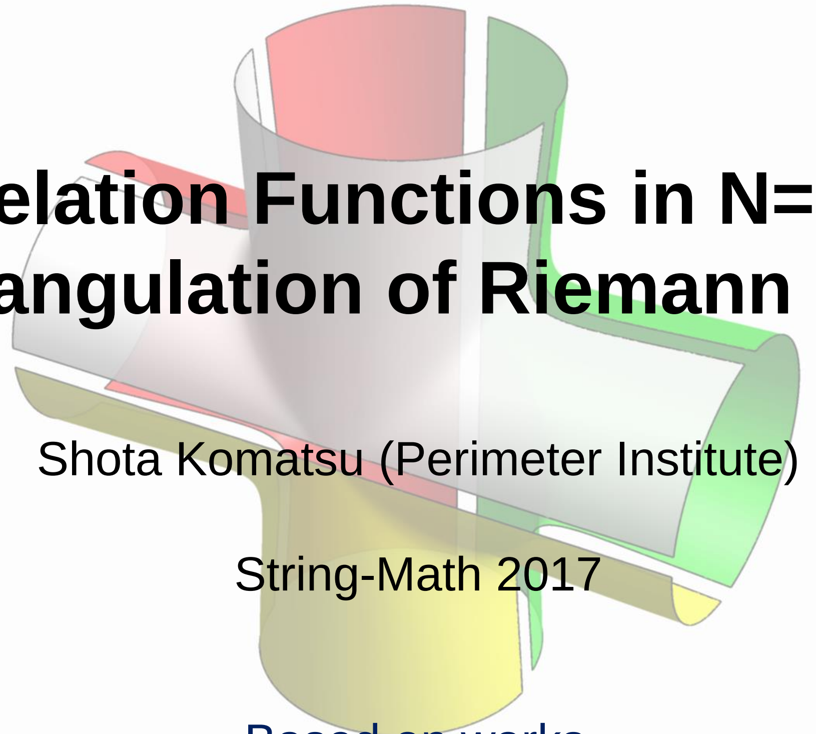
# Correlation Functions in N=4 SYM and Triangulation of Riemann Surfaces



Shota Komatsu (Perimeter Institute)

String-Math 2017

Based on works
with Thiago Fleury (ICTP-SAIFR)

See also [Eden, Sfondrini 2016]

# Goal

Compute the correlation functions of N=4 SYM from **triangulation** of the Riemann surfaces.

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Four-dimensional
SUSY gauge theory

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Two-dimensional

**Why 4d=2d?**

# The answer = AdS/CFT duality

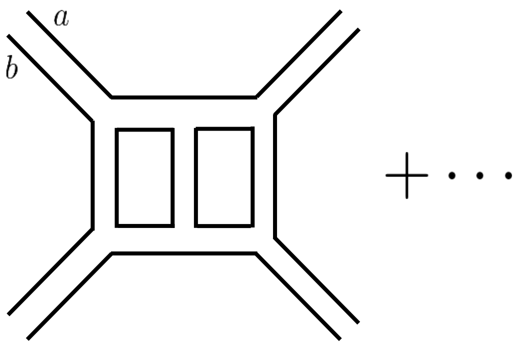
N=4 SYM

Four-dimensional conformal field theory with matrix d.o.f

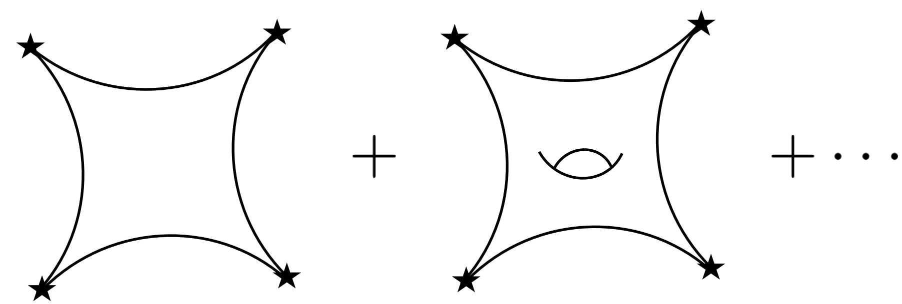
AdS/CFT

String theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$

Ten-dimensional string theory




Ribbon graphs



Riemann surface

- Conjectured through the study of D-branes.
- Precise mechanism of the duality is yet to be understood.

# Large N expansion

**Correlation functions in SU(K) N=4 SYM:**

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim \sum_{m=0}^{\infty} g_s^m \mathcal{G}_m^{(n)}(\lambda)$$

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**N=4 SYM**

$$g_s = \frac{1}{K}$$

$\mathcal{G}_m(\lambda)$ :comes from diagrams
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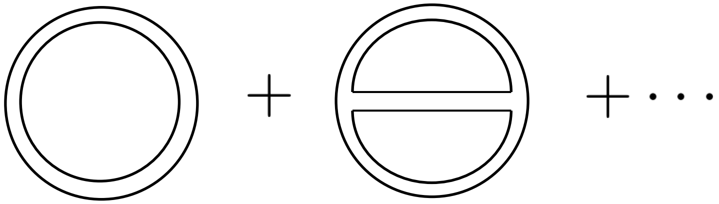

$\lambda \sim$ coupling constant of YM

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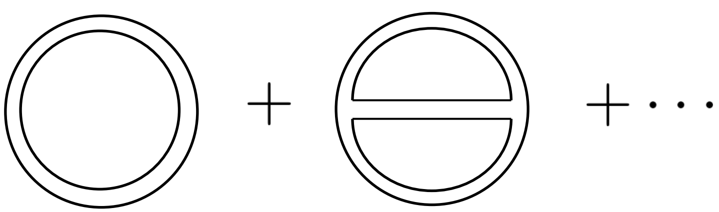


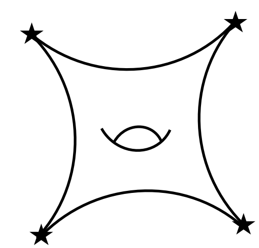
$\lambda \sim$ coupling constant of YM

**String theory in** $\mathrm{AdS}_5 \times \mathrm{S}^5$

$g_s$ :controls the genus expansion of string perturbation

$\mathcal{G}_m(\lambda)$ : comes from genus $m$ worldsheets

$$\int d[\text{moduli}]$$



$\lambda \sim$ size of AdS

# Difficulty

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim \mathcal{G}_0(\lambda) + g_s \mathcal{G}_1(\lambda) + \cdots$$

Today I will only focus on **m=0**.
Even in that case, it is hard to verify the duality because

- N=4 side: Infinitely many diagrams contribute to a single $\mathcal{G}_m(\lambda)$
- String side: Hard to quantize a sigma model w. AdS target space.

# Integrability to the rescue

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \sim \mathcal{G}_0^{(2)}(\lambda) + \cdots$$

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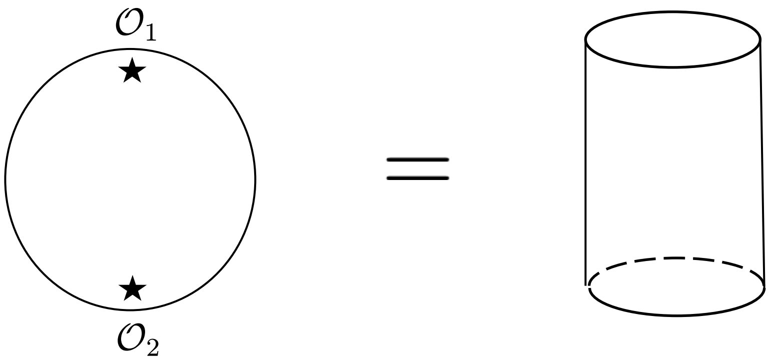
**String theory**

Sigma model in AdS is integrable
(at least classically) [Bena, Polchinski, Roiban 2003]




Integrable models on a cylinder : solved by Thermodynamic Bethe Ansatz

# Integrability to the rescue

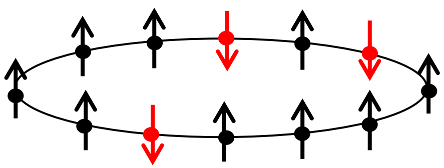
$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \sim \mathcal{G}_0^{(2)}(\lambda) + \cdots$$

## N=4 SYM

Map between "single-trace operators"
And spin chains. [Minahan, Zarembo 2002]

$$\mathcal{O} \sim \mathrm{Tr} \ldots ZYZ \ldots ZYZ \ldots$$

$$=$$



Spin chain turns out to be integrable
(very complicated generalization
of XXX spin chains)

