 2+1d lattice spin model whose ground states reproduce topological $\mathbb{Z}_2$ gauge theory.

Let T be a triangulation of a 2d manifold. Let T_i, $i = 0, 1, 2$ be the set of i-simplices of T.

The Hilbert space is $\otimes_{e \in T_1} V_e$, where $V_e \simeq \mathbb{C}^2$ $\forall e$. The algebra of observables is generated by X_e, Y_e, Z_e, $e \in T_1$.

The Hamiltonian is a sum of commuting projectors:

$$H^T = t \sum_{v \in T_0} \frac{1}{2}(1 - P_v) + t \sum_{f \in T_2} \frac{1}{2}(1 - P_f), \quad t > 0,$$

where

$$P_v = \prod_{e \supset v} X_e, \quad P_f = \prod_{e \subset f} Z_e.$$

The toric code vs. topological $\mathbb{Z}_2$ gauge theory

The space of ground states of the toric code is the image of the projector

$$\Pi_{tot} = \prod_v \frac{1}{2}(1 + P_v) \prod_f \frac{1}{2}(1 + P_f)$$

The 2nd factor projects to states with trivial flux $P_f \in \{+1, -1\}$ for every f, the 1st factor imposes the Gauss law.

Alternatively, one can take the limit $t \to \infty$, or just take t to be finite, but very large. This last option is preferred in condensed matter literature, because then the Hilbert space is a product of local Hilbert spaces.

From the HEP viewpoint, imposing local constraints is fine too.

Excitations of the toric code

The ground states of H^T have zero energy and are in 1-1 correspondence with elements of $H^1(T, \mathbb{Z}_2)$. All excited states have energy Nt for some natural number N.

The basic excitations E and M are localized at vertices and faces, respectively, and have energy t. One can write a modified Hamiltonian whose zero-energy states are forced to have E or M at a particular vertex or face:

$$H^T(v) = H^T + tP_v, \quad H^T(f) = H^T + tP_f.$$

E and M particles are mutually nonlocal, with a π relative statistics. (To see this, need to study operators which move E and M excitations around.) So the composite of E and M behaves as an emergent fermion.

Toric code in the cochain notation

It is convenient to use a cochain notation for the states and operators of the toric code. Let $C^p = C^p(T, \mathbb{Z}_2)$ be the set of $\mathbb{Z}_2$-valued functions on T_p. The coboundary operator $\delta : C^p \to C^{p+1}$ satisfies $\delta^2 = 0$.

V^T has a basis $\left\{|\alpha\rangle,\ \alpha \in C^1\right\}$. The idempotent constraint operators P_v and P_f look as follows in this basis:

$$P_v : |\alpha\rangle \mapsto |\alpha + \delta\Delta_v\rangle, \quad P_f : |\alpha\rangle \mapsto (-1)^{(\delta\alpha)[f]}|\alpha\rangle,$$

where $\Delta_v \in C^0$ is a 0-cochain supported on the vertex v.

Modified constraints in the cochain notation

M-particles naturally live on faces. Let $\beta_2^M \in C^2$ describe an arbitrary distribution of M particles. Define

$$P_f[\beta_2^M] : |\alpha\rangle \mapsto (-1)^{(\delta\alpha - \beta_2^M)[f]}|\alpha\rangle, \quad P_v[\beta_2^M] = P_v.$$

E-particles naturally live on vertices, but we would like to associate them with faces too. Let's fix a branching structure on T, i.e. an orientation of edges such that for every $f \in T_2$ the oriented edges of f do not form a closed loop. Then vertices of every f are ordered from 0 to 2. This allows one to define an associative cup product $C^p \times C^q \to C^{p+q}$.

For any $\beta_2^E \in C^2$ define

$$P_v[\beta_2^E] : |\alpha\rangle \mapsto (-1)^{\int \Delta_v \cup \beta_2^E}|\alpha + \delta\Delta_v\rangle, \quad P_f[\beta_2^E] = P_f.$$

Modified Hamiltonians in the cochain notation

Since the idempotent observables $P_v[\beta_2^E]$ and $P_f[\beta_2^E]$ have finite range, we can define a modified local Hamiltonian which depends on a distribution of E-particles:

$$H^T[\beta_2^E] = t \sum_{v \in T_0} \frac{1}{2}(1 - P_v[\beta_2^E]) + t \sum_{f \in T_2} \frac{1}{2}(1 - P_f[\beta_2^E]).$$

It is a sum of commuting projectors.

Similarly we can define $H^T[\beta_2^M]$ which depends on a distribution of M-particles.

But what we really want is a similar Hamiltonian in the presence of a distribution of emergent fermions EM.

