W-algebras, moduli of sheaves on surfaces, and AGT

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26.7.2017

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- Then the Nekrasov partition function is $\langle \mathbf{1}, \mathbf{1} \rangle$, where $\mathbf{1}$ is the unit cohomology class, which one can uniquely determine based on how the W–algebra acts on it (the Gaiotto state).

The Ext operator

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- Since our construction is purely geometric, it makes sense for sheaves on surfaces S more general than the affine plane $\mathbb{A}^2$

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- Replacing Lie algebras by quantum groups, e.g. $\mathfrak{g} \rightsquigarrow U_q(\mathfrak{g})$ or:

$$\text{Heisenberg} \rightsquigarrow {}_q\text{Heisenberg} = \frac{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]\langle a_n \rangle_{n \in \mathbb{Z} \backslash 0}}{[a_{-m}, a_n] = \delta_m^n \frac{n(1-q_1^n)(1-q_2^n)(1-q^{rn})}{1-q^n}}$$

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- More generally, we will consider the deformed W–algebra of Feigin-Frenkel and Awata-Kubo-Odake-Shiraishi. In type $\mathfrak{gl}_1$, this algebra is ${}_q$Heisenberg, while for $\mathfrak{sl}_2$, it deforms Virasoro.

The deformed W–algebra

- The original definition of the $_q$W–algebra is via free fields:

$$_q\mathcal{W}_r \subset \mathcal{H}_r = \frac{\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]\langle b_n^{(i)} \rangle_{n \in \mathbb{Z} \backslash 0}^{1 \leq i \leq r}}{[b_{-n}^{(i)}, b_m^{(j)}] = \delta_m^n n(1 - q_1^n)(1 - q_2^n)(1 - \delta_{i \neq j} q^{-n\delta_{i<j}})}$$

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- ${}_q\mathcal{W}_r$ is defined as the subalgebra generated by $W_{d,k}$ given by:

$$ W_k(x) = \sum_{1 \leq s_1 \leq \ldots \leq s_k \leq r} : \prod_{i=1}^{k} u_{s_i} \exp \left[\sum_{n=1}^{\infty} \frac{b_{-n}^{(s_i)}}{n x^{-n}} \right] \exp \left[\sum_{n=1}^{\infty} \frac{b_n^{(s_i)}}{n x^n} \right] : $$

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- More intrinsically, ${}_q\mathcal{W}_r$ can be described as the algebra generated by symbols $W_{d,k}$ modulo relations such as:

$$ W_k(x)W_1(y)\zeta\left(\frac{x}{yq^k}\right) - W_1(y)W_k(x)\zeta\left(\frac{y}{xq}\right) = $$

$$ = \frac{(q_1-1)(q_2-1)}{q-1}\left[\delta\left(\frac{x}{yq^k}\right)W_{k+1}(x) - \delta\left(\frac{y}{xq}\right)W_{k+1}(y)\right] $$

The Verma module and the vertex operator

- Let $u_1, \dots, u_r \in \mathbb{C}$. Define the Verma module of ${}_q\mathcal{W}_r$ as:

$$M = {}_q\mathcal{W}_r \Big/ \text{right ideal } \left(W_{d,k}, W_{0,k} - e_k(u_1, \dots, u_r)\right)^{d>0}_{1 \leq k \leq r}$$

and similarly define M' with the parameters u_i replaced by u'_i.

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- as it isn't required to commute with ${}_q\mathcal{W}_r$ on the nose. Instead:

$$[\Phi_m, W_k(x)]_{m^k} \cdot \left(1 - \frac{m^r}{q^{r-k}x} \frac{u_1 \ldots u_r}{u'_1 \ldots u'_r}\right) = 0 \qquad (1)$$

for all $k \geq 1$, where $[A, B]_s = AB - sBA$.

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- **Lemma:** the endomorphism Φ_m is uniquely determined, up to constant multiple, by property (1).

The main Theorem

- **Theorem (N)** There exists an action ${}_q\mathcal{W}_r \curvearrowright K$, for which the latter is isomorphic to the Verma module M.

  Moreover, the Ext operator $A_m : K' \to K$ is equal to the vertex operator $\Phi_m : M' \to M$, up to a simple exponential.

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- We define the action of $W(x, y) = \sum_{d \in \mathbb{Z}}^{k>0} \frac{W_{d,k}}{x^d(-y)^k}$ on K as:

$$W(x, yD_x) = L(x, yD_x) \cdot E(y) \cdot U(x, yD_x)$$

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- and L, E, U are geometric endomorphisms of K that are lower triangular, diagonal, and upper triangular, respectively.

