AdS/CFT and p-Adic Numbers: A Model of Discrete Holography

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String Math 2017 – July 26

This talk is based on:

- MTH, Matilde Marcolli, Ingmar Saberi, Bogdan Stoica, *Tensor networks, p-adic fields, and algebraic curves: arithmetic and the AdS_3/CFT_2 correspondence*, arXiv:1605.07639
- Steven S. Gubser, MTH, Christian Jepsen, Matilde Marcolli, Sarthak Parikh, Ingmar Saberi, Bogdan Stoica, Brian Trundy, *Edge length dynamics on graphs with applications to p-adic AdS/CFT*, arXiv:1612.09580
- Gubser, MTH, Jepsen, Parikh, Saberi, Stoica, Trundy, *Signs of the time: Melonic theories over diverse number systems*, arXiv:1707.01087
- additional work in progress

Overview

- Introduction and motivation
- Informal review of p-adic numbers
- Correspondence for scalar fields
- p-adic field theories
- A bulk action
- Conclusions and future work

Introduction and motivation

- AdS/CFT remarkably successful over the last 20 years; exact equivalance between gravity in Anti-de Sitter space (AdS) to a conformal field theory (CFT) at the AdS boundary.[1]

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- Best understood examples realized by brane configurations in string theory; ex $4d\ N = 4$ SYM $\leftrightarrow$ $\text{AdS}_5 \times S^5$.

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- Possible to understand model independent aspects; $\mathrm{AdS}_3/\mathrm{CFT}_2$, geometry from entanglement (Ryu-Takayanagi), BTZ black hole and Riemann surfaces, renewed interest in $\mathrm{AdS}_2/\mathrm{CFT}_1$ and SYK models, etc.

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- Strong/weak duality, many challenges remain...

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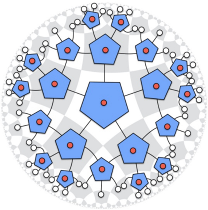

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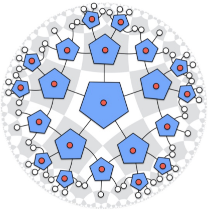

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- Discrete models leave something to be desired; break symmetries, don't have field theory dynamics, full boundary theory unclear

Are there discrete versions of AdS/CFT
that are more similar to continuum models?

Hints from mathematics:

- In low dimensions, the spacetime of the conformal field theory is an algebraic curve over a field;

$$\mathrm{CFT}_2 : \mathbb{P}^1(\mathbb{C})$$
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- It's possible to construct algebraic curves over fields other than $\mathbb{R}$ or $\mathbb{C}$.
- We can complete the rationals $\mathbb{Q}$ with respect to another norm; the *p-adic norm*.

Brief review of p-adic numbers

- As opposed to the usual real norm on $\mathbb{Q}$ (here denoted $|\cdot|_\infty$,) for any $x \in \mathbb{Q}$ we can write

$$x = p^{\nu}(a/b),\ a, b \perp p$$

where p is any prime number.

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- Ex: $|0|_p = 0$, $|p| = p^{-1}$, $|a_n p^n + a_{n+1} p^{n+1} + \dots|_p = p^{-n}$, ...
- Ultrametric: $|x + y|_p \le \max(|x|_p, |y|_p)$

p-adics, continued

- By a theorem of Ostrowski, every nontrivial norm on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ or $|\cdot|_\infty$.

