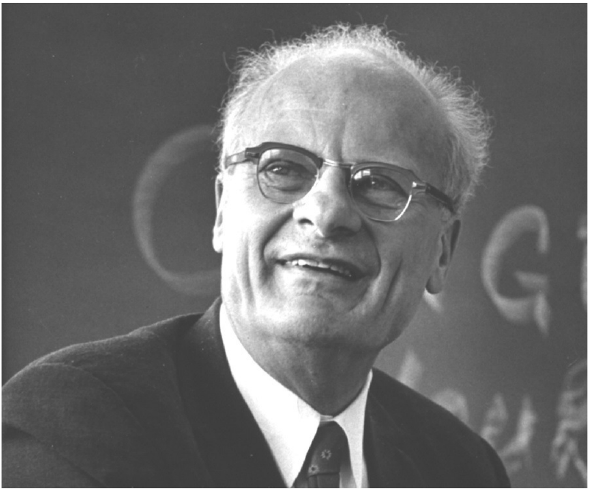
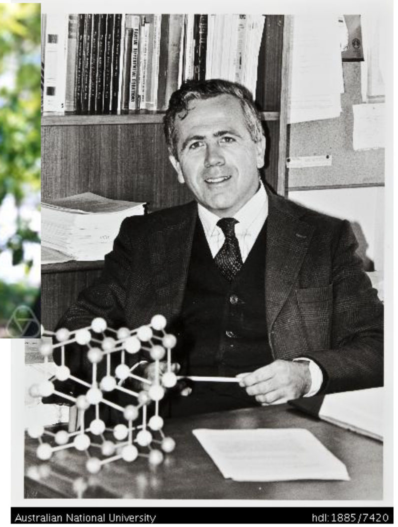
Gauge theories and Bethe eigenfunctions

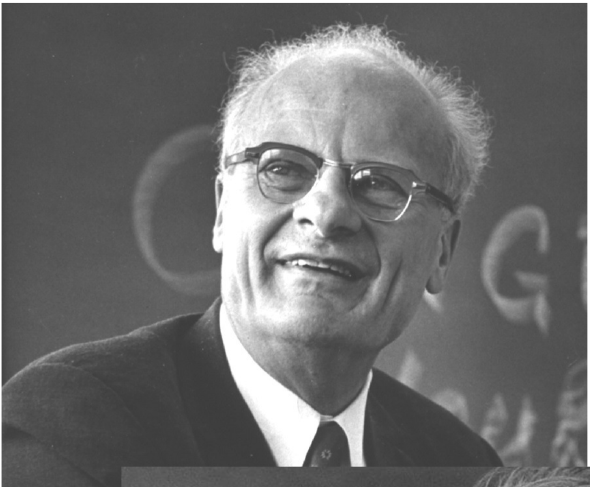
based on *Quasimap counts and Bethe eigefunctions*, Mina Aganagic & A.O., arXiv:1704.08746

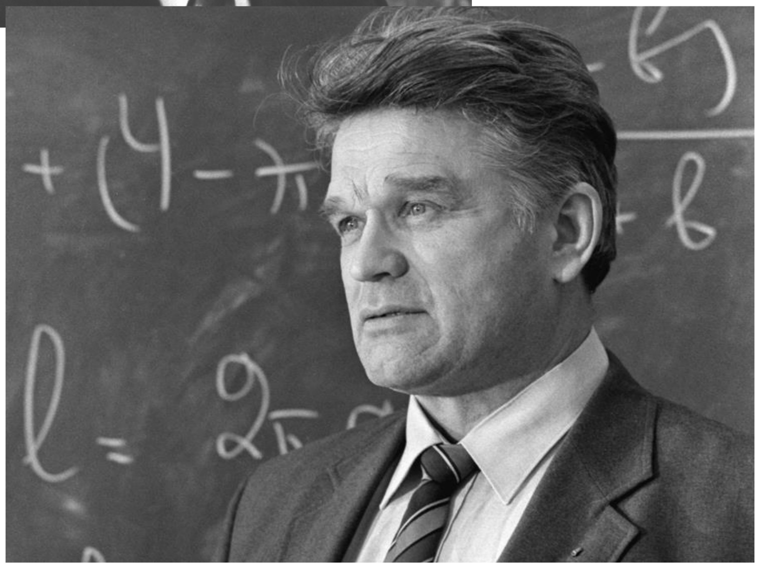
"Bethe Ansatz" is the art and science of finding spectra and eigenfunctions in quantum integrable systems with a quantum group symmetry





"Bethe Ansatz" is the art and science of finding spectra and eigenfunctions in quantum integrable systems with a quantum group symmetry





Quantum groups (quantum loop algebras)

$\widehat{\mathfrak{g}}$

If $\mathfrak{g}$ is a Lie algebra, then so is $\mathfrak{g}[t^{\pm 1}]$ = Laurent poly with values in $\mathfrak{g}$

Quantum groups (quantum loop algebras)

$$= \hat{\mathfrak{g}}$$

If $\mathfrak{g}$ is a Lie algebra, then so is $\mathfrak{g}[t^{\pm 1}]$ = Laurent poly with values in $\mathfrak{g}$

Has representations of the form

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n)$$

a representation of $\mathfrak{g}$

evaluate at $t = a_1$

Quantum groups (quantum loop algebras)

$/\!/ \ \hat{\mathfrak{g}}$

If $\mathfrak{g}$ is a Lie algebra, then so is $\mathfrak{g}[t^{\pm 1}]$ = Laurent poly with values in $\mathfrak{g}$

Has representations of the form

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n)$$

a representation of $\mathfrak{g}$

evaluate at $t = a_1$

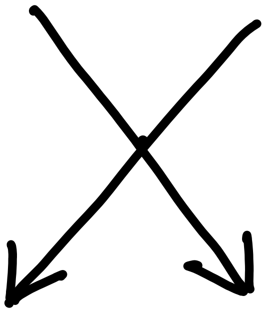
A quantum group $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is a deformation such that

