Spectral networks at marginal stability, BPS quivers, and a new construction of wall-crossing invariants

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In collaboration with: Maxime Gabella, Chan Y. Park and Masahito Yamazaki

Our work is about BPS states of 4d $\mathcal{N} = 2$ quantum field theories, a subject of very fruitful interaction between physics and mathematics, with ties to: Hitchin systems, cluster algebras, quiver representation theory, Teichmueller theory, ...

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In recent years there has been tremendous progress in understanding BPS spectra. Powerful frameworks such as **BPS quivers** and **spectral networks** allow to compute BPS spectra systematically.

But the BPS spectrum is not unique, a theory can have infinitely many different ones. BPS counting is only a part of the puzzle.

A major step forward: the discovery of wall-crossing formulae, describing how BPS spectra change across different regions of moduli space.

An important concept emerged from these developments: the existence of **wall-crossing invariants**.

The importance of a w.c.i. stems from the fact that it encodes all possible BPS spectra of a theory. Nevertheless, in order to construct these invariants one has to know the BPS spectrum in advance, somewhere on the moduli space.

Goal of this talk

Introduce a new kind of "wall-crossing invariant": BPS graphs.
Explain how they establish a link between spectral networks and BPS quivers, and how they provide a new construction of the invariant of Kontsevich and Soibelman, without using information about the BPS spectrum.

A Geometric (re-)View of Wall-Crossing in Class $\mathcal{S}$ Theories

Class $\mathcal{S}$

Four-dimensional $\mathcal{N} = 2$ supersymmetric quantum field theories, classified by

$\mathfrak{g}$ *ADE* Lie algebra, ($\mathfrak{g} = A_n$ in this talk)

C Riemann surface with punctures

D "puncture data"

Arise from twisted compactifications of 6d $(2, 0)$ theory of type $\mathfrak{g}$.

$(\mathfrak{g}, C, D)$ define a Hitchin integrable system

$$F + R^2[\varphi, \overline{\varphi}] = 0 \,, \qquad D_A \varphi = 0 \,.$$

$\mathcal{M}$ moduli space of solutions (modulo gauge) encodes several key features of the low energy dynamics:

- Hitchin fibration $\mathcal{M} \to \mathcal{B}$: Coulomb branch of the moduli space of vacua
- geometry of $\mathcal{B}$ encodes the low energy effective action
- geometry of $\mathcal{M}$ encodes spectrum of excitations over $\mathcal{B}$, BPS states

[Seiberg-Witten, Donagi-Witten, Martinec-Warner, Gorski et al., Klemm et al. Witten, Gaiotto, Gaiotto-Moore-Neitzke]

$$\Sigma_u : \quad \det(\lambda - \varphi(z)) = \lambda^K + \sum_{i=2}^{K} \phi_i \lambda^{K-i} = 0$$

- λ tautological 1-form $\Sigma_u \subset T^*C$
- geometry encoded by meromorphic multi-differentials $\{\phi_i\} \equiv u \in \mathcal{B}$
- λ_j, $j = 1, \ldots, K$ sheets of a $K : 1$ ramified covering $\pi : \Sigma_u \to C$

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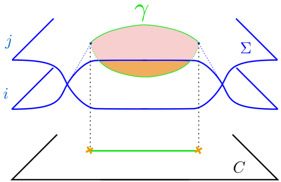
Σ coincides with the Seiberg-Witten curve

- $H_1(\Sigma, \mathbb{Z})$ lattice of charges
- periods $Z_\gamma = \frac{1}{\pi} \oint_\gamma \lambda$ BPS central charge
- minimal area surface $M_\gamma = \frac{1}{\pi} \int_{\pi(\gamma)} |\lambda_j - \lambda_i|$ **[Klemm et al, Mikhailov]**



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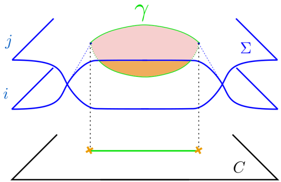

$M_\gamma = |Z_\gamma| \;\Leftrightarrow \lambda_{ij} = e^{i\vartheta} |\lambda_{ij}|$

trajectories on C where

$\lambda_i - \lambda_j$ has fixed phase $\vartheta = Arg\ Z_\gamma$

BPS spectrum determined by geometry of spectral covering map $\Sigma_u \to C$

Systematic "scan" of BPS states: fix $u \in \mathcal{B}$, construct trajectories on C