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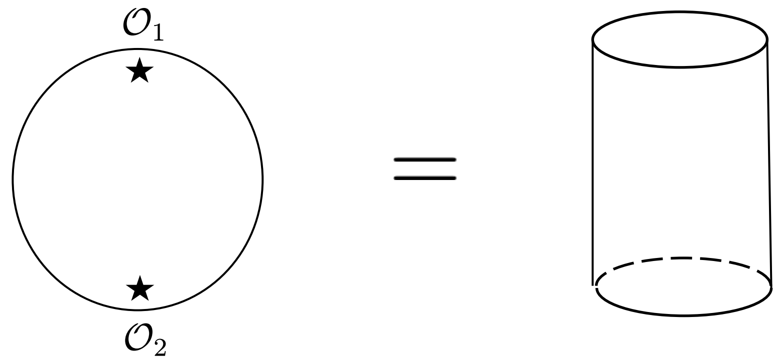


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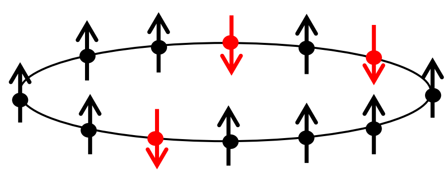

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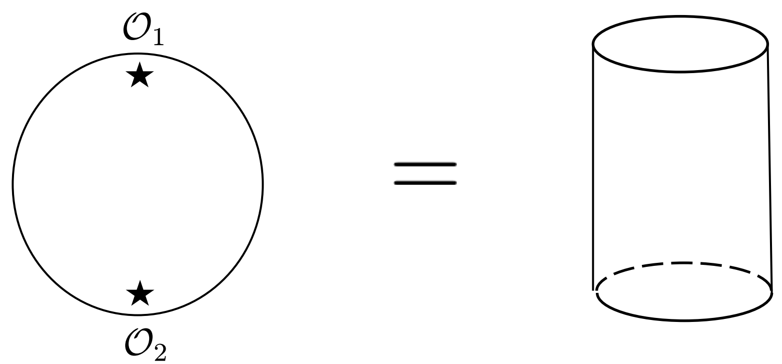


Integrable models on a cylinder : solved by
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Quantum spectral curve

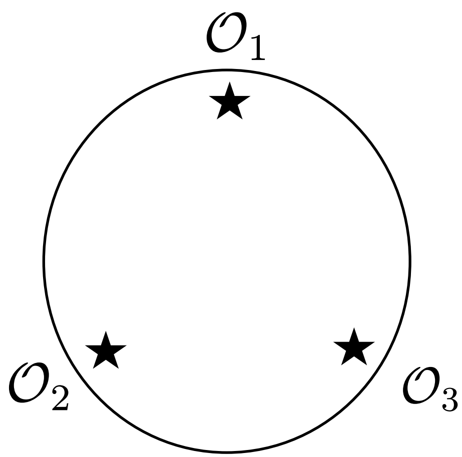
[Gromov, Kazakov, Leurent, Volin 2013]

# Generalization to n=3

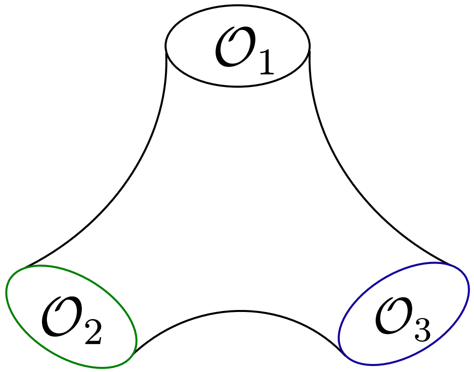
$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$$

# 3pt = a pair of pants

**m=3**

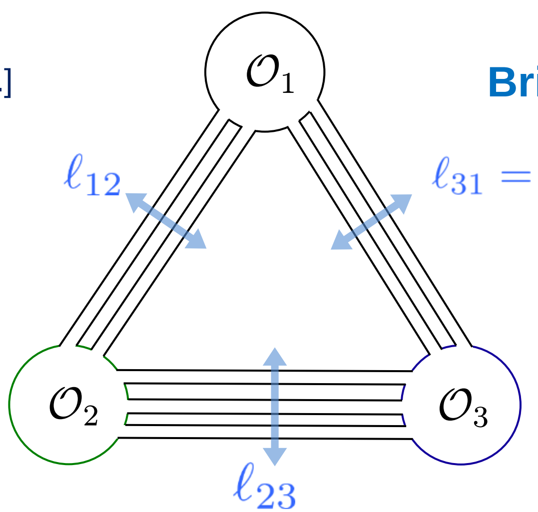



or equivalently




N=4 SYM at zero coupling $\lambda = 0$

[Alday et al], [Okuyama et al.], [Escobedo et al.]
and many others



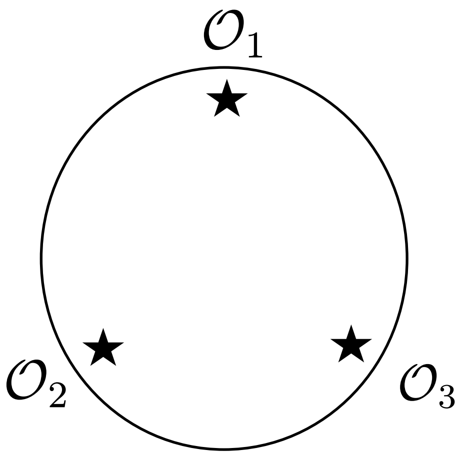

**Bridge length**

$$\ell_{31} = \frac{L_3 + L_1 - L_2}{2}$$

If $\lambda \neq 0$, one has to sum infinitely many diagrams… Can we use integrability?

# 3pt = a pair of pants

**m=3**

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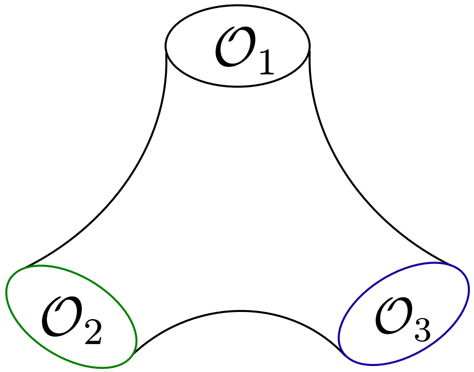
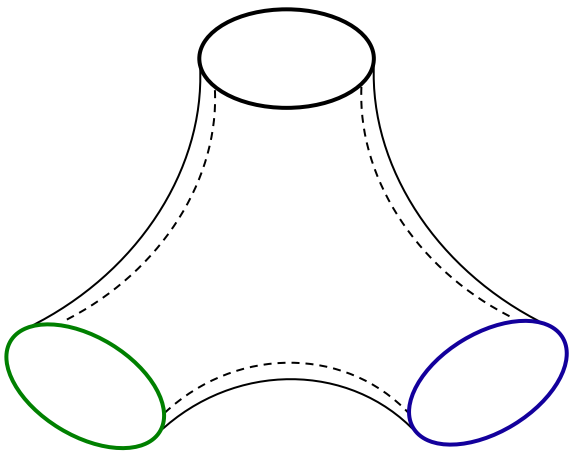



- To use integrability, one has to consider integrable models on a pair of pants.

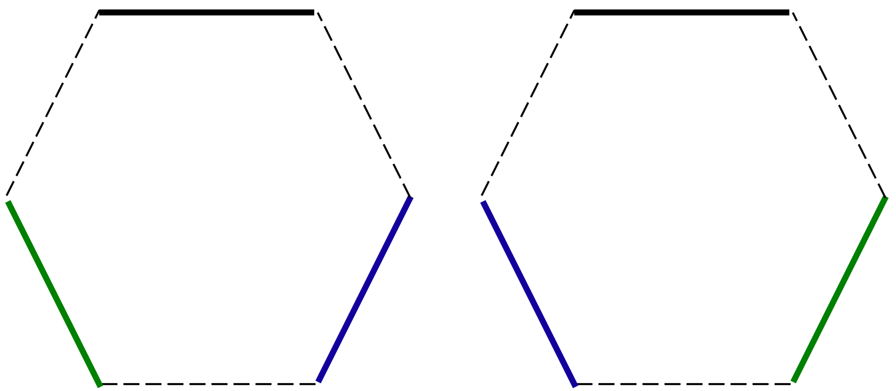
- Never studied before in the literature.