$$V_1(a_1) \otimes V_2(a_2) \not\cong V_2(a_2) \otimes V_1(a_1)$$

R-matrices

For generic a_1/a_2 there is a nontrivial intertwiner

$$V_1(a_1) \otimes V_2(a_2)$$



$$V_2(a_2) \otimes V_1(a_1)$$

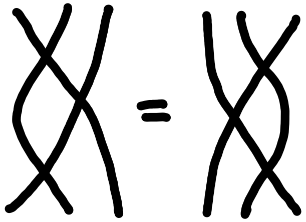
$$R_{V_1,V_2}(a_1/a_2)$$

rational function of a_1/a_2

vertex interaction in integrable vertex models

R-matrices

For generic a_1/a_2 there is a nontrivial intertwiner



YB

$$V_1(a_1) \otimes V_2(a_2)$$

$$V_2(a_2) \otimes V_1(a_1)$$

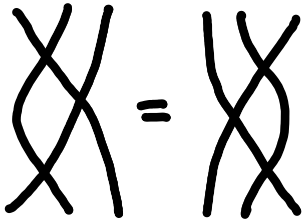
$$R_{V_1, V_2}(a_1/a_2)$$

rational function of a_1/a_2

vertex interaction in integrable vertex models

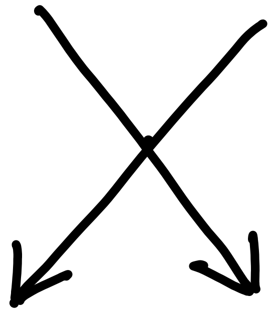
R-matrices

For generic a_1/a_2 there is a nontrivial intertwiner



YB

$V_1(a_1) \otimes V_2(a_2)$



$V_2(a_2) \otimes V_1(a_1)$

$R_{V_1, V_2}(a_1/a_2)$

rational function of a_1/a_2

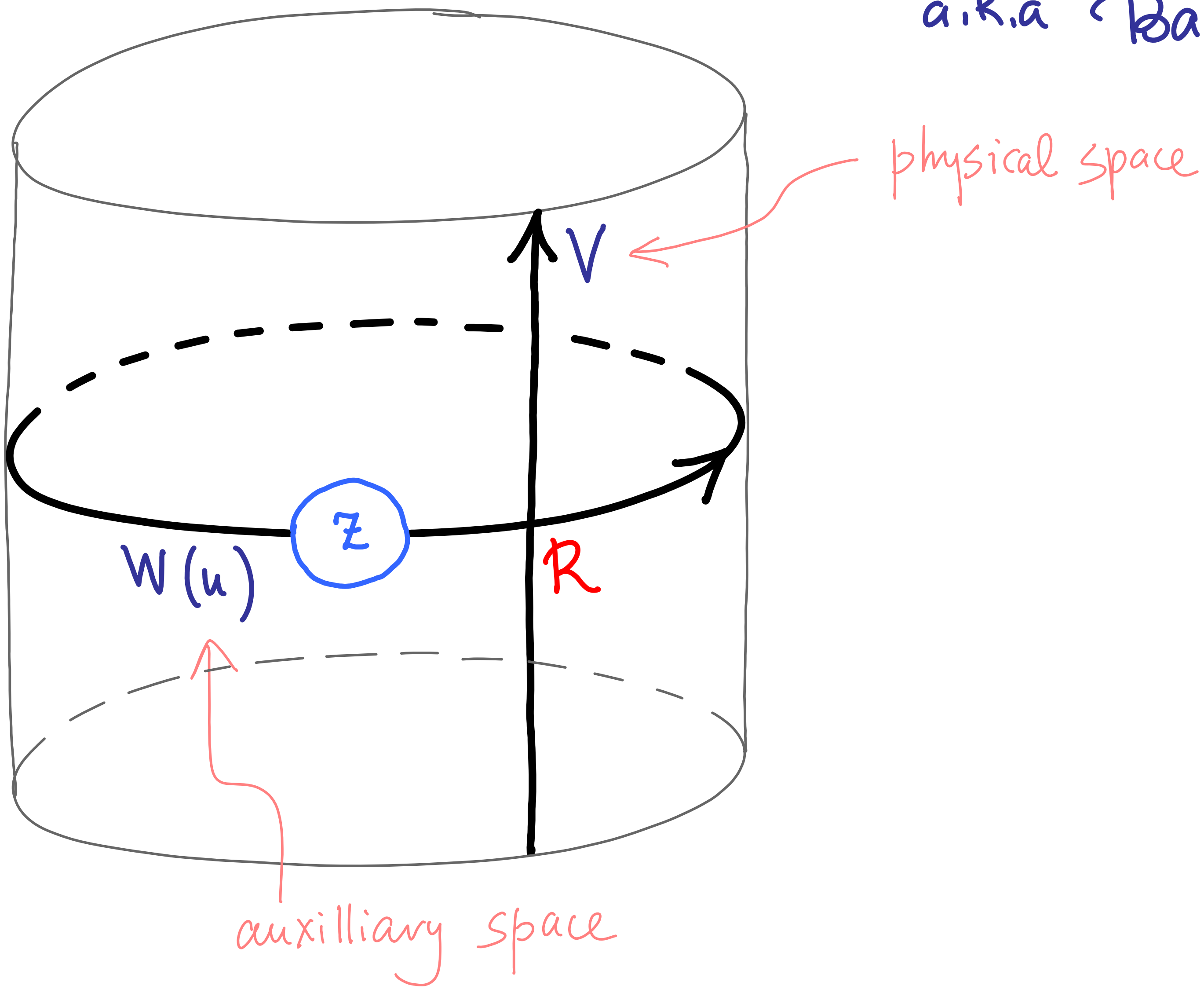
vertex interaction in integrable vertex models

One can reconstruct the whole quantum group from R-matrices

[Faddeev - Reshetikhin - Takhtajan]

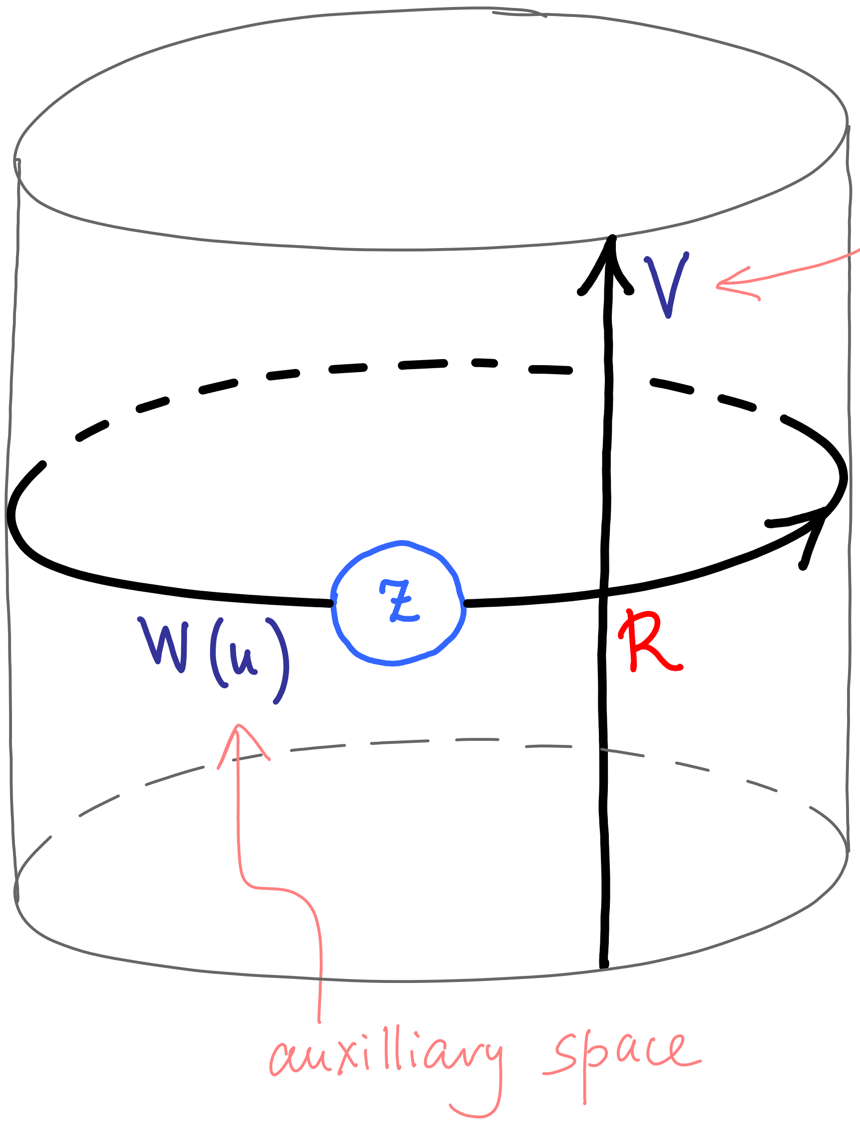
Quantum integrals of motion

a.k.a Baxter Subalgebra



Quantum integrals of motion

a.k.a Baxter Subalgebra in $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$