The moduli space of sheaves

- Explicitly, our $\mathcal{M}$ is the moduli space of framed sheaves:

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- Let K denote the equivariant K–theory of $\mathcal{M}$, whose elements are formal differences of T–equivariant vector bundles on $\mathcal{M}$.
- Since vector bundles can be tensored with T–representations, K is a module over the ring $\mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}, u_1^{\pm 1}, ..., u_r^{\pm 1}]$.

K–theory and partitions

- An r–partition will be a collection of r usual partitions:

$$\boldsymbol{\lambda} = (\lambda^{(1)}, ..., \lambda^{(r)}) \quad \text{where} \quad \lambda^{(k)} = (\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq ...)$$

and we will write $|\boldsymbol{\lambda}|$ for the sum of the sizes of the $\lambda^{(k)}$.

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- T–fixed points of $\mathcal{M}$ are indexed by r–partitions, specifically:

$$\mathcal{F}_{\boldsymbol{\lambda}} = I_{\lambda^{(1)}} \oplus ... \oplus I_{\lambda^{(r)}}$$

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- Given a box $\square$ at coordinates (i,j) in the Young diagram of the constituent partition $\lambda^{(k)}$ of an r–partition $\boldsymbol{\lambda}$, we call:

$$z_\square = u_k q_1^i q_2^j \qquad \text{the } \textbf{weight} \text{ of } \square.$$

The correspondences

- The operator $E(y)$ is multiplication by the exterior algebra of the universal sheaf, and therefore its matrix coefficients are:

$$\langle \boldsymbol{\lambda} | E(y) | \boldsymbol{\mu} \rangle = \delta_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \prod_{i=1}^{r} \left(1 - \frac{u_i}{y}\right) \prod_{\square \in \boldsymbol{\lambda}} \zeta\left(\frac{z_\square}{y}\right)$$

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- For any $d \geq 1$, let us consider the following locus:

$$\mathfrak{Z}_d = \{\mathcal{F}_0 \supset \ldots \supset \mathcal{F}_d \text{ sheaves}, x \in \mathbb{A}^2, \text{ s.t. } \mathcal{F}_{k-1}/\mathcal{F}_k \cong \mathcal{O}_x \ \forall k\}$$

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- and with line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_d$ on $\mathfrak{Z}_d$ that keep track of the length one quotients $\mathcal{F}_0/\mathcal{F}_1, \ldots, \mathcal{F}_{d-1}/\mathcal{F}_d$.

The operators

- $L(x,y)$ and $U(x,y)$ are adjoint, so let's focus on the first one:

$$L(x,y) := \sum_{d=0}^{\infty} \pi_{+*}^{(d)} \left(\frac{x^d}{1 - \frac{y}{\mathcal{L}_1}} \cdot \pi_{-}^{(d)*} \right) \; : \; K \to K[[x, y^{-1}]]$$

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- Therefore, the matrix coefficients of $L(x,y)$ are given by:

$$\langle \boldsymbol{\lambda} | L(x,y) | \boldsymbol{\mu} \rangle = \sum_{\text{tableau of shape } \boldsymbol{\lambda} \backslash \boldsymbol{\mu}}^{T \text{ a standard Young}} \frac{x^{|\boldsymbol{\lambda} \backslash \boldsymbol{\mu}|}}{\left(1 - \frac{y}{z_1}\right) \prod_{i=1}^{|\boldsymbol{\lambda} \backslash \boldsymbol{\mu}|-1} \left(1 - \frac{q z_i}{z_{i+1}}\right)}$$

$$\prod_{1 \leq i < j \leq |\boldsymbol{\lambda} \backslash \boldsymbol{\mu}|} \zeta\left(\frac{z_j}{z_i}\right) \prod_{i=1}^{|\boldsymbol{\lambda} \backslash \boldsymbol{\mu}|} \left[\prod_{\square \in \boldsymbol{\mu}} \zeta\left(\frac{z_i}{z_\square}\right) \prod_{j=1}^{r} \left(1 - \frac{z_i q}{u_j}\right) \right]$$

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$$\prod_{1 \leq i < j \leq |\boldsymbol{\lambda}\backslash\boldsymbol{\mu}|} \zeta\left(\frac{z_j}{z_i}\right) \prod_{i=1}^{|\boldsymbol{\lambda}\backslash\boldsymbol{\mu}|} \left[\prod_{\square \in \boldsymbol{\mu}} \zeta\left(\frac{z_i}{z_\square}\right) \prod_{j=1}^{r} \left(1-\frac{z_i q}{u_j}\right) \right]$$

- where we recall that a standard Young tableau is a labeling of the boxes of $\boldsymbol{\lambda}\backslash\boldsymbol{\mu}$ with the numbers $1, 2, \ldots$ such that the numbers increase as we go up and to the right. Also $z_i = z_{\square_i}$.

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- Finally, the parameter m that defines the Ext bundle and the operator A_m, will now be a K–theory class on S.