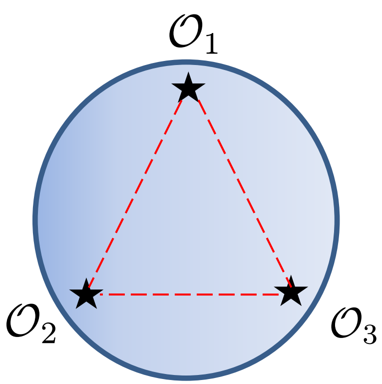
# 3pt = (Hexagon)$^2$



[Basso, SK, Vieira 2015]

$\sim$

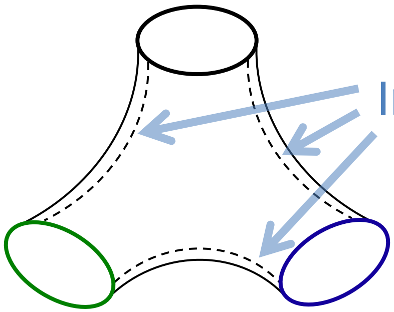




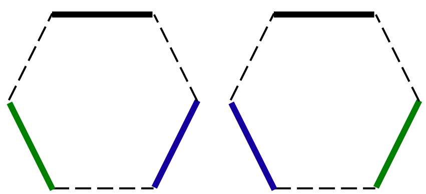

triangulation of the worldsheet

# 3pt = (Hexagon)$^2$

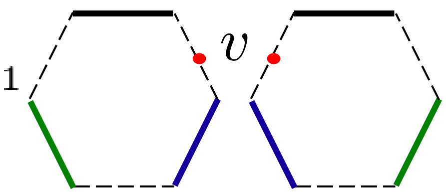
Insert a complete basis of states on the dashed lines.




$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle_{m=0} =$$



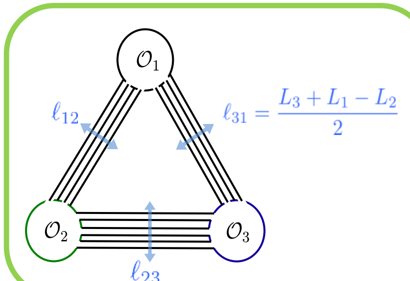
$$+ \int dv\, \mu(v) e^{-\tilde{E}\ell_{31}}$$




$$+ \cdots$$

measure

propagation factor

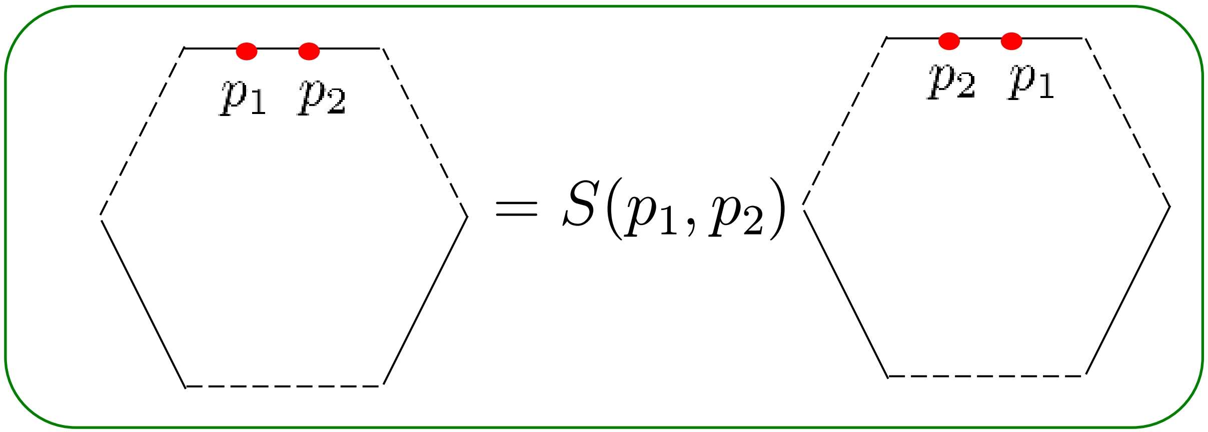



- Similar to sewing construction of 2d CFT.
- The computation of 3pt boils down to the computation of “hexagons”.

# Hexagon and integrability

Symmetry (psl(2|2)) and integrability determine the contributions from each hexagon.

"Form factor bootstrap" Cf. [Smirnov]



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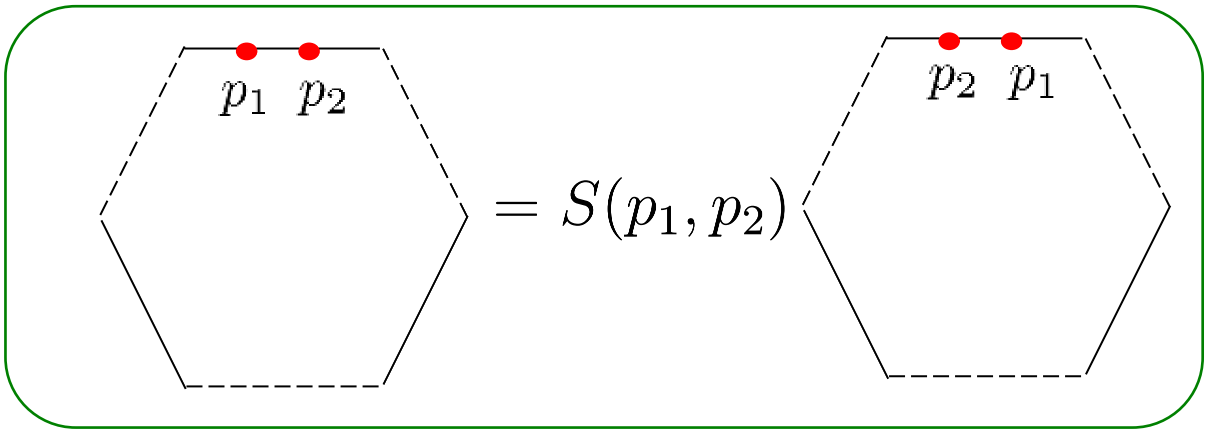


Simplest 2 particle contribution: $h(u_1, u_2) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^+} \frac{1}{\sigma_{12}}$

$$\frac{\lambda}{16\pi^2}\left(x^{\pm} + \frac{1}{x^{\pm}}\right) = u \pm \frac{i}{2}$$

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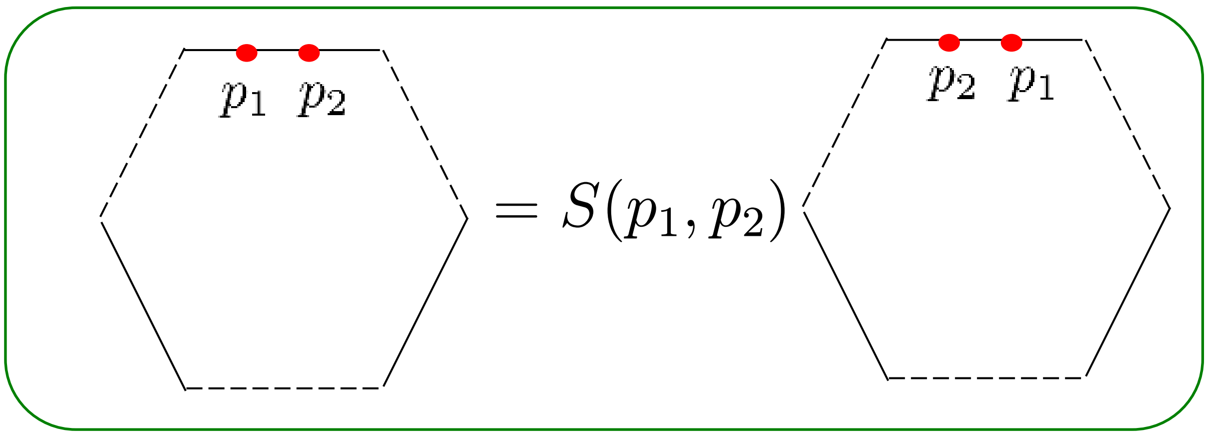


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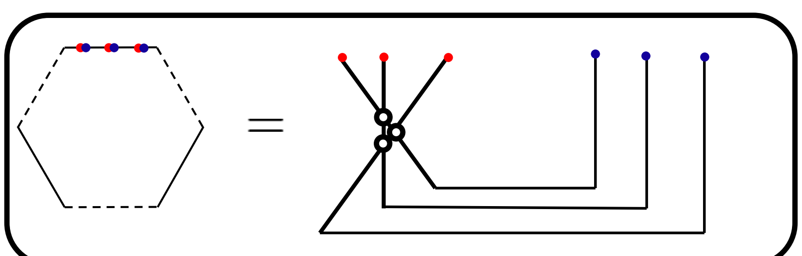
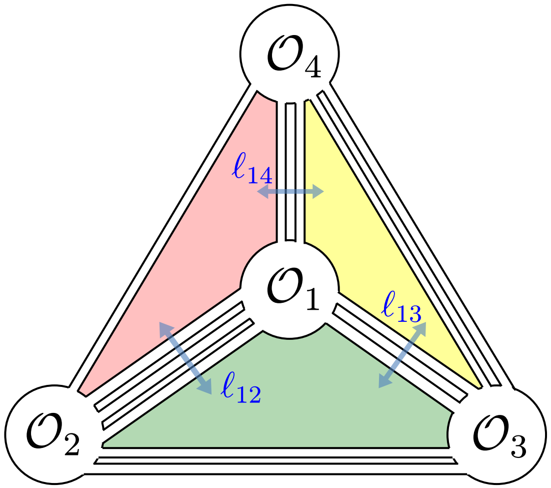
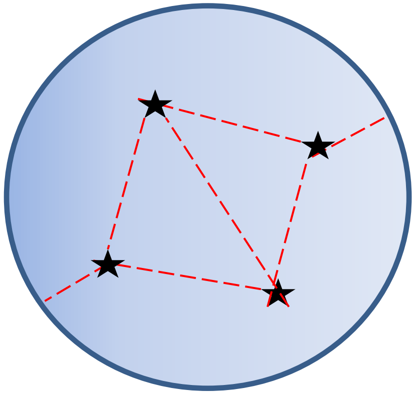



Factorization structure resembling Yang-Baxter.

