

# Outline

## B-model for knot homology

Alexei Oblomkov (joint work with L. Rozansky)

July 26, 2017

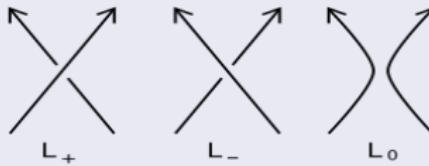
String-Math 2017, University of Hamburg and DESY Hamburg.

- 1 History
- 2 Path to B-model
- 3 KSR model interpretation

# Jones polynomial

Jones polynomial, V. Jones 1984

$$V(O) = q^{1/2} + q^{-1/2},$$
$$q^{-1}V(L_+) - qV(L_-) = (q^{1/2} - q^{-1/2})V(L_0),$$



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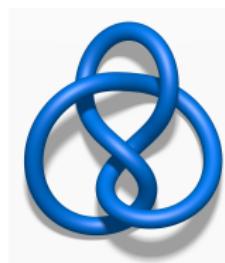
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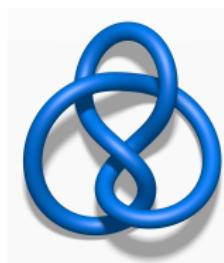
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# Khovanov homology



$$V(4_1) = q^{-5} + q^{-1}(-1) + q^1 + q(-1) + q^5$$

# Khovanov homology



$$\text{Kh}(4_1) = q^{-5}t^{-2} + q^{-1}t^{-1} + q^1t^0 + qt + q^5t^2$$

Theorem (Khovanov 2000)

For every link  $L$  there are graded spaces  $H_{Kh}^*(L)$  such that

$$\sum_i (-1)^i \dim_q(H_{Kh}^i(L)) = V(L).$$

# HOMFLY-PT homology

J. Hoste, A. Ocneanu, K. Millet, P. Freyd, W. Lickorich; J. Przytycki, P. Traczyk; V. Jones 1985

## HOMFLY-PT polynomial

$$P(O) = (a^{-1} - a)/(q^{1/2} - q^{-1/2}),$$
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## Theorem (Khovanov-Rozansky, 2007, 2008)

For every link  $L$  there are doubly graded spaces  $H_{KhR}^*(L)$  such that

$$P(L) = \sum_i (-1)^i \dim_{q,a} H_{KhR}^i(L).$$



# Braids and links

Elements  $\sigma_i$ ,  $i = 1, \dots, n - 1$  generate  $Br_n$

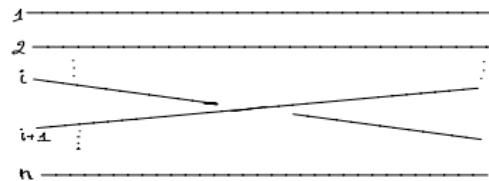


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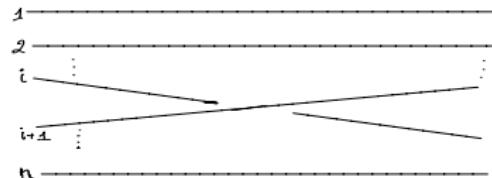


Figure: Generator  $\sigma_i$

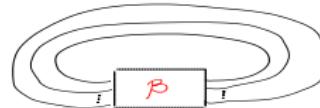


Figure: Closure  $L(\beta)$  of the braid  $\beta$

# Geometric construction for KhR homology

Geometric version of the construction of KhR homology,  
Williamson, Webster 2009

$$\begin{aligned} Br_n \ni \beta &\mapsto \Phi_\beta \in Perv(B_n \backslash GL_n / B_n), \\ H_{KhR}^*(L(\beta)) &= (H^*(GL_n / B_n^\Delta, \Phi_\beta), d_{chr}). \end{aligned}$$

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Here the homology  $H^*(GL_n / B_n^\Delta)$  acquires double grading from two weight filtration and  $d_{chr}$  is the chromatographic differential.