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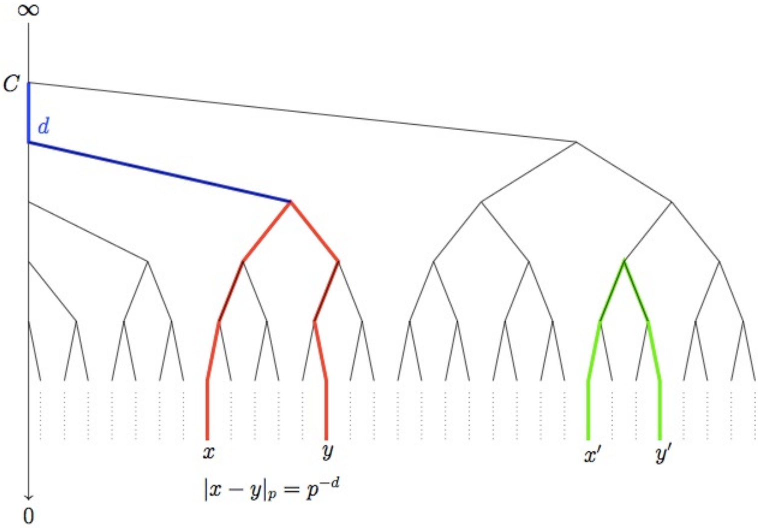
  This is a Cauchy sequence that convergences in the p-adic norm; their set defines the field of the p-adic numbers $\mathbb{Q}_p$.
- (Much of what we do will work for a finite extension of $\mathbb{Q}_p$, such as $\mathbb{Q}_p^n$ or more generally $\mathbb{K}$.)

Visualizing $\mathbb{Q}_p$



Hints from physics:

- p-adic string theory: replace the open bosonic string boundary $\mathbb{P}^1(\mathbb{R})$ with $\mathbb{P}^1(\mathbb{Q}_p)$ (the target space is still smooth.)

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- Veneziano amplitude[2]:

$$A_p^4(k_i) = \int_{\mathbb{Q}_p} dx |x|_p^{k_1 \cdot k_2} |1 - x|_p^{k_1 \cdot k_3}$$

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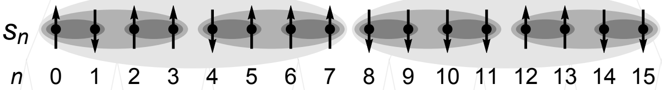
- Hierarchical models, spin blocking, position space RG: Typically consider bi-local stat mech models; hierarchical power law correlations[3]



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Similarities between $\mathbb{R}$ and $\mathbb{Q}_p$

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- Exists a space of well behaved (locally constant) functions on $\mathbb{Q}_p$ spanned by translation eigenfunctions:

$$\chi_p(kx) = e^{2\pi i\{kx\}_p}$$

where $\{\cdot\}_p$ is the fractional part. The Fourier transform is

$$\phi(x) = \int_{\mathbb{Q}_p} e^{2i\pi\{kx\}_p}\hat{\phi}(k)\, dk$$

$\mathbb{R}$ and $\mathbb{Q}_p$: the Adeles

- There exist *adelic* formulas relating functions of reals and functions of p-adics, ex:

$$\prod_p |x|_p = 1$$

(where the product is over all p and ∞.)

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- More interesting examples from Tate's thesis:

$$\zeta_p(s) = \frac{1}{1-p^{-s}}, \quad \zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2)$$

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- Adelic Veneziano amplitudes: $\prod_p A_p^4(k_i) = 1$

Holography for scalar fields: The bulk

- In ordinary AdS/CFT, we can work in Euclidean space, the bulk are cosets:

$$\text{Bulk: } \mathbb{H}^3 = \frac{PGL(2,\mathbb{C})}{SU(2)}, \text{ Boundary: } \mathbb{P}^1(\mathbb{C})$$

$$\text{Bulk: } \mathbb{H}^2 = \frac{PGL(2,\mathbb{R})}{SO(2)}, \text{ Boundary: } \mathbb{P}^1(\mathbb{R})$$