- start from branch point where $\lambda_i(z) = \lambda_j(z)$

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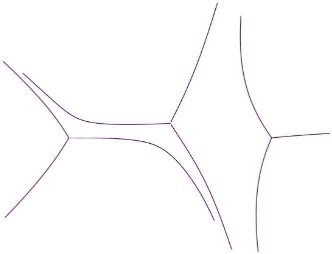

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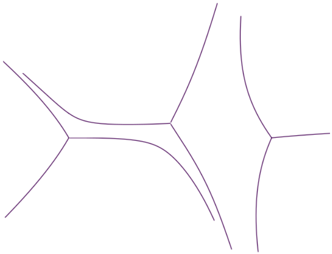

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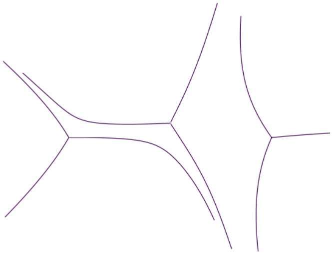

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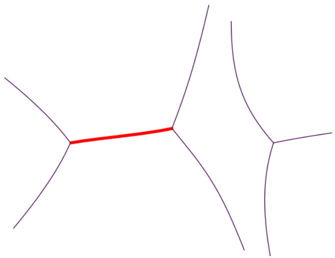

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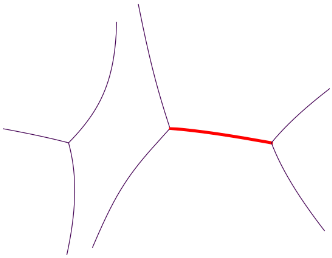

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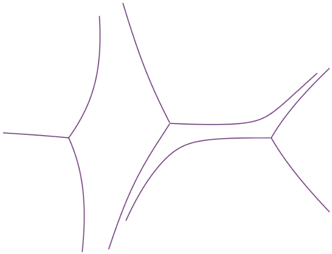


Primitive version of **spectral networks** **[Gaiotto-Moore-Neitzke, Klemm et al.]**
BPS spectrum: finite edges appear at jumps of the spectral network $\vartheta = ArgZ$

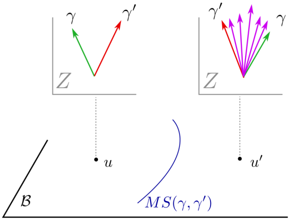
Wall-Crossing

BPS states can interact, and can form **BPS boundstates**

$$E_{bound} = |Z_1 + Z_2| - |Z_1| - |Z_2| \leq 0$$

Marginal stability: at real-codimension one walls in $\mathcal{B}$

$$MS(\gamma, \gamma') := \{u \in \mathcal{B} \mid \mathrm{Arg}\, Z_{\gamma}(u) = \mathrm{Arg}\, Z_{\gamma'}(u)\}$$



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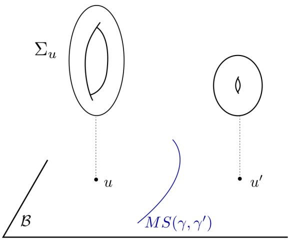

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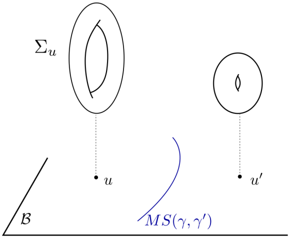

Can we extract invariant information from Σ?

BPS Graphs

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The spectral network at ϑ_c is very special. Several finite edges appear simultaneously. Within the network a **BPS graph** $\mathcal{G}$ emerges.

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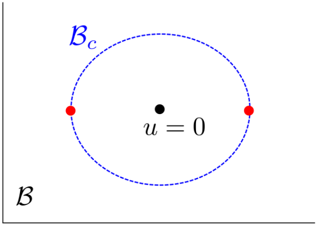
But:

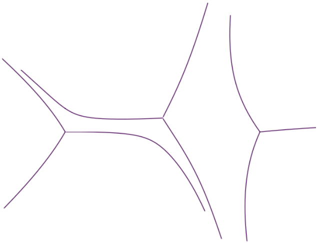
- $\mathcal{B}_c$ is a maximal intersection of walls of marginal stability.
- The BPS spectrum is ill-defined.

It appears that $\mathcal{B}_c$ cannot contain any information about the BPS spectrum.