# A/B model?

$P\text{evr}(B_n \backslash GL_n / B_n) = B_n$ -equivariant constructible sheaves on  $Fl_n =$

$\text{Fuk}_{B_n}(T^*Fl_n) = A - \text{model for } T^*Fl_n$

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## Question

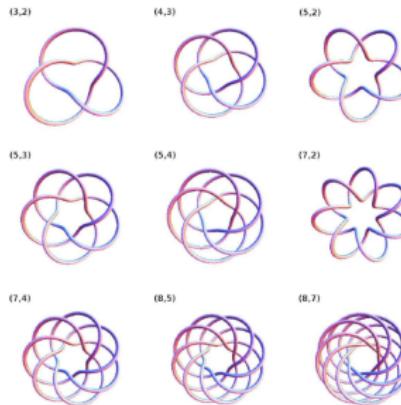
What is the B-model version of the construction? How we can compute KhR homology with coherent instead of constructible sheaves?

# Torus knots

Mathematicians need some examples to make a reasonable guess for B-model. Torus links are in some sense exactly solvable and provide lots of data for a guess.

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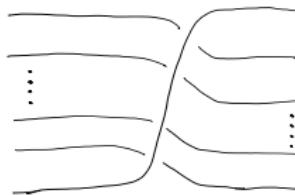
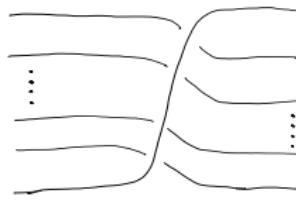


Figure:  $\text{cox}_n \in Br_n$

$$\text{cox}_n = \sigma_1 \cdot \sigma_2 \cdots \sigma_{n-1}.$$

$$T_{m,n} = L(\text{cox}_n^m).$$

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Conjecture (Gorsky, O. Rasmussen, Shende, 2012, Aganagic Shakirov, 2011)

$$H_{KhR}^*(T_{n,1+nk}) = H^0(Z, \Lambda^\bullet \mathcal{B} \otimes L^k), \quad Z \subset \text{Hilb}_n(\mathbb{C}^2).$$

# Hilbert schemes

## Definition

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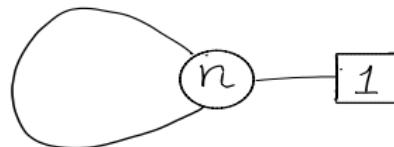
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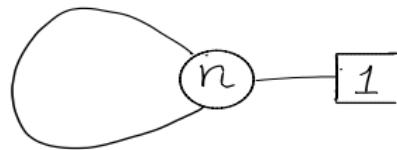
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The  $T_{sc} = \mathbb{C}^* \times \mathbb{C}^*$  action on  $\mathbb{C}^2$  induces the action on  $Hilb_n(\mathbb{C}^2)$ , hence double grading on  $H^i(Z, L^k \otimes \Lambda^m \mathcal{B})$ .

# Quiver



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$Hilb_n(\mathbb{C}^2)$  as quiver variety

$$\mu : T^* \mathfrak{gl}(n) \rightarrow \mathfrak{gl}(n), \quad \mu(X, Y) = XY - YX,$$

$$Hilb_n(\mathbb{C}^2) = \mu^{-1}(0)^{stab}/GL_n.$$

Conjecture (Gorsky,O., Rasmussen, Shende 2012; Gorsky, Negut, Rasmussen 2016)

There is (a canonical) way to construct for  $\beta \in Br_n$  there is  $\mathcal{F}_\beta \in D_{T_{sc}}^{coh}(FHilb_n(\mathbb{C}^2))$  such that

$$H_{KhR}^k(L(\beta)) = H^*(FHilb_n^{dg}, \mathcal{F}_\beta \otimes \Lambda^k \mathcal{B}).$$

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Theorem (O., Rozansky 2016)