# Generalization to m≥4?

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$$



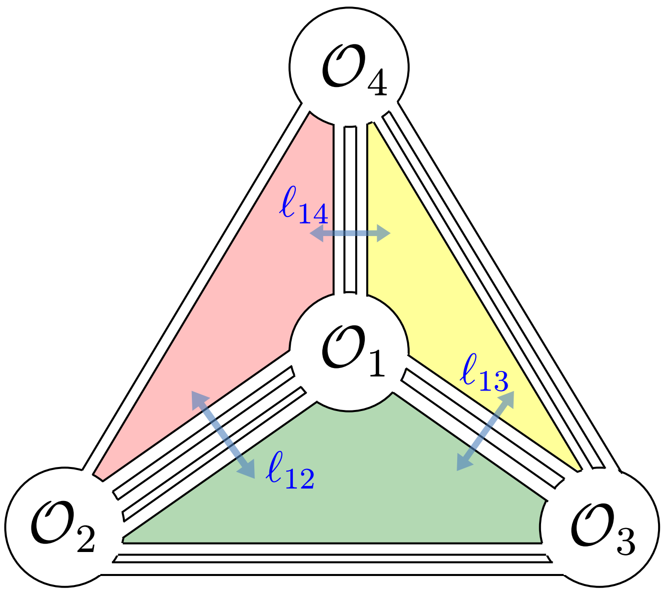
# 4pt = 4 hexagons

At tree level, 4pt can be computed by

1) List up all possible planar graphs.

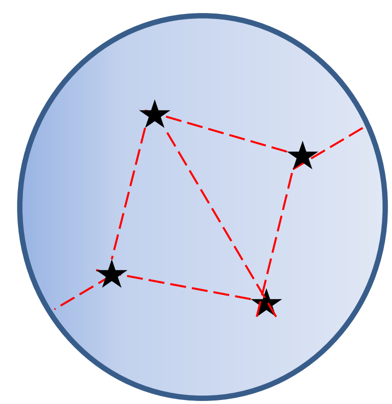
2) Compute contributions from each graph.

$$\sum_{\ell_{ij}}$$



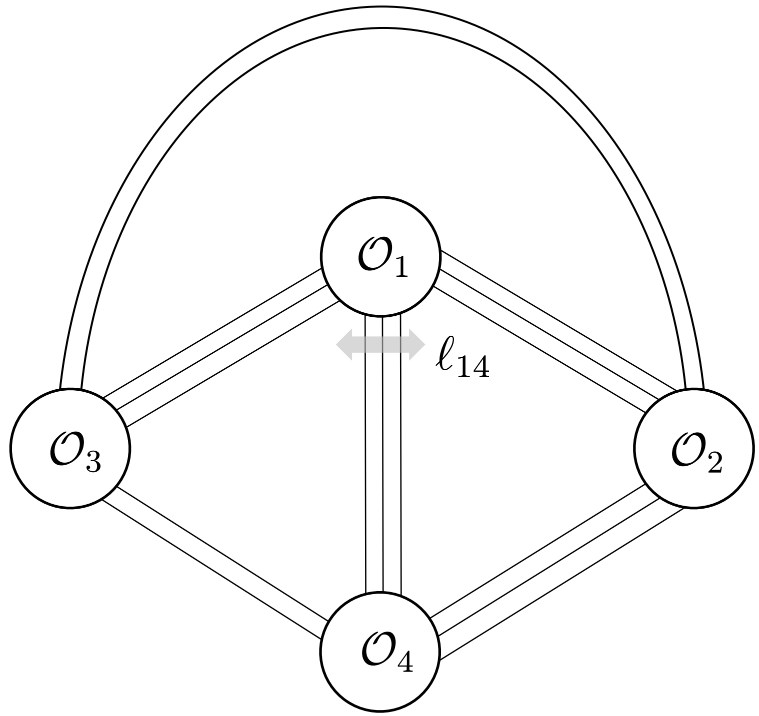

→ 4 hexagons!

triangulation of a 4 punctured sphere



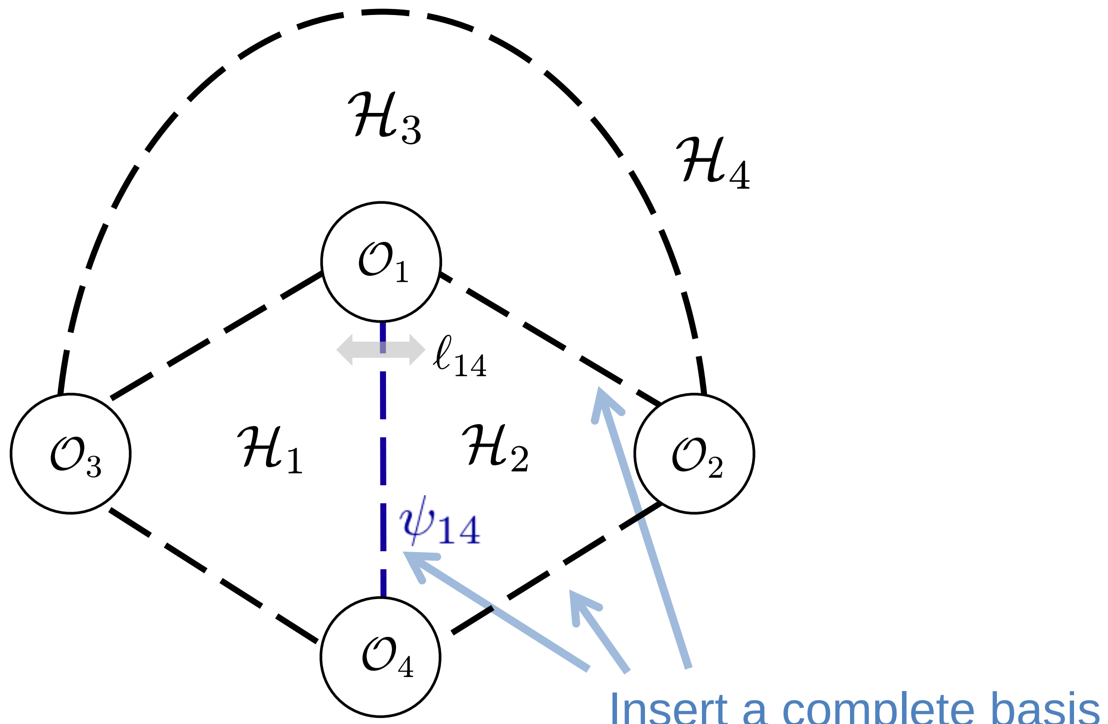
# A natural guess (for 4 BPS op.)