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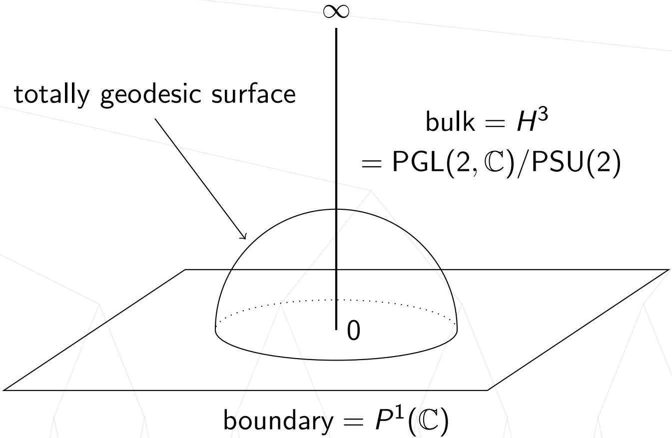
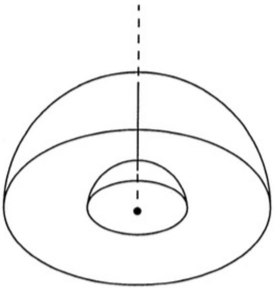
  Boundary conformal transformations $\leftrightarrow$ isometries of bulk.
- Idea is to replace $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{Q}_p$ or some extension.[4]

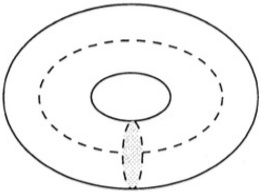
[4] Y. Manin, M. Marcolli, Adv. Theor. Math. Phys. 5 (2001)



∞
totally geodesic surface
bulk $= H^3$
$= \mathrm{PGL}(2, \mathbb{C})/\mathrm{PSU}(2)$
0
boundary $= P^1(\mathbb{C})$

Euclidean black hole solutions can also be understood algebraically, as quotients of empty AdS by *Schottky groups*.



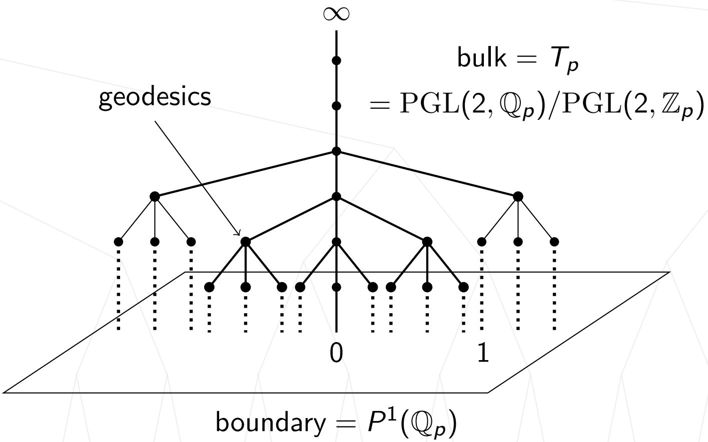


For $\mathbb{Q}_p$, the relevant coset space is

$$T_p = PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$$

This is discrete space known as the *Bruhat-Tits Tree*, the infinite tree of uniform valence $p+1$ we saw earlier.

$(p = 3)$ example:



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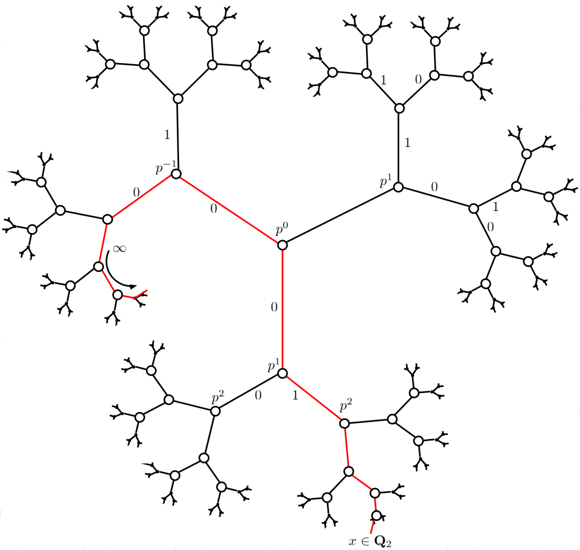
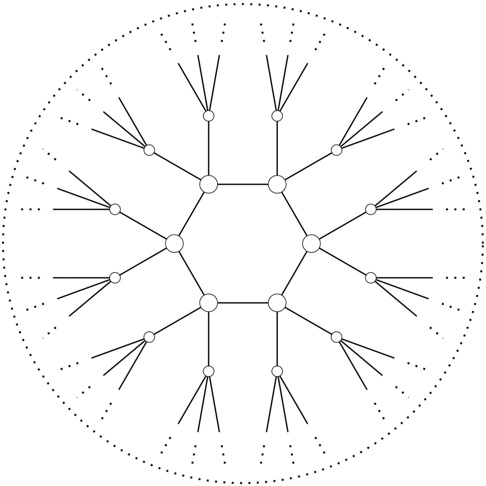


Bruhat–Tits tree of $\mathbb{K} = \mathbb{Q}_2$ and geodesics

Just as before, we can obtain black holes by quotienting this geometry by free subgroups.



p-adic Bañados–Teitelboim–Zanelli black hole with $\mathbb{K} = \mathbb{Q}_3$

Scalar fields on the bulk Bruhat–Tits tree $T_{\mathbb{Q}_p}$

- Study a scalar field on vertices of the tree with action

$$S_{\text{tree}} = \sum_{\langle vv' \rangle} \frac{1}{2}(\phi(v) - \phi(v'))^2 + \sum_{v} V(\phi(v))$$

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$$\Box\phi(v) = \sum_{v'}(\phi(v') - \phi(v)) = m_p^2\phi(v)$$

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- The un-normalized bulk to bulk Green's function is

  $G_\Delta(v, v') = p^{-\Delta d(v,v')}$, where $d(v, v')$ is the integer bulk distance

$$m_p^2 = p^\Delta + p^{1-\Delta} - (p+1) = \frac{-1}{\zeta_p(\Delta - 1)\zeta_p(-\Delta)}$$

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- Bulk to boundary propagator labled by $x \in \mathbb{Q}_p$:

$$K_\Delta(v, x) = p^{-\Delta d_x(v)}$$

with $d_x(v_0) = 0$ at root vertex and $d_x(v) \to -\infty$ as $v \to x$.

[5] Zabrodin, Comm. Math. Phys. (1989)

Scalar fields (cont.)

- For Δ real, min of m_p^2 at $\Delta = 1/2 \to p$-adic BF bound on the mass:

$$m_p^2 \geq -(\sqrt{p} - 1)^2$$

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- Following Witten (1998), for boundary $\phi_0(x)$ we can reconstruct bulk solutions by superposition:

$$\phi(v) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - 1)} \int_{\mathbb{Q}_p} dx\, \phi_0(x) K_\Delta(v, x)$$

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- Holographic ansatz: $\langle \exp \int_{\mathbb{Q}_p} dx \, \phi_0 \mathcal{O} \rangle_{\text{CFT}} = e^{-S_{\text{tree}}(\phi)}|_{\phi \to \phi_0}$

- In this example, possible to integrate out the bulk, giving non-local boundary action:

$$S[\phi] \sim \int_{\mathbb{Q}_p} dx dx' \frac{(\phi_0(x) - \phi_0(x'))^2}{|x - x'|_p^{2\Delta}}$$

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- Bulk fields of mass m_p couple to boundary scalars with two point function:

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- Three and four point functions computed by Gubser et al, detailed agreement with ordinary AdS/CFT.

p-adic conformal field theory

- Global $\mathrm{SL}(2,\mathbb{Q}_p)$-symmetries (primaries), no local conformal algebra (no descendants) ((that we know of))[6]

$$\phi_n(x) \mapsto |cx+d|_p^{2\Delta_n}\phi_n(x)$$

- Correlation function between two primary fields inserted at x and y (scaling dimension Δ_n)

$$\langle \phi_m(x)\phi_n(y)\rangle = \frac{\delta_{n,m}}{|x-y|_p^{2\Delta_n}}$$

(just as we obtained from the bulk.)