Here $z \in e^{\mathfrak{f}} \subset \mathcal{U}_\hbar(\hat{\mathfrak{g}})$

where $\mathfrak{f} \subset \mathfrak{g}$ are diagonal matrices

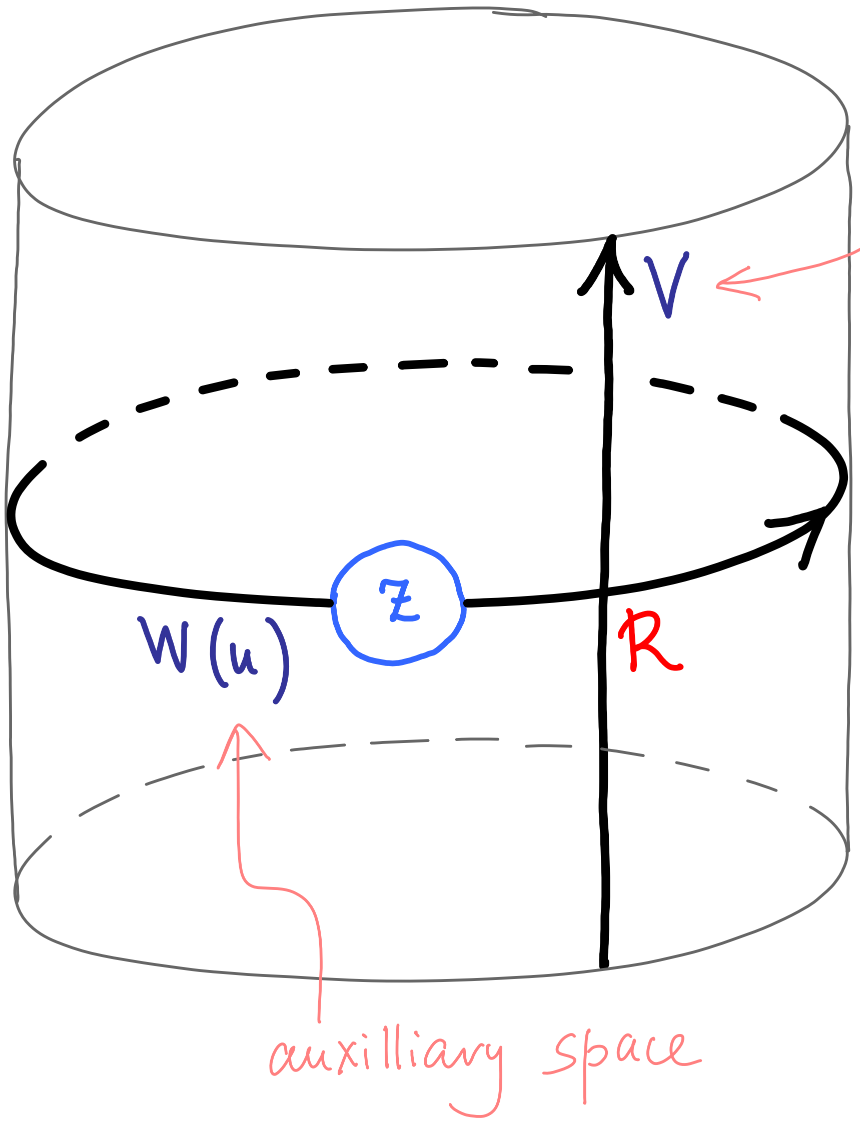
quasiperiodic boundary conditions

Quantum integrals of motion

a.k.a Baxter Subalgebra in

Here $z \in e^{\mathfrak{f}} \subset \mathcal{U}_\hbar(\hat{\mathfrak{g}})$

where $\mathfrak{f} \subset \mathfrak{g}$ are diagonal matrices



Commute for all W and u

for fixed z

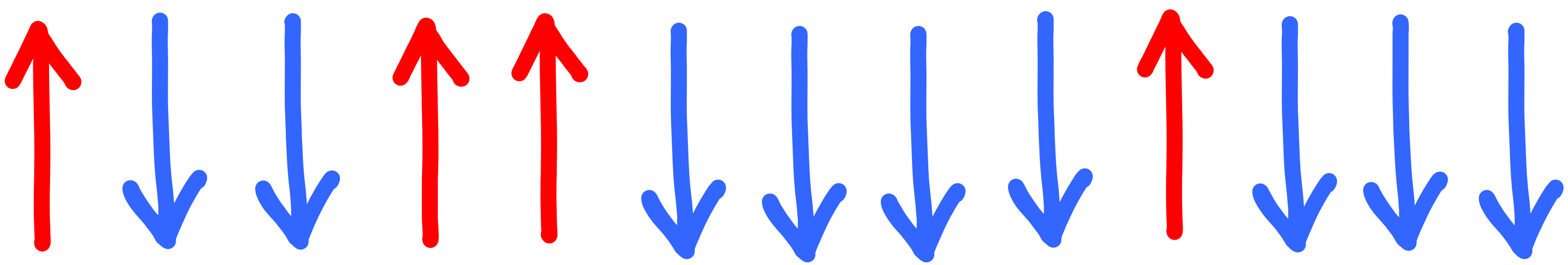
quasiperiodic boundary conditions

The textbook Bethe ansatz diagonalizes these for

$$\mathfrak{g} = \mathfrak{sl}_2$$

$$V = \mathbb{C}^2(a_1) \otimes \cdots\cdots \otimes \mathbb{C}^2(a_n)$$

spin $\frac{1}{2}$ chain of length n



The textbook Bethe ansatz diagonalizes these for

$$\mathfrak{g} = \mathfrak{sl}_2$$

spin $\frac{1}{2}$ chain of length n

$$V = \mathbb{C}^2(a_1) \otimes \cdots\cdots \otimes \mathbb{C}^2(a_n)$$

the challenge is to do it for a very general $\mathfrak{g}$ (including ∞-dimensional)

The textbook Bethe ansatz diagonalizes these for

$$\mathfrak{g} = \mathfrak{sl}_2$$

$$V = \mathbb{C}^2(a_1) \otimes \dots \otimes \mathbb{C}^2(a_n)$$

spin $\frac{1}{2}$ chain of length n

the challenge is to do it for a very general $\mathfrak{g}$ (including ∞-dimensional)

