Spectral networks at marginal stability, BPS quivers, and a new construction of wall-crossing invariants

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String-Math 2017

In collaboration with: Maxime Gabella, Chan Y. Park and Masahito Yamazaki
Our work is about BPS states of 4d $\mathcal{N} = 2$ quantum field theories, a subject of very fruitful interaction between physics and mathematics, with ties to: Hitchin systems, cluster algebras, quiver representation theory, Teichmueller theory, . . .

In recent years there has been tremendous progress in understanding BPS spectra. Powerful frameworks such as BPS quivers and spectral networks allow to compute BPS spectra systematically.

But the BPS spectrum is not unique, a theory can have infinitely many different ones. BPS counting is only a part of the puzzle. A major step forward: the discovery of wall-crossing formulae, describing how BPS spectra change across different regions of moduli space. An important concept emerged from these developments: the existence of wall-crossing invariants. The importance of a w.c.i. stems from the fact that it encodes all possible BPS spectra of a theory. Nevertheless, in order to construct these invariants one has to know the BPS spectrum in advance, somewhere on the moduli space.
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Goal of this talk

Introduce a new kind of “wall-crossing invariant” : BPS graphs. Explain how they establish a link between spectral networks and BPS quivers, and how they provide a new construction of the invariant of Kontsevich and Soibelman, without using information about the BPS spectrum.
A Geometric (re-)View of Wall-Crossing in Class $\mathcal{S}$ Theories
Class $\mathcal{S}$

Four-dimensional $\mathcal{N} = 2$ supersymmetric quantum field theories, classified by

- $\mathfrak{g}$ ADE Lie algebra, ($\mathfrak{g} = A_n$ in this talk)
- $\mathcal{C}$ Riemann surface with punctures
- $\mathcal{D}$ “puncture data”

Arise from twisted compactifications of 6d $(2,0)$ theory of type $\mathfrak{g}$.

$(\mathfrak{g}, \mathcal{C}, \mathcal{D})$ define a Hitchin integrable system

$$F + R^2 [\varphi, \bar{\varphi}] = 0, \quad D_A \varphi = 0.$$ 

$\mathcal{M}$ moduli space of solutions (modulo gauge) encodes several key features of the low energy dynamics:

- Hitchin fibration $\mathcal{M} \rightarrow \mathcal{B}$: Coulomb branch of the moduli space of vacua
- geometry of $\mathcal{B}$ encodes the low energy effective action
- geometry of $\mathcal{M}$ encodes spectrum of excitations over $\mathcal{B}$, BPS states

[Seiberg-Witten, Donagi-Witten, Martinec-Warner, Gorski et al., Klemm et al. Witten, Gaiotto, Gaiotto-Moore-Neitzke]
\[ \Sigma_u : \quad \det(\lambda - \varphi(z)) = \lambda^K + \sum_{i=2}^{K} \phi_i \lambda^{K-i} = 0 \]

- \( \lambda \) tautological 1-form \( \Sigma_u \subset T^* C \)
- geometry encoded by meromorphic multi-differentials \( \{\phi_i\} \equiv u \in B \)
- \( \lambda_j, j = 1, \ldots, K \) sheets of a \( K : 1 \) ramified covering \( \pi : \Sigma_u \rightarrow C \)
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\( \Sigma \) coincides with the Seiberg-Witten curve
- \( H_1(\Sigma, \mathbb{Z}) \) lattice of charges
- periods \( Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda \) BPS central charge
- minimal area surface \( M_\gamma = \frac{1}{\pi} \int_{\pi(\gamma)} |\lambda_j - \lambda_i| \) \[\text{[Klemm et al, Mikhailov]}\]
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\[ M_\gamma = |Z_\gamma| \iff \lambda_{ij} = e^{i\vartheta} |\lambda_{ij}| \]

trajectories on \( C \) where
\( \lambda_i - \lambda_j \) has fixed phase \( \vartheta = \text{Arg} Z_\gamma \)
BPS spectrum determined by geometry of spectral covering map $\Sigma_u \to \mathbb{C}$

Systematic “scan” of BPS states: fix $u \in \mathcal{B}$, construct trajectories on $C$

- start from branch point where $\lambda_i(z) = \lambda_j(z)$

Finite edges appear at topological jumps of the spectral network $W$
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Primitive version of spectral networks [Gaiotto-Moore-Neitzke, Klemm et al.]