$$\sum_{\{\ell_{ij}\}}$$



# A naive guess (for 4 BPS op.)

$$\sum_{\{\ell_{ij}\}} \sum_{\{\psi_{ij}\}}$$



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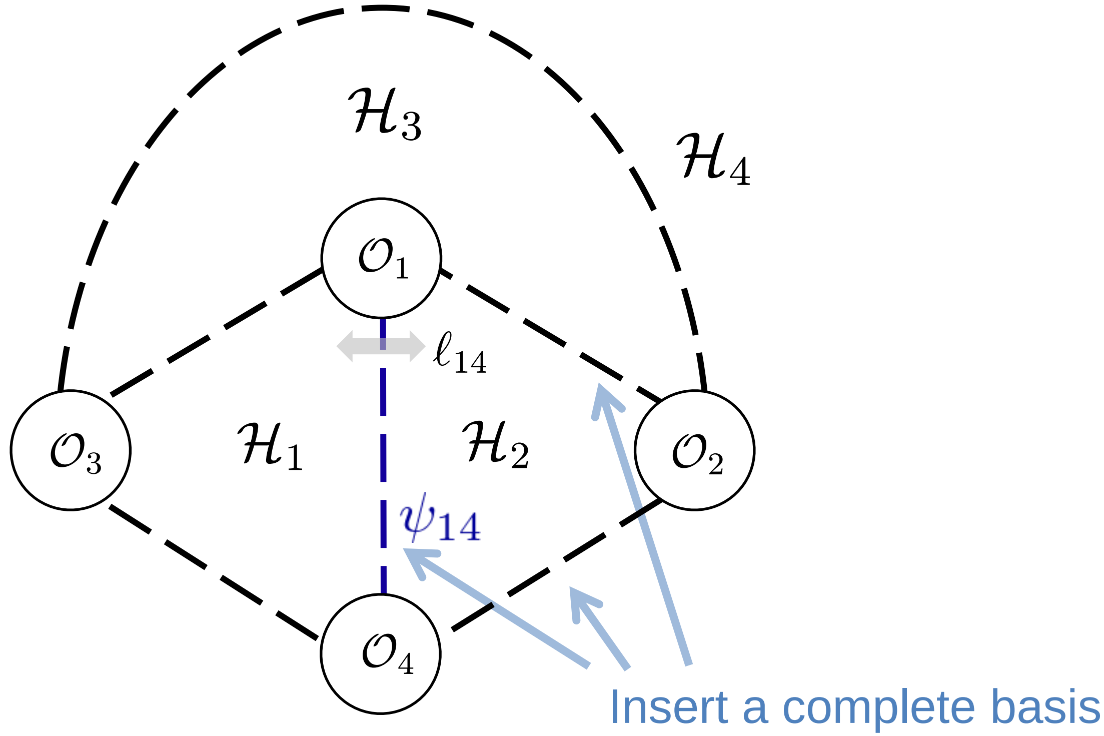


$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{m=0} = \sum_{\text{graphs}} \prod_{(i,j)} \left( \frac{y_i \cdot y_j}{|x_{ij}|^2} \right)^{\ell_{ij}} \left[ \sum_{\psi_{ij}} \prod_{\text{edges}} \mu_{\psi_{ij}} e^{-\tilde{E}_{\psi_{ij}} \ell_{ij}} \prod_{\text{faces}} \mathcal{H}_{\psi_{ij}, \psi_{jk}, \psi_{ki}} \right]$$

measure

propagation factor

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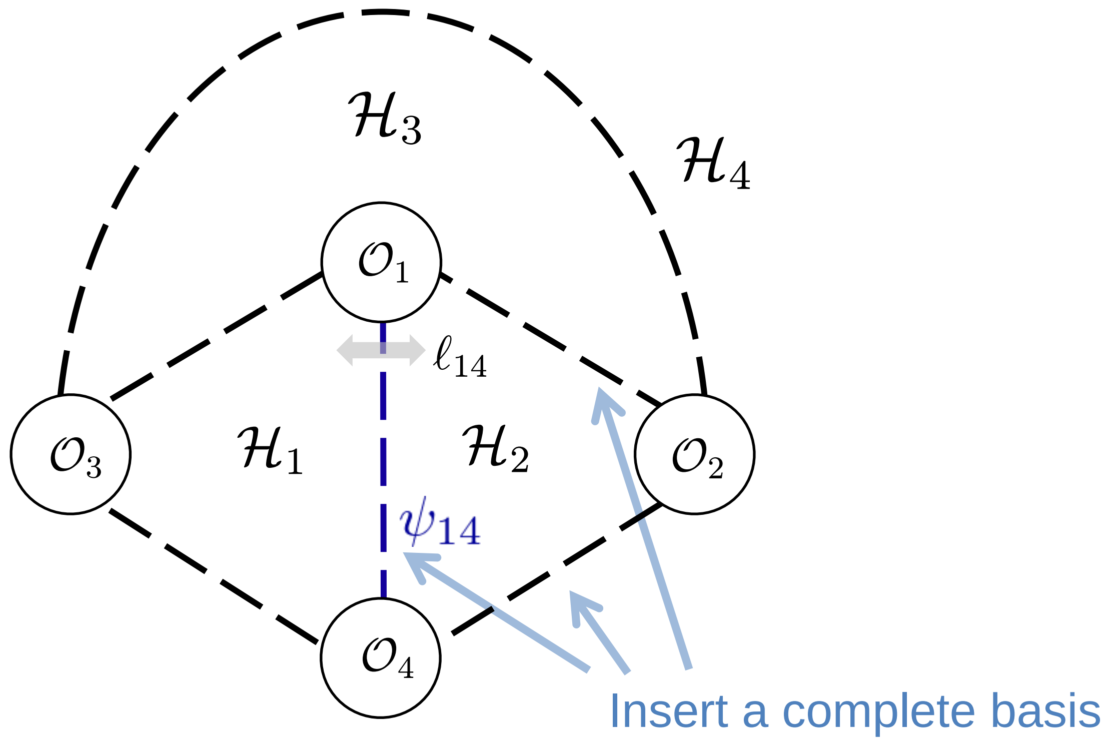


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measure

propagation factor

hexagons

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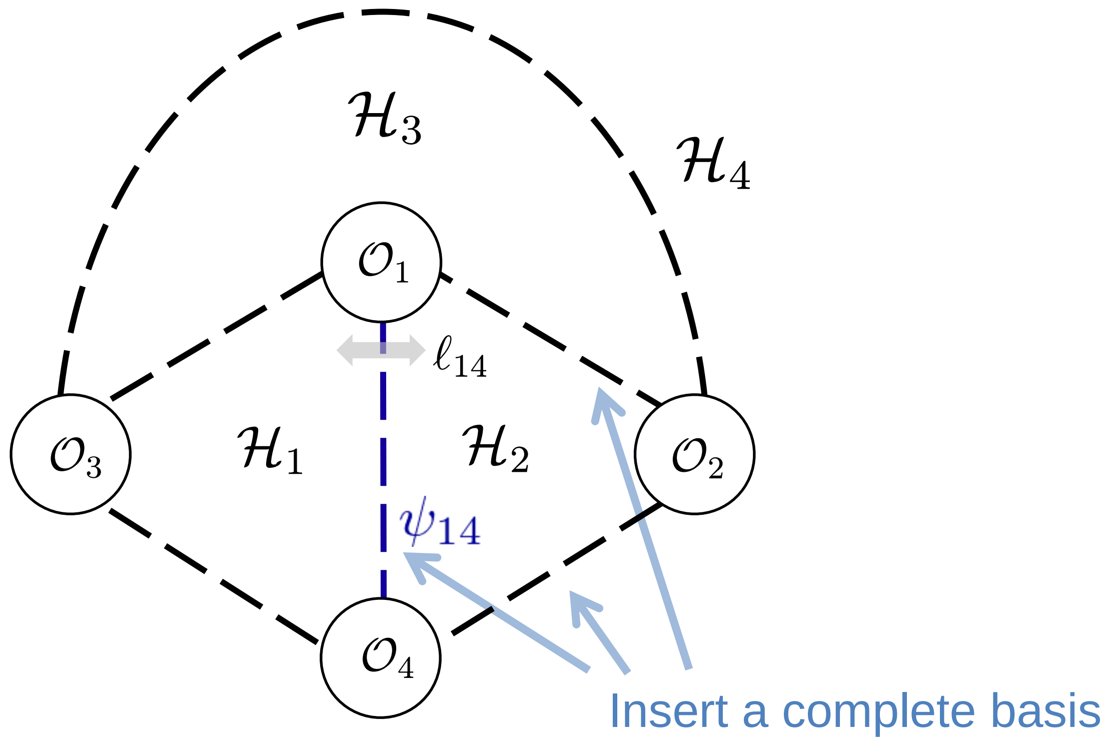