Surprisingly, $\mathcal{G}$ encodes **invariant** information about it!

Example: Argyres-Douglas





$$\lambda^2 - z^3 + z - u = 0$$

$$H_1(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}^2$$

[plots: http://het-math2.physics.rutgers.edu/loom]

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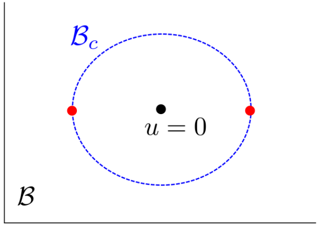

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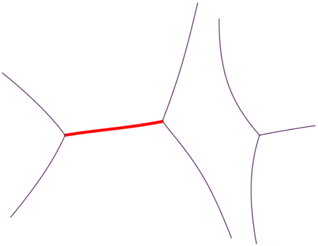

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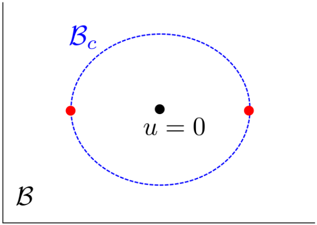

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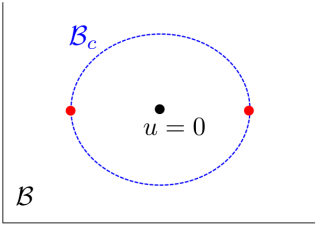

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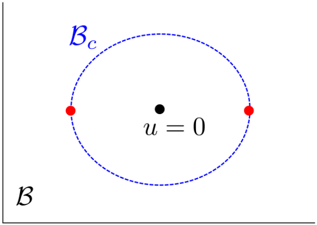

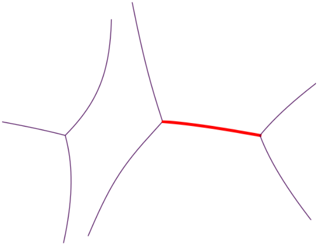

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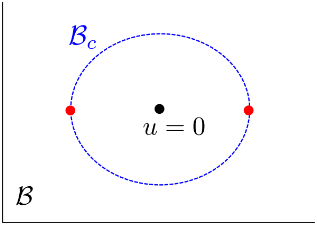

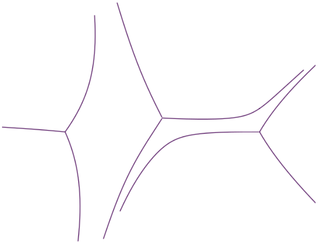

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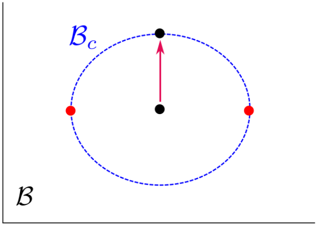

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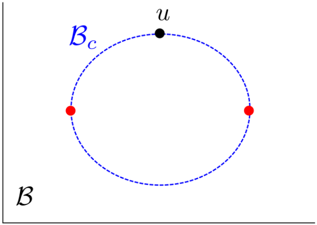

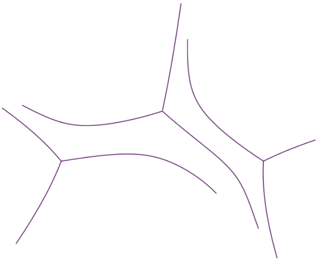

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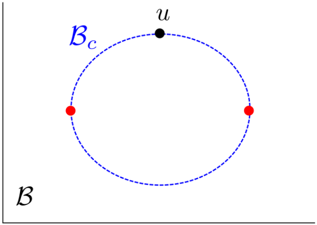

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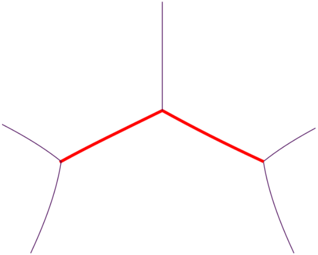

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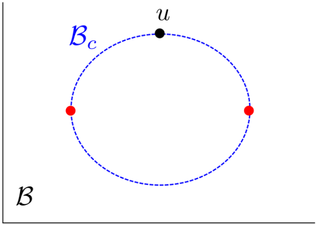