*The conjecture is true if we adjust slightly  $D_{T_{sc}}^{coh}(\dots)$*

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$$D_{T_{sc}}^{per} = \{\dots \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} \mathcal{C}_1 \xrightarrow{d_1} \mathcal{C}_0 \xrightarrow{d_0} \dots | \mathcal{C}_i \text{ are coherent } \}.$$

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Theorem (O., Rozansky 2016)

For every  $\beta \in Br_n$  there is  $\mathcal{F}_\beta \in D_{T_{sc}}^{per}(FHilb_n^{free}(\mathbb{C}))$  such that

$supp(\mathcal{H}^\bullet(\mathcal{F})) \subset FHilb_n(\mathbb{C})$  and

$H^*(\beta) = \mathbb{H}(\mathcal{F}_\beta \otimes \Lambda^* \mathcal{B})$  is HOMFLY-PT homology of  $L(\beta)$ .



# Partial twists



Figure: Element  
 $\delta_i \in Br_n$

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Theorem (O., Rozansky 2017)

$$\Phi_{\beta \cdot \delta_k} = \Phi_\beta \otimes \mathcal{L}_k.$$

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There is an extension of the theorem to the case of

$$cox_S = \overrightarrow{\prod}_{i \in S} \sigma_i, \quad S \subset \{1, \dots, n-1\}.$$

Thus homology of the *Coxeter link*  $L(cox_S \cdot \delta^{\vec{k}})$  is given by the homology of  $[\mathcal{O}_{FZ_S}]^{vir} \otimes \mathcal{L}^{\vec{k}}$ .

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## Powers of the full twist

If  $S = \emptyset$  then  $\text{cox}_S = 1$  and  $L(\prod_i \delta_i^m) = T_{n,mn}$ .

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We expect to have some explicit formulas for links that are obtained from unknot by the Coxeter cabling procedure.

# KSR outline

Kapustin, Saulina and Rozansky proposed a realization of the 3D topological field theory, 2008.

## Three-category $3\mathcal{C}at_{sym}$

$Obj(3\mathcal{C}at_{sym}) = \{\text{holomorphic symplectic manifolds}\}$

$Hom(X, Y) = \{(F, L, f : F \rightarrow L), L \subset X \times Y \text{ is Lagrangian}\}.$

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$(F, L, f) \in \mathit{Hom}(X, Y)$   $(G, L', g) \in \mathit{Hom}(Y, W)$  compose to

$(H, L'', h), \quad H := (F \times W) \times_{X \times Y \times W} (X \times G)$

and  $h : H \rightarrow X \times W, L'' = h(H)$ .

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For our purposes we will need smaller three-category  $3Cat_{man}$ .

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$$(Z, w) \circ (Z', w') = (Z \times Y \times Z', w' - w) \in Hom(X, W).$$

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For  $(Z, w), (Z', w') \in Hom(X, Y)$  we have

$$Hom((Z, w), (Z', w')) = MF(X \times Z \times Z' \times Y, w' - w).$$

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Matrix Factorizations, Eisenbud 1980

$$MF(\mathbb{C}^n, W) = \{\dots \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1 \dots\}$$

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Example

If  $n = 1$  and  $W = x^4$  then following is an element of  $MF(\mathbb{C}, W)$ :

$$\dots \mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x] \xrightarrow{x^3} \mathbb{C}[x] \xrightarrow{x} \dots$$



# $3\mathcal{C}at_{sym}$ vs $3\mathcal{C}at_{man}$

Functor  $3\mathcal{C}at_{man} \rightarrow 3\mathcal{C}at_{sym}$

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$$(Z, w) \mapsto (F_w, L_w, \pi)$$

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Let impose condition on  $(Z_i, w_i)$ :  $Crit_w \subset \{w = 0\}$ , then we have

$$MF(X \times Z_1 \times Z_2 \times Y, w_1 - w_2) \rightarrow D^{per}(F_{w_1} \times_{T^*(X \times Y)} F_{w_2}).$$

# Main example

3Cat  $\mathfrak{gl}$

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$\mathfrak{gl} \rightarrow 3Cat_{man}$