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Tree level
Wick contraction

measure

propagation
factor

hexagons

# Q: How do the cross-ratios appear in the formulae?

$$\langle \mathcal{O}\mathcal{O} \rangle = \frac{1}{|x_{12}|^{2\Delta}}$$

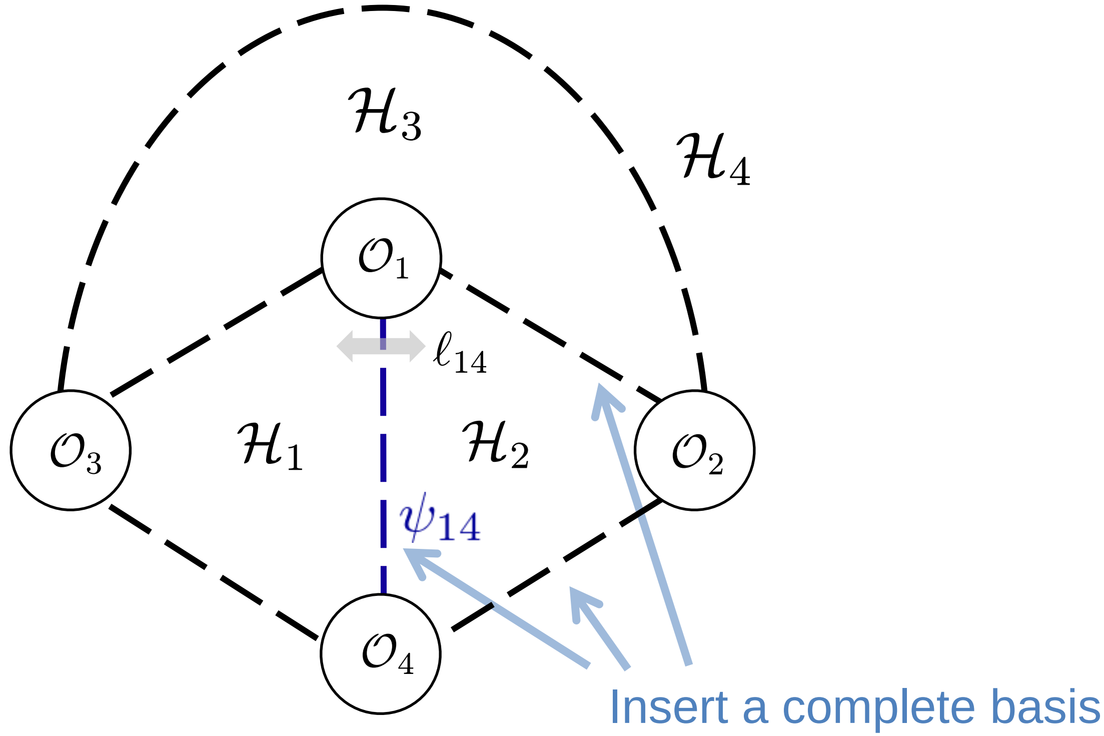
$$\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle = \frac{C_{123}}{|x_{12}|^{\Delta}|x_{23}|^{\Delta}|x_{31}|^{\Delta}}$$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \frac{G(z,\bar{z})}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}$$

$$z\bar{z} = \frac{|x_{12}|^2|x_{34}|^2}{|x_{13}|^2|x_{24}|^2}, \qquad (1-z)(1-\bar{z}) = \frac{|x_{14}|^2|x_{23}|^2}{|x_{13}|^2|x_{24}|^2}$$

# A proposal (for 4 BPS op.)

$$\sum_{\{\ell_{ij}\}} \sum_{\{\psi_{ij}\}}$$

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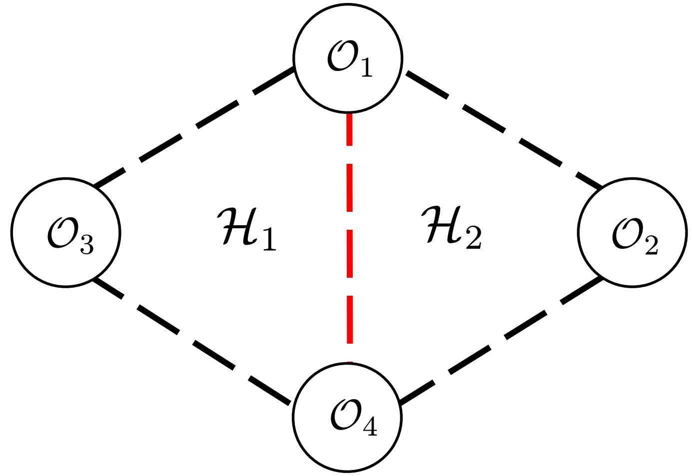
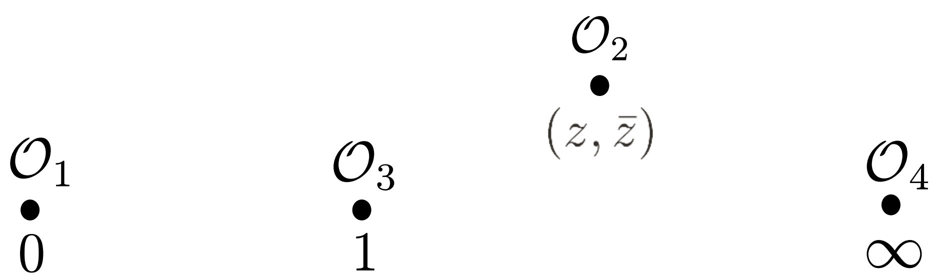
Weight factor (cross-ratio dependent) **New!**

# Weight factor from symmetry

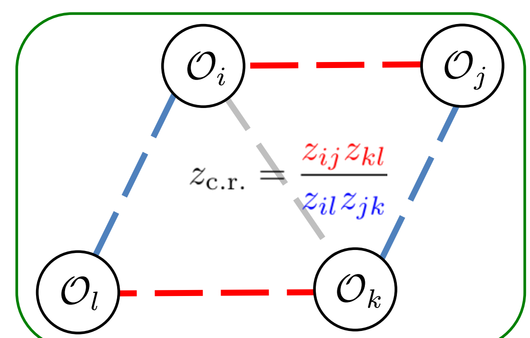
- Consider gluing of the edge 14:



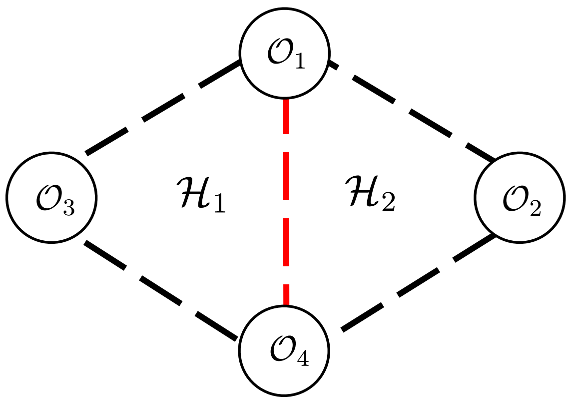

- Perform the conformal transformation to map it to




- In this frame, $\mathcal{H}_1 : (0, 1, \infty)$ ← canonical
  $\mathcal{H}_2 : (0, (z, \bar{z}), \infty)$ ← "rotated"



# Weight factor from symmetry

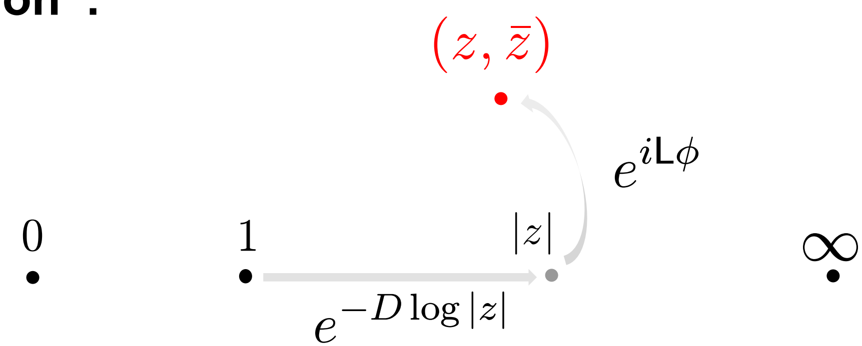



$\mathcal{H}_1 : (0, 1, \infty)$ ← canonical

$\mathcal{H}_2 : (0, (z, \bar{z}), \infty)$ ← "rotated"

$$\mathcal{H}_2 = g^{-1}\, \mathcal{H}_1\, g$$

**"Rotation":**




$$g = e^{-D \log |z| + i\mathsf{L}\phi}$$

$$e^{i\phi} := \sqrt{\frac{z}{\bar{z}}}$$

**Gluing:**

$$\cdots \mathcal{H}_2 e^{-\tilde{E}\ell_{14}} \mathcal{H}_1 \cdots \to \sum_{\psi} \mu_\psi \cdots \mathcal{H}_1 |\psi\rangle e^{-\tilde{E}\ell_{14}} \underline{\langle\psi|g|\psi\rangle} \langle\psi|\mathcal{H}_1 \cdots$$

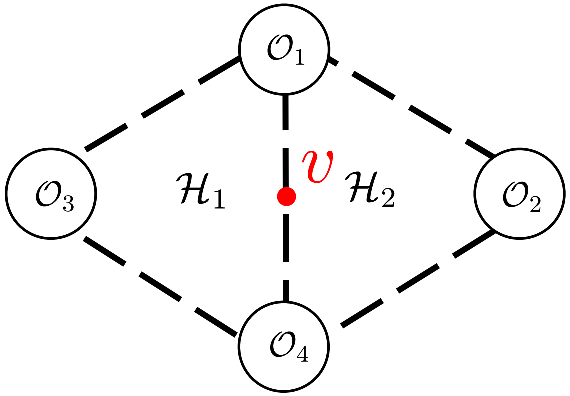
Weight factor (cross-ratio dependent)

# Weak coupling check

At $O(\lambda)$, only 1-particle states contribute to the final answer.