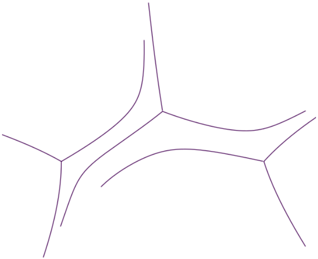

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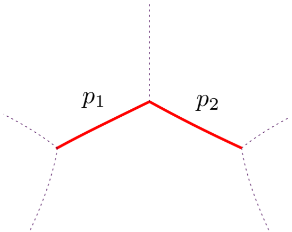
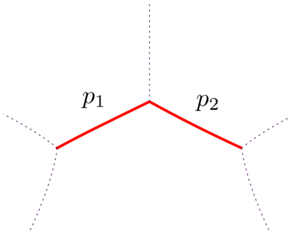


General feature: $\mathcal{G}$ is much simpler than the whole network.



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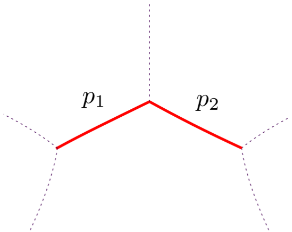
What this graph tells:

- $[\pi^{-1}(p_1)] = \gamma_1$, $[\pi^{-1}(p_2)] = \gamma_2$ are homology cycles.
- Both γ_1, γ_2 are BPS states (hypermultiplets) in any nearby chamber.
- They are a positive-integral basis for $\Gamma_+ := Z_{u_c}^{-1}(e^{i\vartheta_c}\mathbb{R}^+) \subset H_1(\Sigma, \mathbb{Z})$.
- Intersection $\langle\gamma_1, \gamma_2\rangle = -1$.



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$\mathcal{G}$ encodes the BPS quiver: $\gamma_1 \bigcirc \longrightarrow \bigcirc \gamma_2$

Quiver-Graph Correspondence

BPS quiver Q: an oriented graph composed of **nodes** Q_0 and **arrows** Q_1, with a **superpotential** $W \in R\langle Q\rangle$ (formal sum of cycles in the path algebra).

BPS states: zero-modes of supersymmetric quantum mechanics encoded by Q, subject to additional stability conditions. **[Fiol, Denef, Cecotti-Vafa, Alim et al, Cecotti-del Zotto]**

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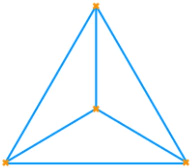
General quiver-graph correspondence: **[Gabella-L-Park-Yamazaki]**

- $\mathcal{G}$ admits a natural decomposition into **elementary webs** $\to$ Q_0.



BPS quiver Q: an oriented graph composed of **nodes** Q_0 and **arrows** Q_1, with a **superpotential** $W \in R\langle Q\rangle$ (formal sum of cycles in the path algebra).

BPS states: zero-modes of supersymmetric quantum mechanics encoded by Q, subject to additional stability conditions. **[Fiol, Denef, Cecotti-Vafa, Alim et al, Cecotti-del Zotto]**

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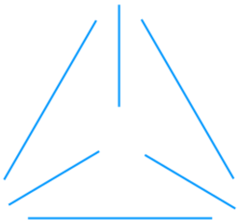

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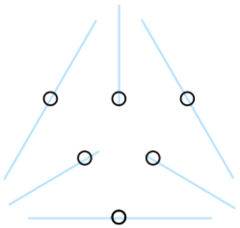

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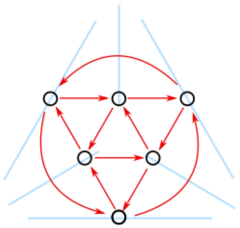

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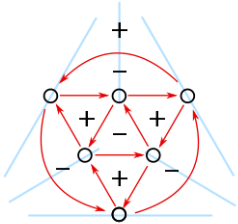

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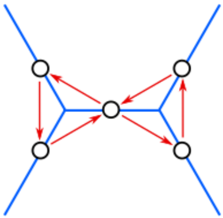
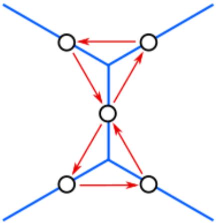
- edge flip $\to$ quiver mutation.





A_1 Theories

$$\mathcal{B} = \{\text{quadratic diff. with presc. poles}\} \supset \{\text{Strebel diff.}\}_0 = \mathcal{B}_c$$

$\mathcal{G}$ is the critical graph of a Strebel diff., or a "maximally contracted Fenchel-Nielsen network" **[Hollands-Neitzke]**. Existence guaranteed **[Strebel, Liu]**.