A more general problem is to solve certain q-difference eq. for

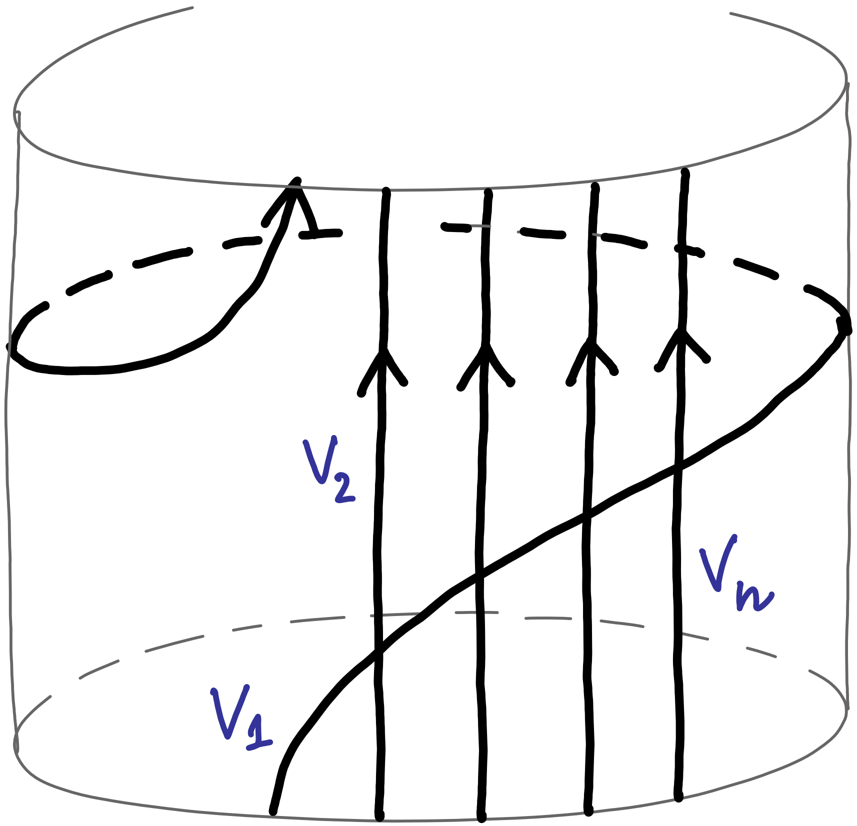
$$\Psi(a_1, \dots, a_n) \in V_1(a_1) \otimes \dots \otimes V_n(a_n)$$

unrelated to any other variables used so far concretely, the <u>Quantum Knizhnik-Zamolodchikov eq.</u> for

$$\Psi(a_1, \dots, a_n) \in V_1(a_1) \otimes \dots \otimes V_n(a_n)$$

reads

$$\Psi(q a_1, \dots, a_n) = (z \otimes 1 \otimes \dots \otimes 1) R_{V_1, V_n} \cdots R_{V_1 V_2} \Psi$$



[I. Frenkel – N. Reshetikhin]

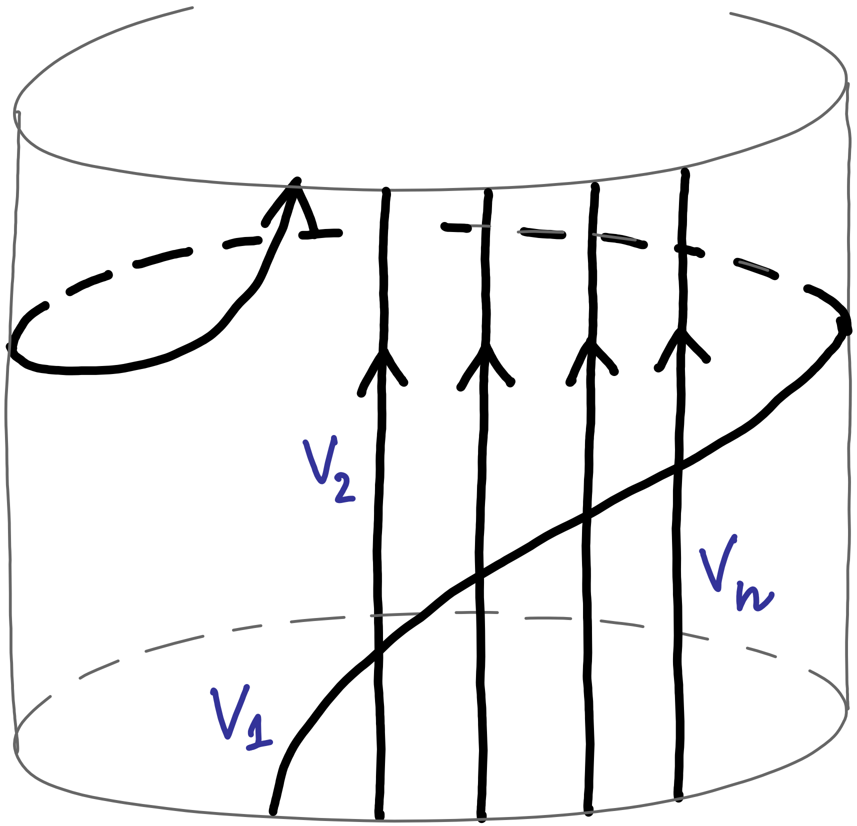
[F. Smirnov]

concretely, the Quantum Knizhnik-Zamolodchikov eq. for

$$\Psi(a_1, \dots, a_n) \in V_1(a_1) \otimes \dots \otimes V_n(a_n)$$

reads

$$\Psi(q a_1, \dots, a_n) = (z \otimes 1 \otimes \dots \otimes 1) R_{V_1, V_n} \cdots R_{V_1 V_2} \Psi$$



there are commuting "dynamical" equations in z

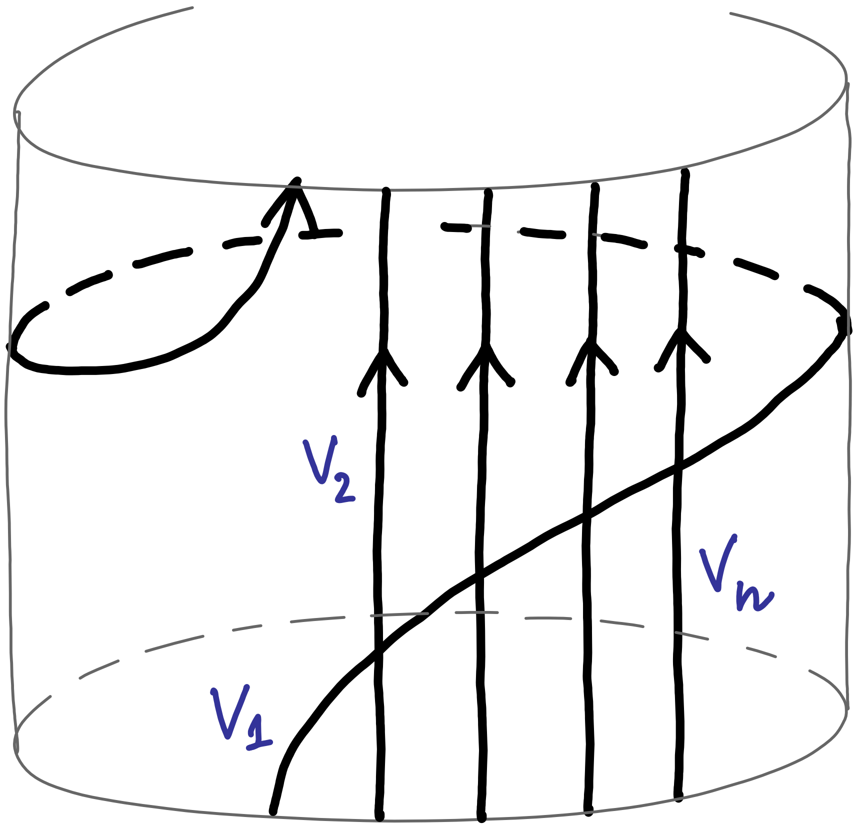
[Etingof, Felder, Tarasov, Varchenko, ...]