$Hom(\mathfrak{gl}_n, \mathfrak{gl}_m) \ni Z \mapsto (Z, w(x, z, y) = \mu_n(z)(x) - \mu_m(z)(y)).$

Moment maps:  $\mu_n : Z \rightarrow \mathfrak{gl}_n^*$ ,  $\mu_m : Z \rightarrow \mathfrak{gl}_m^*$

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Let's take  $Z = T^*Fl$ .  $T^*Fl = \{(g, Y) \in GL_n \times \mathfrak{n}\}/B$

$$\mu(g, Y) = Ad_g(Y), \quad \phi = (Z, Tr(XAd_g Y)) \in Hom(\mathfrak{gl}_n, \mathfrak{gl}_0).$$

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$Hom(\phi, \phi)$

$$MF_n = MF_{B^2}(\mathfrak{gl}_n \times G^2 \times \mathfrak{n}^2, W),$$

$$W(X, g_1, Y_1, g_2, Y_2) = Tr(X(Ad_{g_1} Y_1 - Ad_{g_2} Y_2)).$$

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Theorem (O.-Rozansky, 2017)

For any  $n$  there is group homomorphism:

$$\Psi : Br_n^{aff} \rightarrow (MF_n, \star).$$



# Affine Braid group

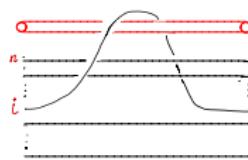


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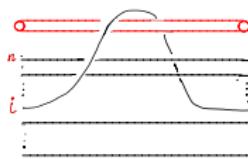


Figure: New element  $\Delta_i \in Br_n^{aff}$

Forgetful map:  $Br_n^{aff} \rightarrow Br_n$

$$\text{forget} : \Delta_n \mapsto 1, \quad \Delta_k \mapsto \delta_k.$$

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$$fr : X^{fr} \rightarrow X, \quad fr^* : MF_n \rightarrow MF_n^{fr}.$$

## Theorem

$$fr^* \circ \Psi = \Psi \circ \text{for}.$$



# Knot homology

Embedding  $j : FHilb^{free} \rightarrow X^{fr}$

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## Theorem (O.-Rozansky 2016)

*The triply graded vector space*

$\mathbb{H}(\mathcal{F}_\beta \otimes \Lambda^* \mathcal{B})$  is an isotopy invariant of  $L(\beta)$ .



# Where is $FHilb$ ?

Knorrer reduction:  $MF_n = MF_{B_n^2}(\mathfrak{b} \times G \times \mathfrak{n}, \bar{W})$ ,  
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$\bar{j}^*(\bar{\mathcal{F}}_\beta) \in MF_B(FHilb_n^{free}, 0)$  has homology support in  $FHilb_n$

# $gl(m|n)$ homology

$$\mathcal{FHilb}_n^{\text{free}}(\mathbb{C}) = \{(X, Y, v) \in \mathfrak{b} \times \mathfrak{n} \times V | \mathbb{C}\langle X, Y \rangle v = V\} / GL_n,$$
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$$\phi_{m|n} \in H^0(FHilb_n^{free}(\mathbb{C}), \mathcal{B}^\vee), \quad \phi_{m|n}(X, Y, v) = X^m Y^n v$$

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Let  $d_{\mathcal{F}}$  be the differential of  $\mathcal{F}_\beta \in D^{per}(FHilb_n^{free}(\mathbb{C}))$  and

$$\mathbb{H}_{m|n}(\beta) := H(\mathcal{F}_\beta \otimes \Lambda^\bullet \mathcal{B}, d_{\mathcal{F}} + d_{m|n})$$

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Conjecture [O., Rozansky 2016]

$$\mathbb{H}(\beta) = H_{gl(m|n)}(L(\beta)).$$

# Virtual sheaves

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$$[O_{FZ}]^{vir} = K(X_{ii} - X_{i+1,i+1}, [X, Y]_{ij}).$$