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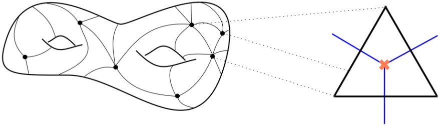
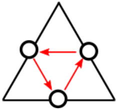
- BPS graphs are **dual to ideal triangulations** of C



- Quivers from triangulated surfaces **[Bridgeland-Smith, Fomin-Shapiro-Thurston, Labardini Fragoso, Alim et al., Fock-Goncharov]**:
  - $Q_0 \leftrightarrow$ edges, $Q_1 \leftrightarrow$ faces
  - canonical superpotential $\leftrightarrow$ face-loops and puncture-loops
  - mutations $\leftrightarrow$ flips



A_{N-1} Theories

$$\mathcal{B} = \{\text{meromorphic } k\text{-diff. w/ prescribed poles at punctures, } k = 2, \ldots, N\}$$

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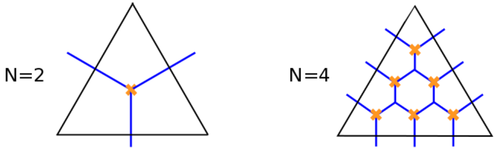
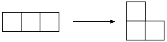
Conjecture: **restrict** to k-differentials with poles of order k (**full punctures**), then $\mathcal{B}_c$ exists and BPS graphs are dual to ideal **N-triangulations**



- $\binom{N}{2}$ branch points in each Δ, connected by elementary webs.
- Candidate BPS graphs for A_{N-1} theories with full punctures.
- Motivated by "N-lift construction" **[Gaiotto-Moore-Neitzke]**.
- Found explicit examples, pass nontrivial checks with known BPS quivers.

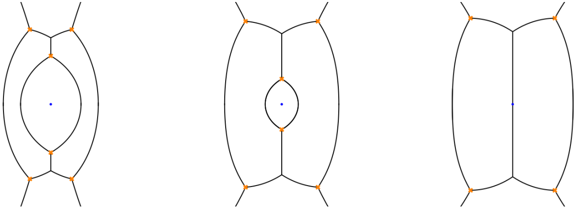
Beyond full punctures

Nice feature of theories with full punctures: obtain many class $\mathcal{S}$ theories with **partial punctures**, by tuning "puncture data D" **[Gaiotto, Chacaltana-Distler-Tachikawa]**

Example: $\mathfrak{g} = A_2$ 

$$\varphi \sim \frac{1}{z}\begin{pmatrix} m_1 & & \\ & m_2 & \\ & & -m_1 - m_2 \end{pmatrix} \quad \rightarrow \quad \frac{1}{z}\begin{pmatrix} m & & \\ & m & \\ & & -2m \end{pmatrix}$$

- Pole in $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)^2 \sim 1/z^6$ becomes milder $\sim 1/z^4$.
- Zeroes Δ coincide with **branch points**, get **absorbed** by the puncture.
- Spectral curve undergoes a **topological transition**, reflected by $\mathcal{G}$.



Kontsevich-Soibelman Invariants

Kontsevich-Soibelman Wall-Crossing Formula

Jumps of BPS spectrum are controlled by an $\mathrm{Arg}\, Z_\gamma$-ordered product of quantum dilogarithms

$$\prod_{\gamma,m}^{\mathrm{Arg} Z(u)\nearrow} \Phi((-y)^m X_\gamma)^{a_m(\gamma,u)} = \prod_{\gamma',m'}^{\mathrm{Arg} Z(u')\nearrow} \Phi((-y)^{m'} X_{\gamma'})^{a_{m'}(\gamma',u')}$$

- $a_m(\gamma, u)$ counts $|\gamma, m\rangle$
  (Laurent coeff. of "protected spin character" / motivic DT invariants)
- Quantum torus algebra: $X_\gamma X_{\gamma'} = y^{\langle\gamma,\gamma'\rangle} X_{\gamma+\gamma'}$

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In addition to geometric data, the full definition of spectral networks includes **combinatorial data**, it is entirely determined by the topology of the network.

The network data defines a **coordinate system** $\{\mathcal{X}_\gamma\}$ for $\mathcal{M}$, viewed as $\mathcal{M}_{\text{flat}}(C, GL(K))$. Conjecturally part of a cluster atlas.