- Full 1-particle integral

$$\sum_{a=1}^{\infty} 2\cos\phi \frac{\sin a\phi}{\sin\phi} \int_{-\infty}^{\infty} \frac{dv}{2\pi} \tilde{\mu}_a(v) e^{-i\tilde{p}_a(v)\log|z|}$$



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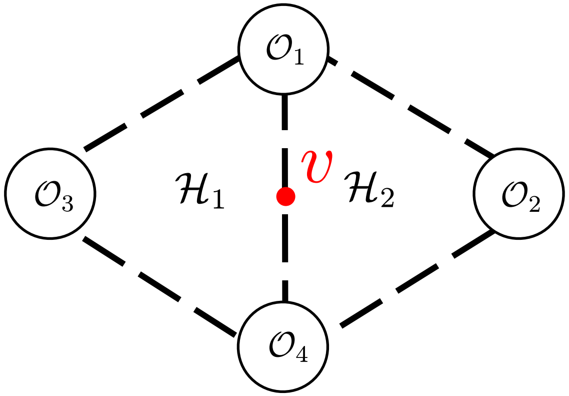


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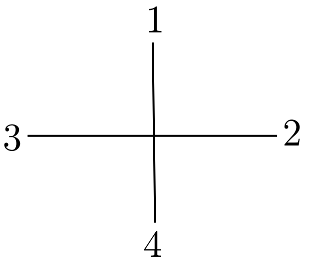
- Leading order at weak coupling

$$\sum_{a=1}^{\infty} 2\cos\phi \frac{\sin a\phi}{\sin\phi} \int_{-\infty}^{\infty} \frac{dv}{2\pi} \frac{a}{(v^2+a^2/4)^2} e^{-2iv\log|z|}$$

$$\propto \frac{2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log z\bar{z} \log\frac{1-z}{1-\bar{z}}}{z-\bar{z}}$$

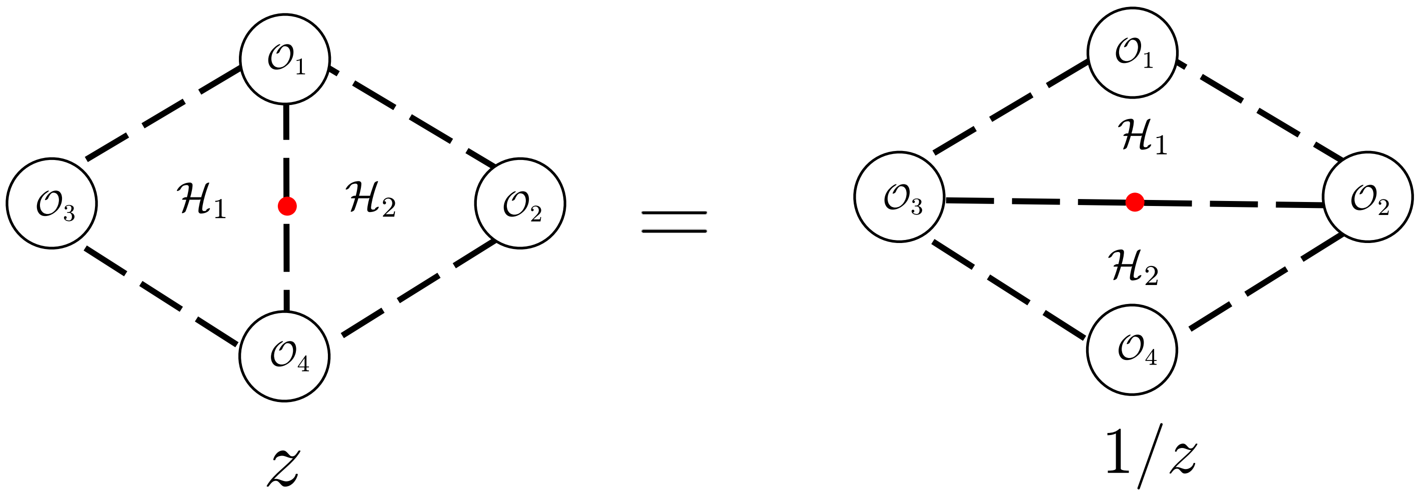



1-loop conformal integral
(Bloch-Wigner function)

Reproduces the known answer!

# Flip invariance

We can glue two hexagons in 2 different ways.




Our construction is "flip-invariant". (We don't have a proof, but we can check case by case.)

This guarantees the crossing symmetry of the four-point function!

# ***Conclusion & Prospects***

# Conclusion

- Proposed an integrability-based framework to study correlation functions in N=4 SYM. (works also for correlators involving non-BPS operators.)

- Deep relation to triangulation of the Riemann surface.

# Prospect

- More points, higher genus, etc?

- 2d CFT correlator (such as minimal models) from the triangulation of the Riemann surface?

- String field theory from triangulation?

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- Proposed an integrability-based framework to study correlation functions in N=4 SYM. (works also for correlators involving non-BPS operators.)

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- More points, higher genus, etc?

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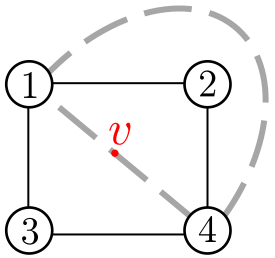
*Thank you!*

# Higher loops

- Full 1-particle integral

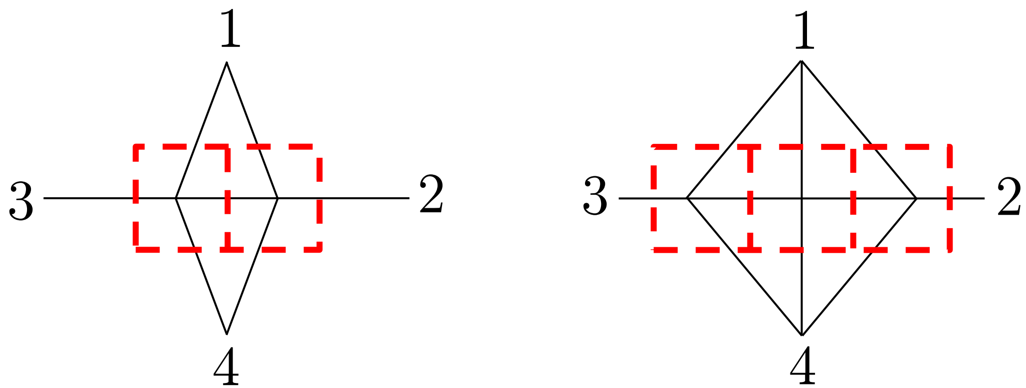
$$\sum_{a=1}^{\infty} 2(\cos\phi - \cos\theta)\frac{\sin a\phi}{\sin\phi}\int \frac{dv}{2\pi}\tilde{\mu}_a(v)e^{-i\tilde{p}_a(v)\log|z|}$$




One can expand this integral at arbitrary order at weak coupling

**Surprise:**

L-th subleading term ⟶ L-loop conformal ladder integral



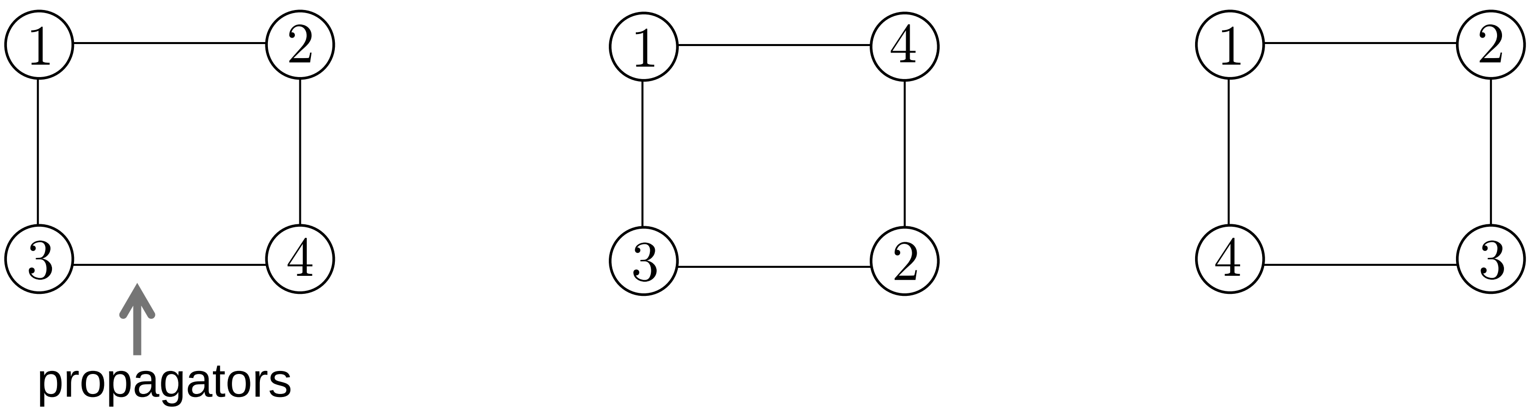

$$\sum_{k=0}^{L}\frac{(-1)^k(2L-k)!}{L!(L-k)!k!}\log^k(z\bar{z})(\mathrm{Li}_{2L-k}(z)-\mathrm{Li}_{2L-k}(\bar{z}))$$

Full 1-particle integral resums all the ladder integrals!