BPS spectrum: finite edges appear at jumps of the spectral network $\vartheta = \text{Arg}Z$
BPS states can interact, and can form **BPS boundstates**

\[ E_{\text{bound}} = |Z_1 + Z_2| - |Z_1| - |Z_2| \leq 0 \]

Marginal stability: at real-codimension one walls in \( B \)

\[ MS(\gamma, \gamma') := \{ u \in B \mid \text{Arg } Z_\gamma(u) = \text{Arg } Z_{\gamma'}(u) \} \]
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Change in geometry of \( \Sigma_u \) \( \Rightarrow \) transition of BPS spectrum

Can we extract invariant information from \( \Sigma \)?
BPS Graphs
Let $\mathcal{B}_c \subset \mathcal{B}$ be a locus where $Z_\gamma$ of all BPS states have the same phase

$$\mathcal{B}_c := \{ u \in \mathcal{B} , \ \text{Arg} \ Z_\gamma(u) = \text{Arg} \ Z_\gamma'(u) \equiv \vartheta_c(u) \}$$

The spectral network at $\vartheta_c$ is very special. Several finite edges appear simultaneously. Within the network a BPS graph $\mathcal{G}$ emerges.
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But:

- $\mathcal{B}_c$ is a maximal intersection of walls of marginal stability.
- The BPS spectrum is ill-defined.

It appears that $\mathcal{B}_c$ cannot contain any information about the BPS spectrum. Surprisingly, $\mathcal{G}$ encodes invariant information about it!
Example: Argyres-Douglas

\[ \lambda^2 - z^3 + z - u = 0 \]

\[ H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^2 \]

[plots: http://het-math2.physics.rutgers.edu/loom]
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What this graph tells:

- $[\pi^{-1}(p_1)] = \gamma_1$, $[\pi^{-1}(p_2)] = \gamma_2$ are homology cycles.
- Both $\gamma_1$, $\gamma_2$ are BPS states (hypermultiplets) in any nearby chamber.
- They are a positive-integral basis for $\Gamma_+ := Z_{u_c}^{-1}(e^{i\vartheta_c}[R^+]) \subset H_1(\Sigma, \mathbb{Z})$.
- Intersection $\langle \gamma_1, \gamma_2 \rangle = -1$. 
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$\mathcal{G}$ encodes the BPS quiver: $\gamma_1 \overset{\circ}{\longrightarrow} \overset{\circ}{\longrightarrow} \gamma_2$
Quiver-Graph Correspondence
BPS quiver $Q$: an oriented graph composed of nodes $Q_0$ and arrows $Q_1$, with a superpotential $W \in R\langle Q \rangle$ (formal sum of cycles in the path algebra).

BPS states: zero-modes of supersymmetric quantum mechanics encoded by $Q$, subject to additional stability conditions.  

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General quiver-graph correspondence: [Gabella-L-Park-Yamazaki]

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- $\mathcal{G}$ admits a natural decomposition into elementary webs $\rightarrow Q_0$.
- common branch points of elementary webs $\rightarrow Q_1$. 

![Diagram of a BPS quiver with nodes and arrows]
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- $G$ admits a natural decomposition into **elementary webs** $\rightarrow Q_0$.
- common branch points of elementary webs $\rightarrow Q_1$.
- $\sum$(face-loops) - $\sum$(vertex-loops) $\rightarrow W$. 

![Diagram of a quiver graph with nodes connected by arrows, showing cycles and face-loops, vertex-loops, and superpotential.](image-url)
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- common branch points of elementary webs $\to Q_1$.
- $\sum$(face-loops) - $\sum$(vertex-loops) $\to W$.
- edge flip $\to$ quiver mutation.
$\mathcal{B} = \{\text{quadratic diff. with presc. poles}\} \supset \{\text{Strebel diff.}\}_0 = \mathcal{B}_c$

$\mathcal{G}$ is the critical graph of a Strebel diff., or a “maximally contracted Fenchel-Nielsen network” [Hollands-Neitzke]. Existence guaranteed [Strebel, Liu].
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- BPS graphs are **dual to ideal triangulations** of \( C \)

- Quivers from triangulated surfaces [Bridgeland-Smith, Fomin-Shapiro-Thurston, Labardini Fragoso, Alim et al., Fock-Goncharov]:
  - \( Q_0 \leftrightarrow \text{edges}, \ Q_1 \leftrightarrow \text{faces} \)
  - canonical superpotential \( \leftrightarrow \text{face-loops and puncture-loops} \)
  - mutations \( \leftrightarrow \text{flips} \)
\( A_{N-1} \) Theories

\[ \mathcal{B} = \{ \text{meromorphic } k\text{-diff. w/ prescribed poles at punctures, } k = 2, \ldots, N \} \]