concretely, the Quantum Knizhnik-Zamolodchikov eq. for

$$\Psi(a_1, \dots, a_n) \in V_1(a_1) \otimes \dots \otimes V_n(a_n)$$

reads

$$\Psi(q a_1, \dots, a_n) = (z \otimes 1 \otimes \dots \otimes 1) R_{V_1, V_n} \cdots R_{V_1 V_2} \Psi$$



there are commuting "dynamical" equations in z

as $q \to 1$ become an eigenvalue problem a generalization of Bethe Ansatz is the search for integral solutions of the q - difference equations

depends on the representation

$$\Psi_\alpha = \int f_\alpha(x_1, x_2, \ldots, a)\, \mathbb{K}(x, z, a, \ldots, q)$$

integration variables

fixed kernel function

index in the physical space $V = \bigotimes V_i(a_i)$

$$\mathbb{K} \sim e^{\frac{1}{\ln q} S(x,z,a) + \ldots}$$

as $q \to 1$

studied by [Tarasov-Varchenko,...] for $\mathfrak{g} = \mathfrak{gl}(n)$

in the $q \to 1$ limit, we get

spectrum of the problem

$$\frac{\partial}{\partial x_i} S = 0$$

Bethe equations for "Bethe roots" $x_1, x_2, \ldots$

in the $q \to 1$ limit, we get

spectrum of the problem

$$\frac{\partial}{\partial x_i} S = 0$$ ← Bethe equations for "Bethe roots" $x_1, x_2, \ldots$

$$a_i \frac{\partial}{\partial a_i} S \rightsquigarrow$$ eigenvalues of the qKZ operators, etc.

in the $q \to 1$ limit, we get

Spectrum of the problem

$$\frac{\partial}{\partial x_i} S = 0$$ Bethe equations for "Bethe roots" $x_1, x_2, \ldots$

$$a_i \frac{\partial}{\partial a_i} S = \text{eigenvalues of the qKZ operators, etc.}$$

and the map

$$\text{Hilbert space} \ni \alpha \mapsto f_\alpha \Big|_{\text{Bethe}} \in \text{functions on the spectrum}$$

is the diagonalization!

So, the main problem is to find

functions

$$\text{Hilbert space } V \ni \alpha \longmapsto f_\alpha(x_1, x_2, \dots)$$

"off-shell Bethe function"

name introduced by Babujian

So, the main problem is to find

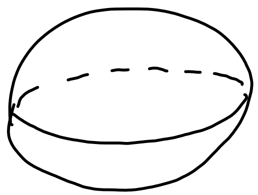
functions

$$\text{Hilbert space } V \ni \alpha \longmapsto f_\alpha(x_1, x_2, \ldots)$$

"off-shell Bethe function"

and this is the problem we solve in the setup discovered by Nekrasov and Shatashvili

It embeds the problem in 3d susy gauge theories on

$M^3 =$  $\times S^1$

Riemann surface C

NS correspondence

records the weight of α, e.g. the number of $\uparrow$

$$\text{gauge group} = \prod_{i=1}^{\text{rank of}} U(v_i)$$

$$\uparrow\ \downarrow\ \downarrow\ \uparrow\ \uparrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \uparrow\ \downarrow\ \downarrow\ \downarrow$$

NS correspondence

records the weight of α,
e.g. the number of $\uparrow$

$$\text{gauge group} = \prod_{i=1}^{\text{rank of}} U(v_i)$$

the Bethe roots

$$\begin{pmatrix} x_{i1} & & & 0 \\ & x_{i2} & & \\ & & \ddots & \\ 0 & & & x_{iv_i} \end{pmatrix}$$

live in the maximal torus of the gauge group

NS correspondence

records the weight of α, e.g. the number of $\uparrow$

gauge group $= \prod_{i=1}^{\text{rank of}} U(v_i)$

the Bethe roots
$$\begin{pmatrix} x_{i1} & & & 0 \\ & x_{i2} & & \\ & & \ddots & \\ 0 & & & x_{iv_i} \end{pmatrix}$$
live in the maximal torus of the gauge group

integration over $\nearrow$ is just a projection onto gauge-invariant states by H. Weyl

NS correspondence

records the **weight** of α,
e.g. the number of $\uparrow$

$$\text{gauge group} = \prod_{i=1}^{\text{rank of}} U(v_i)$$

here act $\begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \end{pmatrix}$

$$\text{matter} = \bigoplus \mathbb{C}^{v_i} \otimes \text{flavor space } W_i$$

NS correspondence

records the weight of α,
e.g. the number of $\uparrow$

rank of

gauge group $= \prod_{i=1} U(v_i)$

here act $\begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \end{pmatrix}$

matter $= \bigoplus \mathbb{C}^{v_i} \otimes$ flavor space W_i

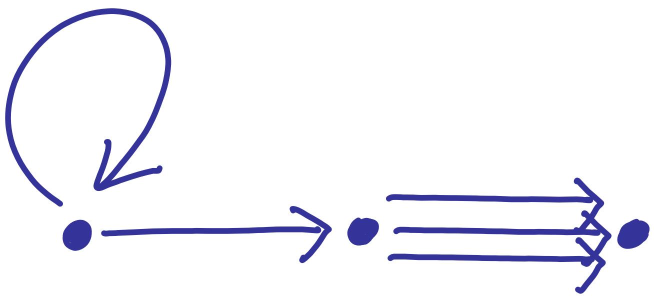
$\bigoplus$ bifundamental $(\mathbb{C}^{v_i})^* \otimes \mathbb{C}^{v_j}$

$i \longrightarrow j$

sum over arrows in a

quiver = Dynkin* diagram of $\mathfrak{g}$

arbitrary graph



NS correspondence

records the weight of α, e.g. the number of $\uparrow$

rank of

$$\text{gauge group} = \prod_{i=1} U(v_i)$$

here act $\begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \end{pmatrix}$

$$\text{matter} = \bigoplus \mathbb{C}^{v_i} \otimes \text{flavor space } W_i$$

$$\bigoplus \text{bifundamental } (\mathbb{C}^{v_i})^* \otimes \mathbb{C}^{v_j}$$

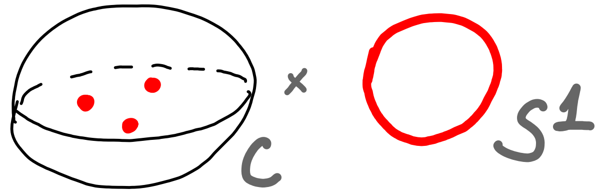
$$i \longrightarrow j$$

sum over arrows in a

quiver = Dynkin* diagram of $\mathfrak{g}$

$\bigoplus$ duals, for more susy

NS correspondence

$M^3 =$ 

Hilbert space of the quantum integrable system $=$ line operators

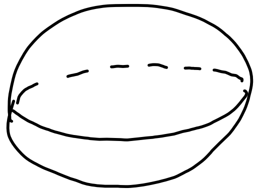
$=$ equivariant K-theory of

$\mathcal{M}_{\text{Higgs}}$

Higgs branch of the moduli of vacua

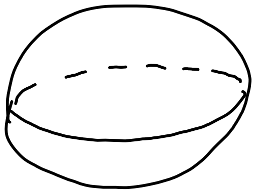
Nakajima quiver variety

already is a module over a certain (smaller) quantum group by the original construction of Nakajima mathematically, the susy indices for