- Ultrametricity strongly constrains the form of 3-point and 4-point functions; determined *exactly* by operator product expansion (OPE) coefficients. This has a nice interpretation in terms of geodesic Witten diagrams.

[6] E. Melzer, *Non-Archimedean conformal field theories*, Int. Journal of Modern Physics (1989)

The free boson

- Quadratic action for a scalar field with $\Delta = 0$:

$$S_p[\phi] = -\int_{\mathbb{Q}_p} dx dx' \frac{(\phi(x) - \phi(x'))^2}{|x - x'|_p^2}$$

- $\frac{dxdx'}{|x-x'|_p^2}$ is invariant under $SL(2, \mathbb{Q}_p)$ acting on x, x'.
- Pathl integral: $\mathbb{C}$-valued fields so usual form $Z_p = \int \mathcal{D}\phi \; e^{-S_p[\phi]}$

$$Z_p[J] = \int \mathcal{D}\phi \; \exp\left(-S_p[\phi] + \int_{\mathbb{Q}_p} J(x')\phi(x')dx'\right)$$

Green functions

- Green functions for the non-local (Vladimirov) derivative $\partial_{(p)} G(x-x') = -\delta(x-x')$
- Momentum space: $\widetilde{G}(k) = -\frac{\chi(ky)}{|k|_p}$

$$G(x-y) = -\int_{\mathbb{Q}_p} \frac{\chi(k(x'-x))}{|k|_p} dk = -\int_{\mathbb{Q}_p} \frac{\chi(ku)}{|k|_p} dk$$

- Regularization at $k \to 0$ and p-adic Gamma function $\Gamma_p(\alpha) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}$:

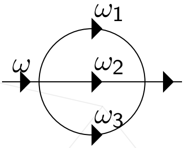
$$\lim_{\alpha\to 0} \int_{\mathbb{Q}_p} \chi(ku)|k|_p^{\alpha-1} dk = \lim_{\alpha\to 0} \Gamma_p(\alpha)|u|_p^{-\alpha}$$

- obtain 2-point function behavior (with $a \to 0$)

$$\langle 0|\phi(x)\phi(x')|0\rangle \sim \log \left| \frac{x-x'}{a} \right|_p$$

Non-renormalization

- p-adic field theories have strong non-renormalization properties due to ultrametricity. A la Wilson's discrete RG, we can integrate out one shell of loop momentum:



$$I_2(\omega) = \int_{\mathcal{O}_K^\times} d\omega_1 d\omega_2 d\omega_3 \, \delta(\omega_1 + \omega_2 + \omega_3 - \omega) G(\omega_1) G(\omega_2) G(\omega_3) \, ,$$

where the $G(\omega) \sim \frac{1}{|\omega|_p^2}$ are momentum space propagators. We can u substitute $\tilde{\omega}_1 = \omega_1 - \omega$ to get:

$$I_2(\omega) = I_2(0) \to \text{Kinetic term never renormalized.}$$

- Can use this to compute anomalous dimensions in $O(N)$.[7]

[7] Gubser, Jepsen, Parikh, Trundy; arXiv:1703.04202

Sign characters and SYK-like models

- To define fermions, we need Grassmann variables as well as $\mathrm{sgn}(t)$ characters; quadratic characters of $\mathbb{Q}_p$ which square to the identity. These generalize the ordinary sign function of the reals needed for time ordered correlators of fermions.
- Considered p-adic versions of Klebanov-Tarnopolsky-Witten models with fermions in $O(N)^3$ flavor group (SYK-like):

$$S_{\text{free}} = \int_{\mathbb{Q}_p} d\omega\, \frac{1}{2}\psi^{abc}(-\omega)|\omega|_p^s\, \mathrm{sgn}(\omega)\psi^{abc}(\omega)$$

$$S_{\text{int}} = \int_{\mathbb{Q}_p} dt\, g\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c}$$

Pairs of indices are contracted either with δ or with a fixed antisymmetric matrix Ω; and the sign character may be either "odd" or "even." (Differs from the real sign character.)