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Conjecture: restrict to $k$-differentials with poles of order $k$ (full punctures), then $\mathcal{B}_c$ exists and BPS graphs are dual to ideal **N-triangulations**

- \( \binom{N}{2} \) branch points in each $\Delta$, connected by elementary webs.
- Candidate BPS graphs for $A_{N-1}$ theories with full punctures.
- Motivated by “$N$-lift construction” [Gaiotto-Moore-Neitzke].
- Found explicit examples, pass nontrivial checks with known BPS quivers.
Beyond full punctures

Nice feature of theories with full punctures: obtain many class $S$ theories with **partial punctures**, by tuning “puncture data $D$” [Gaiotto, Chacaltana-Distler-Tachikawa]

Example: $g = A_2$

$$\varphi \sim \frac{1}{z} \begin{pmatrix} m_1 & m_2 \\ -m_1 - m_2 & \end{pmatrix} \rightarrow \frac{1}{z} \begin{pmatrix} m \\ m \\ -2m \end{pmatrix}$$

- Pole in $\Delta = \prod_{i<j} (\lambda_i - \lambda_j)^2 \sim 1/z^6$ becomes milder $\sim 1/z^4$.
- Zeroes $\Delta$ coincide with branch points, get absorbed by the puncture.
- Spectral curve undergoes a topological transition, reflected by $G$.
Kontsevich-Soibelman Invariants
Jumps of BPS spectrum are controlled by an $\text{Arg} \, Z_\gamma$-ordered product of quantum dilogarithms

\[
\begin{align*}
\text{Arg}Z(u) \uparrow
\prod_{\gamma,m} \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)} = \prod_{\gamma',m'} \Phi((-y)^{m'} X_{\gamma'})^{a_{m'}(\gamma',u')}
\end{align*}
\]

- $a_m(\gamma, u)$ counts $|\gamma, m\rangle$
  (Laurent coeff. of “protected spin character” / motivic DT invariants)
- Quantum torus algebra: $X_\gamma X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}$
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- Quantum torus algebra: $X_{\gamma} X_{\gamma'} = y^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}$
In addition to geometric data, the full definition of spectral networks includes **combinatorial data**, it is entirely determined by the topology of the network. The network data defines a **coordinate system** \( \{ \mathcal{X}_\gamma \} \) for \( \mathcal{M} \), viewed as \( \mathcal{M}_{\text{flat}}(C, GL(K)) \). Conjecturally part of a cluster atlas.
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Key property: At the phase of a BPS state \( \vartheta = \text{Arg}Z \) the topology of \( \mathcal{W} \) jumps, inducing a (quantum) “change of coordinates”

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\mathcal{X}'_{\eta} = \left[ \prod_m \Phi((-y)^m \mathcal{X}_\gamma)^{a_m(\gamma)} \right] \mathcal{X}_\eta \left[ \prod_m \Phi((-y)^m \mathcal{X}_\gamma)^{a_m(\gamma)} \right]^{-1}
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[Gaiotto-Moore-Neitzke, Galakhov-L-Moore]
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[Gaiotto-Moore-Neitzke, Galakhov-L-Moore]

BPS spectrum (at fixed \( u \)) controls \( \vartheta \)-transition functions of coordinate charts

Coordinates at \( (\vartheta, u) \) with \( (\vartheta + \pi, u) \) are related by \( X'_{\gamma} = U X_{\gamma} U^{-1} \)
At the Critical Locus

\[ X_\gamma \text{ exhibits a single jump at } \vartheta_c \text{ captured by } \mathbb{U} \]

\[ X'_\gamma = \mathbb{U} \cdot X_\gamma \cdot \mathbb{U}^{-1} \]
At the Critical Locus

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Take this as a new definition of $\mathbb{U}$. 
At the Critical Locus

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- Key fact: the change of coordinates is entirely determined by the topology of the degenerate sub-network, that is $\mathcal{G}$. 
At the Critical Locus

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  \]

- Take this as a new definition of $\mathbb{U}$.

- Key fact: the change of coordinates is entirely determined by the topology of the degenerate sub-network, that is $\mathcal{G}$.