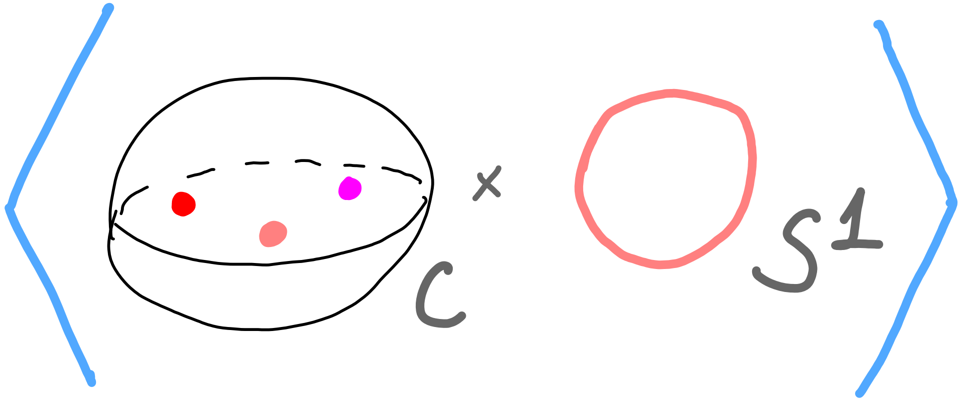
$$M^3 = \quad$$  $$\times S^1$$

LGSM

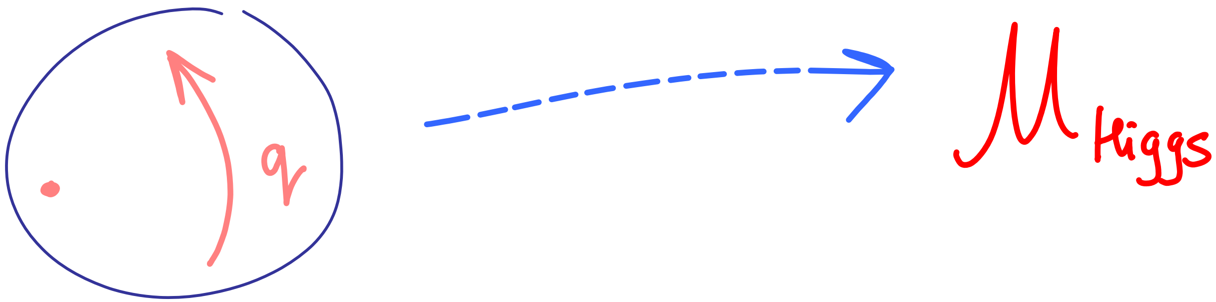
are integrals in K-theory of the space of (quasi-)maps

 $$\xrightarrow{g} \mathcal{M}_{\text{Higgs}}$$

weighted by $z^{\deg g}$ ⟵ multiindex, a subject both formally and conceptually related to other kinds of curve counting such as Gromov-Witten theories (⟵ top strings) in particular, the subject has the quantum K-theory ring, with structure constants given by



as well as the quantum q-difference equations, which record the response to twisting the geometry over $\mathbb{P}^1$



One of the key insights of NS :

quantum K-theory ring = symm. polynomials in x_{ij} / Bethe equations

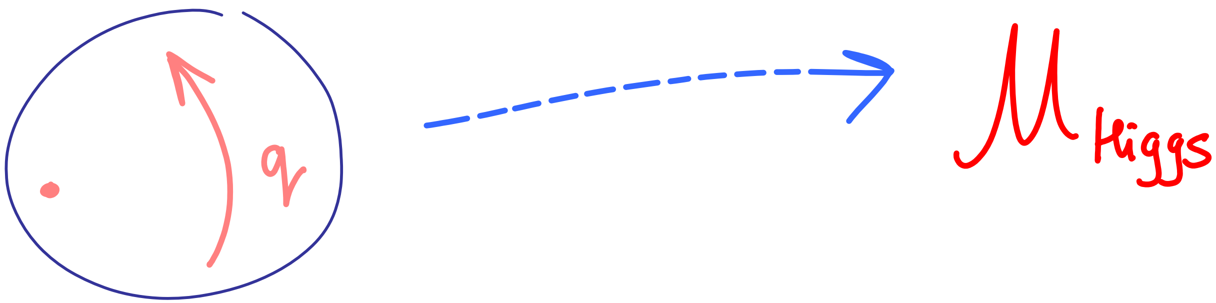
standard generators, Chern roots of universal bundles

See [Pushkar-Smirnov-Zeitlin] for a discussion aimed at mathematicians

Baxter's Q-operators

the problem may be studied from a geometric representation theory angle [Maulik-O.] and one of the end results of this analysis is

Theorem [A.O., A. Smirnov-A.O.] qKZ + dynamical eq + ... are the quantum difference equations for $\mathcal{M}_{\text{Higgs}}$ where the step q is the equivariant variable that rotates the domain of the quasimap



Now back to the main problem of Bethe anzatz:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{the Hilbert space of a QIS} \ni \alpha \arrow[r, "?"] \arrow[d, equal, "{[NS]}"'] & f_\alpha(x_1, x_2, \dots) \arrow[d, equal, "{[NS]}"] \\
K_{eq}(\mathcal{M}_{\text{Higgs}}) \arrow[r, "!\ {[AO]}"] & \text{representations of the gauge group}
\end{tikzcd}
$$

"off-shell Bethe function"

supply of observables

observables

Now back to the main problem of Bethe anzatz:

"off-shell Bethe function"

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{the Hilbert space of a QIS} \ni \alpha \arrow[r, "?"] \arrow[d, equal, "{[NS]}"'] & f_\alpha(x_1, x_2, \ldots) \arrow[d, equal, "{[NS]}"] \\
K_{eq}(\mathcal{M}_{Higgs}) \arrow[r, shift left, "{!\ [AO]}"] & \text{representations of the gauge group} \arrow[l, shift left]
\end{tikzcd}$$

supply of observables

observables

depends on z and q

in contrast to

Smirnov has very nice formulas for the natural map in this direction

Now back to the main problem of Bethe anzatz:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{the Hilbert space of a QIS} \ni \alpha \arrow[r, "?"] \arrow[d, equal, "\text{[NS]}"'] & f_\alpha(x_1, x_2, \dots) \arrow[d, equal, "\text{[NS]}"] \\
K_{eq}(\mathcal{M}_{Higgs}) \arrow[r, "!\ \text{[AO]}"] & \text{representations of the gauge group}
\end{tikzcd}$$

"off-shell Bethe function"