In the limit of large N, with g^2N^3 fixed, the leading-order Schwinger-Dyson equation is

$$G = F + \sigma_\Omega(g^2N^3)G \star G^3 \star F.$$

Solve in the IR to obtain universal limiting behavior:

$$G(t) = b\frac{\text{sgn}(t)}{|t|^{1/2}}, \quad |t| \gg (g^2N^3)^{1/(2-4s)}$$

where

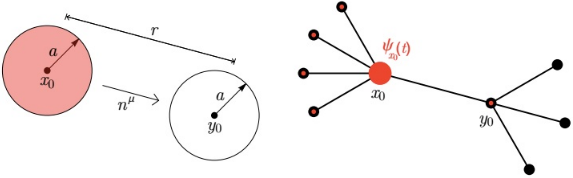
$$\frac{1}{b^4g^2N^3} = -\sigma_\Omega\Gamma(\pi_{-1/2,\text{sgn}})\Gamma(\pi_{1/2,\text{sgn}}).$$

(Γ is the Gelfand-Graev Gamma function.) Scaling in the IR limit is completely independent of the spectral parameter of the UV theory! Universal Kitaev IR.

Dynamics of Edge Lengths

First attempt at bulk gravity: graph curvature as defined in

- Y. Lin, L. Lu, and S.-T. Yau, *Ricci curvature of graphs*, Tohoku Math. J. (2) 63 (2011), no. 4 605–627



Ricci without Riemann tensor in terms of transport distance (Wasserstein distance of measures) between nearby balls: for graphs probability distribution $\psi_{x_0}(t)$ (small t)

$$\psi_x(t) = \begin{cases} 1 - \frac{d_J(x_0)}{D_{x_0}} t & x = x_0 \\ \frac{a_{x_0 x}^{-1}}{D_{x_0}} t & x \sim x_0 \\ 0 & \text{otherwise} \end{cases}$$

D_{x_0} lapse function and $d_J(x_0) = \sum_{x \sim x_0} a_{x_0 x}^{-1}$ normalization factor

Ricci curvature on a graph with edge lengths a_{xy}

$$\kappa_{xy} = \frac{1}{D_x a_{xy}} \left(\frac{1}{a_{xy}} - \sum_i \frac{1}{a_{xx_i}} \right) + \frac{1}{D_y a_{xy}} \left(\frac{1}{a_{xy}} - \sum_i \frac{1}{a_{yy_i}} \right)$$

- For uniform tree of valence $p+1$, e.o.m. solved by config with all edges of equal length $a_e = a$ and $D_x = d_J(x) = (p+1)/a^2$; we get:

$$\kappa_{xy} = -2\frac{p-1}{p+1}$$

  independent of scale a. Constant negative curvature on shell!
- Linearizing the above action gives a massless mode corresponding to small edge length fluctuations.

Further work (in progress)

- Precise pair of bulk/boundary theories; bulk interpretation of p-adic $O(N)$, SYK models, connection to p-adic string?
- Reconstructing real AdS/CFT from p-adic through adelic formulas
- Holographic entanglement entropy, bulk geodesics agree with R-T formula
- Better connection to holographic codes, algebraic curves over $\mathbb{P}^1(\mathbb{F}_p)$
- Higher dimensional cases, Bruhat-Tits buildings, Drinfeld plane, symmetric spaces