**The topology of the BPS graph determines $\mathbb{U}$.** [L]
The graph has 2 edges, each contributes an equation

\[ F'_p = \cup F_p \cup^{-1} \]

with

\[
F_{p_1} = 1 + y^{-1}X_{\gamma_1} + y^{-1}X_{\gamma_1+\gamma_2}
\]
\[
F_{p_2} = 1 + y^{-1}X_{\gamma_2}
\]
\[
F'_{p_1} = 1 + y^{-1}X_{\gamma_1}
\]
\[
F'_{p_2} = 1 + y^{-1}X_{\gamma_2} + y^{-1}X_{\gamma_1+\gamma_2}
\]

Together, they determine the monodromy

\[
\cup = 1 - \frac{y}{(y)_1} (X_{\gamma_1} + X_{\gamma_2}) + \frac{y^2}{(y)_1} X_{\gamma_1+\gamma_2} + \frac{y^2}{(y)_2} (X_{2\gamma_1} + X_{2\gamma_2}) + \ldots
\]
\[
= \Phi(X_{\gamma_1})\Phi(X_{\gamma_2})
\]
The graph has three edges $p_1, p_2, p_3$; each contributes one equation

$$F_p' = \cup F_p \cup^{-1}$$

with

$$F_{p_1} = \frac{1 + x_{\gamma_1} + (y+y^{-1})x_{\gamma_1+\gamma_3} + x_{\gamma_1+2\gamma_3} + (y+y^{-1})x_{\gamma_1+\gamma_2+2\gamma_3} + x_{\gamma_1+2\gamma_2+2\gamma_3} + x_{2\gamma_1+2\gamma_2+2\gamma_3}}{(1-x_{2\gamma_1+2\gamma_2+2\gamma_3})^2}$$

$$F_{p_1}' = \frac{1 + x_{\gamma_1} + (y+y^{-1})x_{\gamma_1+\gamma_2} + x_{\gamma_1+2\gamma_2} + (y+y^{-1})x_{\gamma_1+2\gamma_2+\gamma_3} + x_{\gamma_1+2\gamma_2+2\gamma_3} + x_{2\gamma_1+2\gamma_2+2\gamma_3}}{(1-x_{2\gamma_1+2\gamma_2+2\gamma_3})^2}$$

$F_{p_2,3}$ & $F_{p_2,3}'$ are obtained by cyclic shifts of $\gamma_1, \gamma_2, \gamma_3$.

The solution:

$$\cup = \left( \prod_{n \geq 0} \Phi \left( x_{\gamma_1+n(\gamma_1+\gamma_2)} \right) \right)$$

$$\times \Phi \left( x_{\gamma_3} \right) \Phi \left( (-y)x_{\gamma_1+\gamma_2} \right)^{-1} \Phi \left( (-y)^{-1}x_{\gamma_1+\gamma_2} \right)^{-1} \Phi \left( x_{2\gamma_1+2\gamma_2+\gamma_3} \right)$$

$$\times \left( \prod_{n \geq 0} \Phi \left( x_{\gamma_2+n(\gamma_1+\gamma_2)} \right) \right)$$
Remark 1

The BPS graph can have some **symmetries**. They are inherited by $\mathcal{U}$. Hidden by the factorization $\mathcal{U} = \prod \Phi(X)$, but **manifest on the BPS graph** (Ex. $\mathbb{Z}_3$ symmetry in $\mathcal{N} = 2^*$).

Reflect basic properties of the Schur index [Cecotti-Neitzke-vafa, Iqbal-vafa, Cordova-Gaiotto-Shao], computed as the correlator of a TQFT on $C$ [Gadde-Pomoni-Rastelli-Razamat]: it is a symmetric function of the flavor fugacities.

Remark 2

How to make sense of $\mathcal{U}$ physically at $\mathcal{B}_c$? BPS spectrum is ill-defined!

Rich physics in the background. Key idea is to use **surface defects**.

- Induce a new sector of “2d-4d” BPS states.

- **(framed) 2d-4d wall-crossing**: creation/decay of 2d-4d states is controlled by 4d BPS spectrum. Unification of Cecotti-Vafa and Kontsevich-Soibelman wall-crossing. [Gaiotto-Moore-Neitzke]

- Key to computing $\mathcal{U}$ via 2d-4d wall-crossing: unlike 4d BPS states, **stability** of 2d-4d spectrum is **well-defined at $\mathcal{B}_c$**.
Conclusions
1. We introduce a new object: the **BPS graph** $\mathcal{G}$ of a theory of class $\mathcal{S}$. $\mathcal{G}$ lives on the UV curve, and emerges from degenerate spectral networks at $\mathcal{B}_c$.

2. Link between spectral networks and **BPS quivers**. Correctly encodes known quivers. Approach to obtain many new ones, by moduli-deformation of K-lifts.

3. A new construction of Kontsevich-Soibelman invariants based on $\mathcal{G}$. **Manifestly** wall-crossing invariant. Exhibits **symmetries** of $\mathcal{U}$. 
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2. Link between spectral networks and **BPS quivers**. Correctly encodes known quivers. Approach to obtain many new ones, by moduli-deformation of K-lifts.

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Thank You.