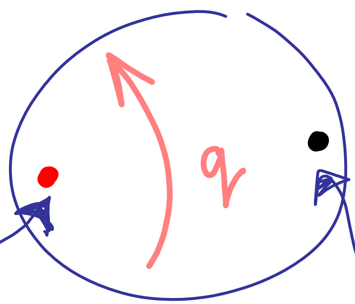
supply of "descendent" observables

observables

in our paper, we give several equivalent descriptions of this map, some of which make no explicit reference to gauge theories mathematically, the dictionary

$$K_{eq}(\mathcal{M}_{Higgs}) \rightleftarrows \text{representations of the gauge group}$$

is given by 2-point functions with



$\dashrightarrow \mathcal{M}_{Higgs}$

relative insertion at 0

descendent (i.e. from universal bundles) insertion at ∞

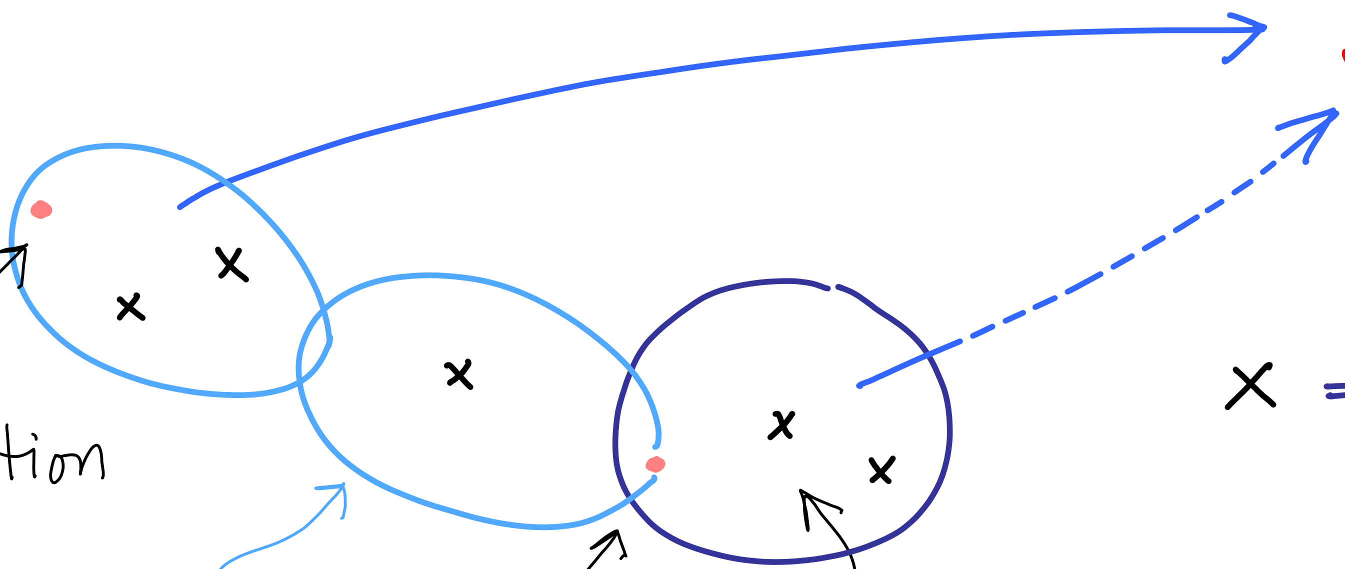
ambient stack

$K_{eq}(\mathcal{M}_{Higgs})$

representations of the gauge group $= K_{eq}\left(\left[\text{matter}/\text{gauge}\right]\right)$

relative insertions:

$\mathcal{M}_{Higgs}$



new location

here we zoom in (i.e. form a new bubble) every time a × approaches

old location of the insertion

original space on which gauge theory lived

× = singularities of the quasi-map (cf. point-like instantons)

Description #1

$f_\alpha(x)$ is the descendent observable that corresponds to the relative observable α

Description #2 $f_\alpha(x)$ is the stable envelope

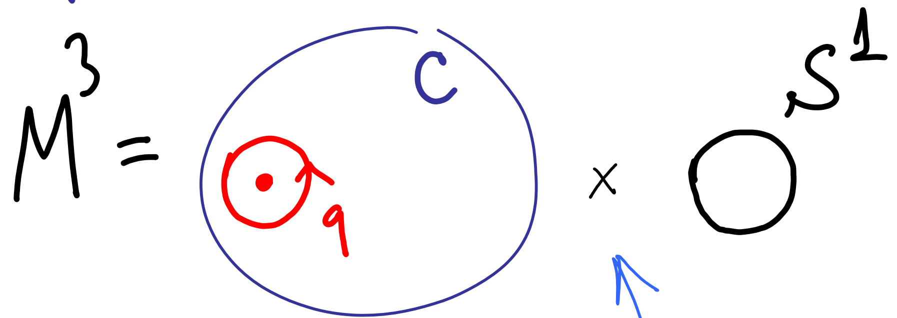
ambient stack

$$K_{eq}\left(\mathcal{M}_{Higgs}\right) \longrightarrow K_{eq}\left(\left[\text{matter}/\text{gauge}\right]\right)$$

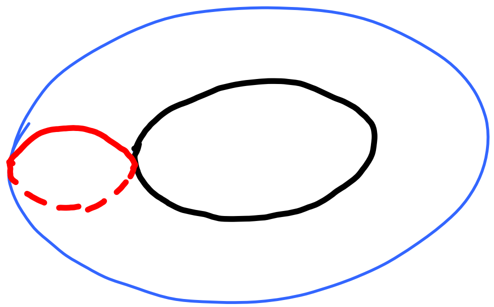
no stability

in the style of Halpern-Leistner - Maulik - A.O.

This is the $q \to 0$ limit of the elliptic stable envelopes of [Aganagic-O.], which are boundary conditions for gauge theories in question



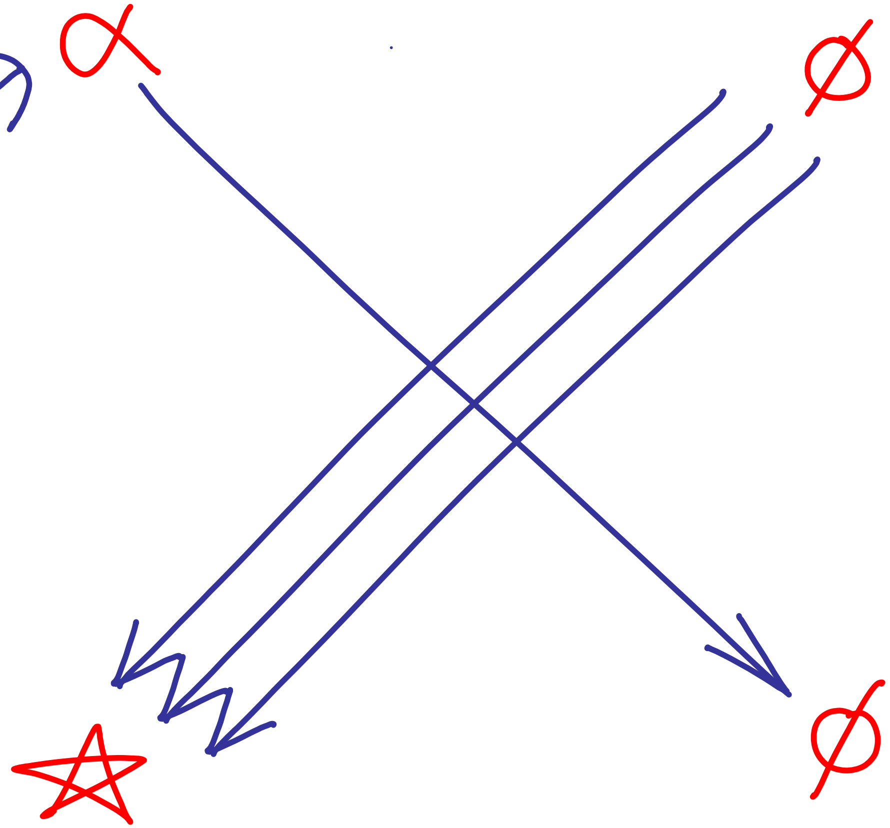
$\partial M^3 = \text{torus}$



Description #3

$f_\alpha(x)$ is a matrix element of R-matrix

our representation of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$



$\emptyset \in \bigotimes_{i,j} F_i(x_{ij})$

fundamental representations

vacuum

specific eigenvector of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$

this specializes to the

$B(x_1)B(x_2)\ldots\ |\emptyset\rangle$

formula for $\mathfrak{g} = \mathfrak{sl}_2$

Description #4

Explicit abelianization formula in the style of Shenfeld-Smirnov—....