Inserting emergent fermions

For any $\beta_2^{EM} \in C^2$ define

$$P_v[\beta_2^{EM}] : |\alpha\rangle \mapsto (-1)^{\int \Delta_v \cup \beta_2^{EM}} |\alpha + \delta\Delta_v\rangle,$$

$$P_f[\beta_2^{EM}] : |\alpha\rangle \mapsto (-1)^{(\delta\alpha - \beta_2^{EM})[f]} |\alpha\rangle$$

These are commuting idempotents, which are also finite-range observables. So we can define a local Hamiltonian which is a sum of commuting projectors and depends on a distribution of EM particles:

$$H^T[\beta_2^{EM}] = t \sum_{v \in T_0} \frac{1}{2}(1 - P_v[\beta_2^{EM}]) + t \sum_{f \in T_2} \frac{1}{2}(1 - P_f[\beta_2^{EM}]).$$

Spins on a 2d lattice

Consider a system of bosonic spins living on the faces of T. The Hilbert space is $V^S = \otimes_{f \in T_2} V_f$, where $V_f \simeq \mathbb{C}^2$. The algebra of observables is

$$End(V^S) = \otimes_{f \in T_2} End(V_f),$$

and is generated by X_f, Y_f, Z_f, $f \in T_2$. A natural basis for V^S is labeled by $\beta_2^S \in C^2$:

$$Z_f |\beta_2^S\rangle = (-1)^{\beta_2^S[f]} |\beta_2^S\rangle.$$

Consider the total spin operator $S = \prod_f Z_f$. The algebra of observables commuting with S is generated by Z_f, $f \in T_2$, and "hopping" operators

$$S_e = X_{L(e)} X_{R(e)}, \quad e \in T_1,$$

where $L(e)$ is the face to the left of an oriented edge e, and $R(e)$ is the face to the right of e (here we use orientation on T).

Fermions on a 2d lattice

Similarly, let's place the fermions on faces of T. The Hilbert space is $W = \widehat{\otimes}_{f \in T_2} W_f$, where $W_f \simeq \mathbb{C}^{1|1}$. The algebra of observables is

$$\widehat{\otimes}_{f \in T_2} End(W_f),$$

and is generated by γ_f, γ'_f, $f \in T_2$, where γ_f, γ'_f are generators of $\mathrm{Cl}(2)$.

The algebra of even observables is generated by $(-1)^{F_f} = i\gamma_f \gamma'_f$, $f \in T_2$ (fermion parity on the face f) and operators

$$S_e^F = i\gamma_{L(e)}\gamma'_{R(e)}, \quad e \in T_1.$$

Comparing bosonic and fermionic algebras

Similarities:

- All these operators square to 1
- Hopping operators S_e anti-commute with Z_f for $f = L(e)$ and $f = R(e)$ and commute for all other f. The same applies to S_e^F and $(-1)^{F_f}$.

Differences:

- S_e and S'_e commute for all e and e', while S_e^F and $S_{e'}^F$ sometimes commute and sometimes anti-commute
- For a fixed v, we have $\prod_{e \supset v} S_e = 1$, but $\prod_{e \supset v} S_e^F = c(v) \prod_{f \supset v} (-1)^{F_f}$, where $c(v)$ is a c-number sign.

Emergent fermion attachment

To fix this, we need to attach to every "spin-down" state at a face f an emergent fermion EM from the toric code.

This means that we consider the product Hilbert space $V^S \otimes V^T$ with the Hamiltonian

$$H_0 = t \sum_{v \in T_0} (1 - P_v[\frac{1}{2}(1 - Z)]) + t \sum_{f \in T_2} (1 - P_f[\frac{1}{2}(1 - Z)]),$$

where $\frac{1}{2}(1 - Z)$ is a 2-cochain with values in operators on V^S whose value on face f is $\frac{1}{2}(1 - Z_f)$.
Then we define hopping operators for emergent fermions U_e as follows:

$$U_e : |\alpha\rangle \mapsto (-1)^{\int \alpha \cup \Delta_e} |\alpha + \Delta_e\rangle,$$

where $\Delta_e \in C^1$ is the 1-cochain supported on e.

Properties of the emergent fermion hopping operator

- $U_e^2 = 1$.
- U_e anti-commutes with $(-1)^{\delta\alpha[f]}$ if $e \subset f$ and commutes with it otherwise.
- U_e and $U_{e'}$ commute for some pairs e, e' and anti-commute for others, and the rule is the same as for S_e^F and $S_{e'}^F$.
- On the image of $H^T[\beta_2^{EM}]$, one has $\prod_{e\supset v} U_e = \tilde{c}(v) \prod_{f\supset v} (-1)^{\beta_2^{EM}[f]}$.

This is very similar to properties of S_e^F, except that F_f (the number of physical fermions modulo 2) is replaced with β_2^{EM} (the number of emergent fermions modulo 2), and the c-number sign $\tilde{c}(v)$ is different.