# Four-point functions of length-2 operators

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle_{\rm con} = \sum_{\rm graphs} \left(\prod_{(i,j)} (d_{ij})^{\ell_{ij}}\right) \left[\sum_{\psi_{ij}} \prod_{(i,j)} \mu_{\psi_{ij}} e^{-\tilde{E}_{\psi_{ij}}\ell_{ij}} \mathcal{W}_{\psi_{ij}} \prod_{(i,j,k)} \mathcal{H}_{\psi_{ij},\psi_{jk},\psi_{ki}}\right]$$

1. List all tree-level diagrams



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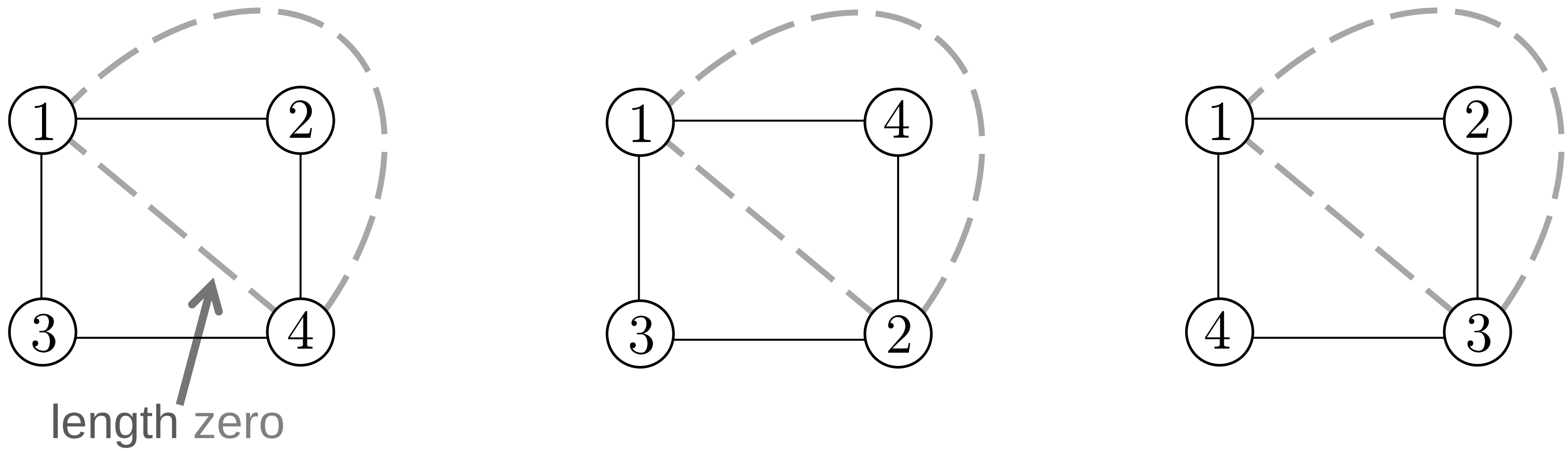
1. List all tree-level diagrams, and cut them into hexagons.



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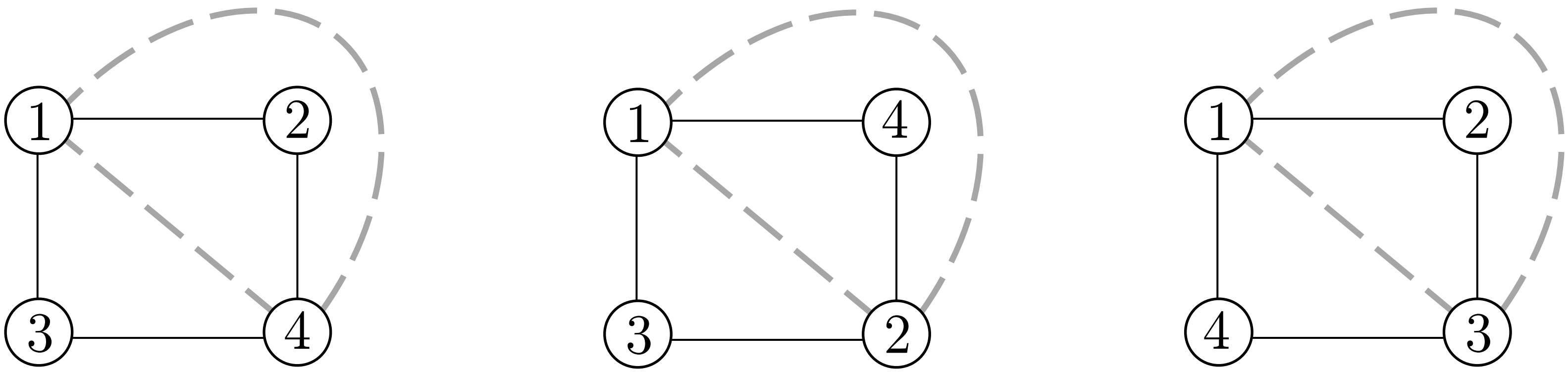
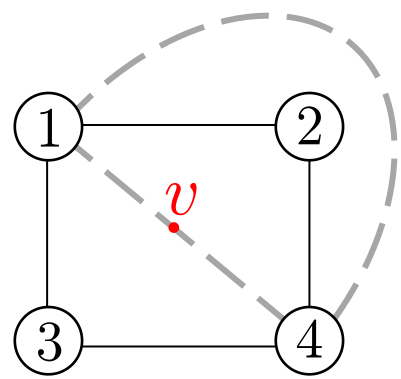



2. Decorate them with magnons.

$$\mu_N \sim O(\lambda^N)\,, \qquad e^{-\tilde{E}\ell} \sim O(\lambda^\ell)$$

N-particle

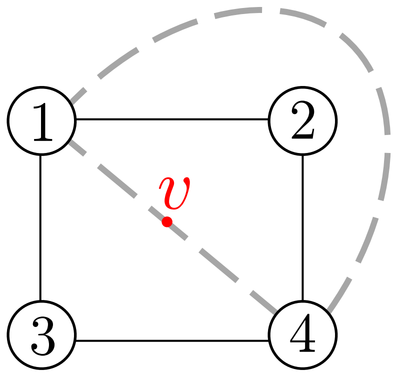



At one loop, only 1 particle on zero-length channel!

# Four-point functions of length-2 operators

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{\rm con} = \sum_{\rm graphs} \left( \prod_{(i,j)} (d_{ij})^{\ell_{ij}} \right) \left[ \sum_{\psi_{ij}} \prod_{(i,j)} \mu_{\psi_{ij}} e^{-\tilde{E}_{\psi_{ij}} \ell_{ij}} \mathcal{W}_{\psi_{ij}} \prod_{(i,j,k)} \mathcal{H}_{\psi_{ij},\psi_{jk},\psi_{ki}} \right]$$

- 1-particle state: $(v, \quad a, \quad \text{flavor})$
  - $v$: rapidity
  - $a$: bound states (KK modes)
  - flavor: derivative, scalar, fermion etc.




- flavor sum = character

$$\sum_{\rm flavor} \mathcal{W}_{\psi_a} = \mathrm{Str}_a [g] = 2(\cos\phi - \cos\theta) \frac{\sin a\phi}{\sin\phi} e^{-i\tilde{p}_a(v)\log|z|}$$

$\mathcal{W}_{\psi_a}$: $\langle \psi | g | \psi \rangle$; Str: su(2|2); $a$: a-th anti-sym rep.

$$e^{i\phi} := \sqrt{\frac{z}{\bar{z}}} \qquad e^{i\theta} := \sqrt{\frac{\alpha}{\bar{\alpha}}} \qquad \alpha\bar{\alpha} := \frac{(\vec{Y}_1 \cdot \vec{Y}_2)(\vec{Y}_3 \cdot \vec{Y}_4)}{(\vec{Y}_1 \cdot \vec{Y}_3)(\vec{Y}_2 \cdot \vec{Y}_4)}$$