arXiv.org > math-ph > arXiv:1704.08746

Mathematical Physics

Quasimap counts and Bethe eigenfunctions

Mina Aganagic, Andrei Okounkov

(Submitted on 27 Apr 2017)

We associate an explicit equivalent descendent insertion to any relative insertion in quantum K-
explicit formula for off-shell Bethe eigenfunctions for general quantum loop algebras associated
the corresponding quantum Knizhnik-Zamolodchikov and dynamical q-difference equations.

Subjects: **Mathematical Physics (math-ph)**; High Energy Physics - Theory (hep-th); Algebraic Geometry (m
Cite as: arXiv:1704.08746 **[math-ph]**
(or arXiv:1704.08746v1 **[math-ph]** for this version)

Section 3.2

as a T-module. In particular, the A-weights in V are given by minus contents of the boxes. As a polarization, we may take

$$
\begin{aligned}
T^{1/2} &= V + (t_1 - 1)\,\mathrm{Hom}(V,V) \\
&= \sum x_i + (t_1 - 1) \sum_{i,j} x_i/x_j
\end{aligned}
$$

where $\{x_i\}$ are the Chern roots of V. A fixed point is specified by the assignment of x_i to the boxes of λ, up to permutation.

If we take t_1 to be a *repelling* weight for A then

$$
T^{1/2}_{\gtrless} = \sum_{c(i)\gtrless 0} x_i + t_1 \sum_{c(i)\gtrless c(j)+1} x_i/x_j - \sum_{c(i)\gtrless c(j)} x_i/x_j
$$

where

$$
T^{1/2}_{>} = T^{1/2}_{\text{attracting}}\,, \quad T^{1/2}_{<} = T^{1/2}_{\text{repelling}}\,,
$$

and $c(i)$ is the content of the box in λ assigned to x_i. Therefore, up to an $\hbar$ multiple, we have

$$
\mathsf{f}_\lambda = \text{symmetrization of } \frac{\Pi_1 \Pi_2}{\Pi_3}
$$

where

$$
\Pi_1 = \prod_{c(i)<0} (1 - x_i) \prod_{c(i)>0} (t_1 t_2 - x_i)
$$

and

$$
\begin{aligned}
\Pi_2 &= \prod_{c(i)<c(j)+1} (x_j - t_1 x_i) \prod_{c(i)>c(j)+1} (t_2 x_j - x_i) \\
\Pi_3 &= \prod_{c(i)<c(j)} (x_j - x_i) \prod_{c(i)>c(j)} (t_1 t_2 x_j - x_i)\,.
\end{aligned}
$$

These are formulas for K-theoretic stable envelopes for $\mathrm{Hilb}(\mathbb{C}^2, n)$ with the polarization and slope as in Proposition 7. They are a direct K-theoretic generalization of the formulas from [48, 50].

Note that in all cases treated by the formula (79) the slope is near an integral line bundle. Much more interesting functions appear at fractional slopes, but they seem to be not required in the context of Bethe Ansatz.

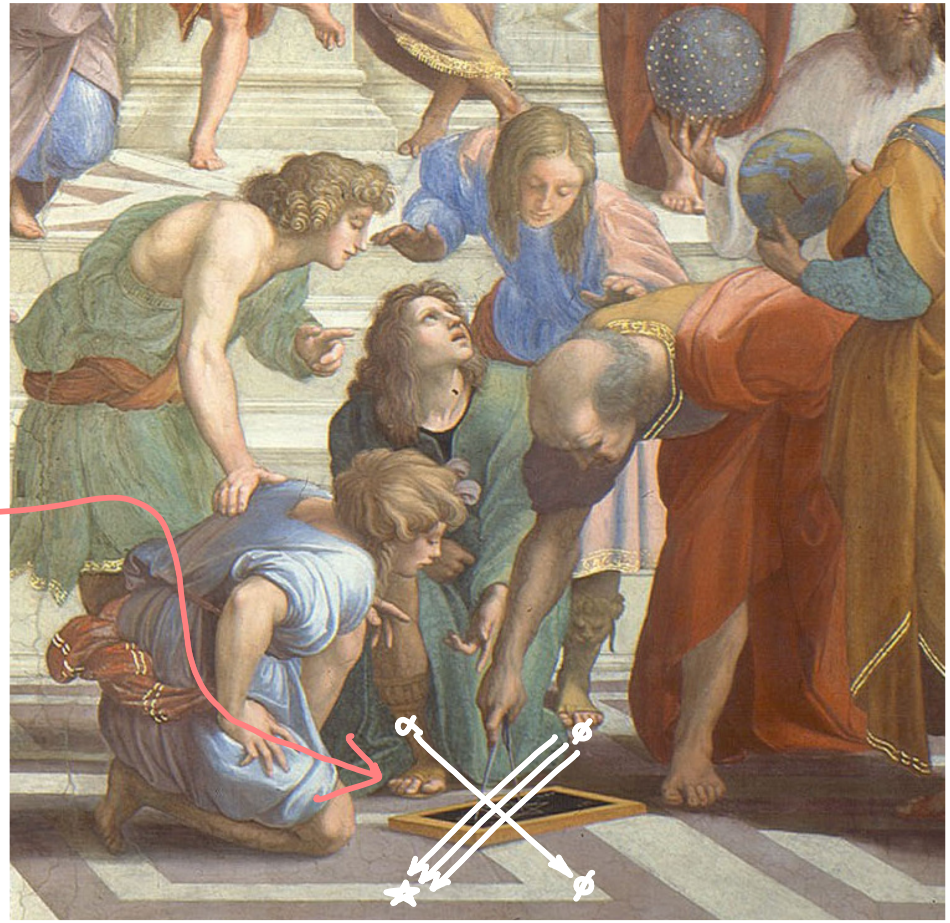
3.2.7

The proof of Proposition 7 takes several steps. As a first step, we clarify the geometric meaning of the formula (79).

We separate the numerator and denominator in (79) by writing

$$
\left(T^{1/2}\right)_{\text{repelling}} \oplus \hbar \left(T^{1/2}\right)_{\text{attracting}} = \rho_+ - \rho_-
$$

why is geometry effective in proving explicit formulas like ?



for a number of both theoretical and very practical reasons such as:

- it lets one be inexplicit about many features of R, or $U_\hbar(\hat{\mathfrak{g}})$, or ....
- it automatically selects the right contour of integration
- it is awfully good at showing that poles cancel
- ....