The 2d analog of the Jordan-Wigner transformation

Consider the following map:

$$(-1)^{F_f} \mapsto Z_f, \quad S_e^F \mapsto d(e) U_e S_e,$$

where $d(e)$ is some c-number sign depending on the edge. We claim that provided a certain condition on $d(e)$ is satisfied, this map induces a homomorphism of algebras after projecting to the zero-energy states of the Hamiltonian H_0.

The condition on $d(e)$ has the form $\prod_{e \supset v} d(e) = c(v)\tilde{c}(v)$. That is, $d(e)$ is a 1-chain with values in $\mathbb{Z}_2$ which is a trivialization of the 0-chain $c(v)\tilde{c}(v)$. If it has a solution, one can get other solutions by adding a closed 1-chain, i.e. a 1-cycle.

Further, a 1-cycle which is a boundary of a 2-cycle can be absorbed into S_e by redefining some X_f and Y_f by a sign. Hence distinct possibilities for $d(e)$ can be labeled by elements of $H_1(T, \mathbb{Z}_2)$ (if a solution exists at all).

When does a solution for $d(e)$ exist?

It turns out that $c(v)\tilde{c}(v)$ is Poincare-dual to a 2-cocycle representing the 2nd Stiefel-Whitney class $w_2 \in H^2(T, \mathbb{Z}_2)$. If the space is orientable manifold, this class is always trivial (in two dimensions), and a solution $d(e)$ exists.

In general, w_2 is an obstruction to having a spin structure (provided w_1 vanishes). So its trivialization defines a spin structure. Thus the 1-cycle $d(e)$ is a lattice representation of a spin structure.

Thus the 2d Jordan-Wigner transformation always exists, but is not unique: distinct transformations are labeled by spin structures.

NB. The inverse of the above bosonization map was first described by Bhardwaj, Gaiotto, AK, 2016 in the context of lattice TQFT. The bosonization map in the above form was worked by Yu-An Chen and AK.

Bosonization in general dimensions

It was argued in Gaiotto and AK, 2015 that bosonization exists in all dimensions.

The idea is the same: find a topological system with emergent fermions and attach these emergent fermions to spin excitations.

While it is well-known that 2+1d TQFTs can have emergent fermions, it is less obvious how to engineer them in higher dimensions.

For example, Jackiw and Rebbi, 1976, showed how fermions can arise in a purely bosonic 3+1d system, but their example is very far from being a TQFT (involves Yang-Mills gauge fields and dynamical scalars).

A TQFT with emergent fermions

Nevertheless, there is a universal solution for this problem in all dimensions. Consider a Euclidean lattice system in $d+1$ dimensions whose only "field" is a $(d-1)$-cocycle B with values in $\mathbb{Z}_2$.

Postulate also that $B \mapsto B + \delta\lambda$, where λ is a $(d-2)$-cocycle, is a gauge symmetry. Thus B is a topological $(d-1)$-form gauge field with values in $\mathbb{Z}_2$.

Now recall that there exist operations $Sq^q : H^p(X, \mathbb{Z}_2) \mapsto H^{p+q}(X, \mathbb{Z}_2)$ called Steenrod squares and consider the action

$$S_X(B) = \frac{1}{2} \int_X Sq^2 B \in \mathbb{R}/\mathbb{Z},$$

so that the partition function is

$$Z_X \sim \sum_{[B] \in H^{d-1}(X, \mathbb{Z}_2)} \exp(2\pi i S(B)).$$

We claim that this TQFT has an excitation which is an emergent fermion.

A $d = 3$ example

Here $Sq^2 B = B \cup B$, so we get

$$S_X(B) = \frac{1}{2} \int_X B \cup B.$$

Now we recall that on a closed orientable manifold $B \cup B = w_2 \cup B$. Also, impose the constraint $\delta B = 0$ using a Lagrange multiplier $a \in C^1(X, \mathbb{Z}_2)$. We get:

$$S'_X(a, B) = \frac{1}{2} \int_X (\delta a + w_2) \cup B.$$

Thus the Wilson loop $\exp(i\pi \int_\gamma a)$ is not a topological operator (because $\delta a \neq 0$ on-shell). But if $w_2 = \delta\eta$ for some 1-cochain η, then

$$W_\gamma = \exp(i\pi \int_\gamma (a + \eta)),$$

where γ is a 1-cycle, is a topological observable.

This loop operator represents a worldline of an excitation which requires a trivialization of w_2 (i.e. a spin structure) for its definition.

Jordan-Wigner in general dimensions

To get a $d = 3$ Jordan-Wigner transformation from this, one needs to rewrite the above TQFT in the Hamiltonian form, construct "hopping" operators for the emergent fermion, and show that they obey the same algebra as hopping operators for fermions.

The case of general d is similar: one just needs to use Wu's formula $Sq^2 B = w_2 \cup B$ valid on any closed orientable $(d+1)$-manifold and any $(d-1)$-cocycle B.

The details are being worked out.