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Key property: At the phase of a BPS state ($\vartheta = ArgZ$) the topology of $\mathcal{W}$ jumps, inducing a (quantum) "change of coordinates"

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[Gaiotto-Moore-Neitzke, Galakhov-L-Moore]

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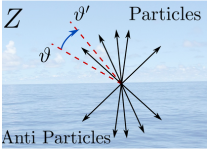
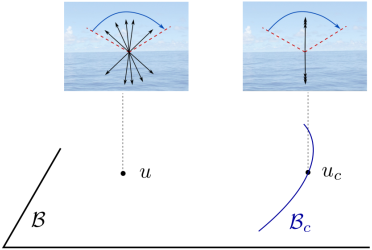
[Gaiotto-Moore-Neitzke, Galakhov-L-Moore]

BPS spectrum (at fixed u) controls ϑ-transition functions of coordinate charts



Coordinates at (ϑ, u) with $(\vartheta + \pi, u)$ are related by $X'_\gamma = \mathbb{U} X_\gamma \mathbb{U}^{-1}$

At the Critical Locus



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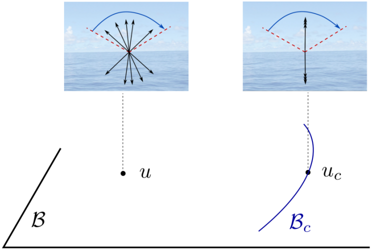

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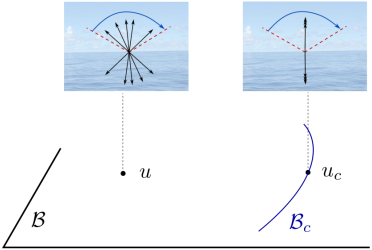

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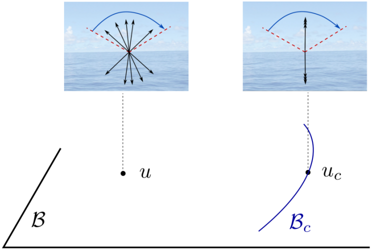

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The topology of the BPS graph determines $\mathbb{U}$. [L]

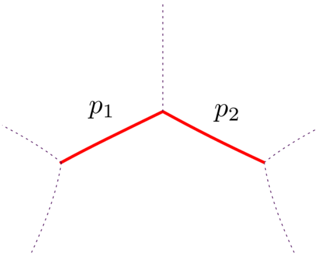
Argyres-Douglas

The graph has 2 edges, each contributes an equation

$$F'_p = \mathbb{U}\, F_p\, \mathbb{U}^{-1}$$

with

$$\begin{aligned} F_{p_1} &= 1 + y^{-1} X_{\gamma_1} + y^{-1} X_{\gamma_1+\gamma_2} \\ F_{p_2} &= 1 + y^{-1} X_{\gamma_2} \\ F'_{p_1} &= 1 + y^{-1} X_{\gamma_1} \\ F'_{p_2} &= 1 + y^{-1} X_{\gamma_2} + y^{-1} X_{\gamma_1+\gamma_2} \end{aligned}$$



Together, they determine the monodromy

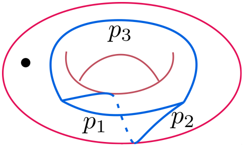
$$\begin{aligned} \mathbb{U} &= 1 - \frac{y}{(y)_1}\left(X_{\gamma_1} + X_{\gamma_2}\right) + \frac{y^2}{(y)_1^2} X_{\gamma_1+\gamma_2} + \frac{y^2}{(y)_2}\left(X_{2\gamma_1} + X_{2\gamma_2}\right) + \dots \\ &= \Phi(X_{\gamma_1})\Phi(X_{\gamma_2}) \end{aligned}$$

$SU(2)\ \mathcal{N}=2^*$

The graph has three edges p_1, p_2, p_3;
each contributes one equation

$$F'_p = \mathbb{U}\, F_p\, \mathbb{U}^{-1}$$

with



$$F_{p_1} = \frac{1+X_{\gamma_1}+\left(y+y^{-1}\right)X_{\gamma_1+\gamma_3}+X_{\gamma_1+2\gamma_3}+\left(y+y^{-1}\right)X_{\gamma_1+\gamma_2+2\gamma_3}+X_{\gamma_1+2\gamma_2+2\gamma_3}+X_{2\gamma_1+2\gamma_2+2\gamma_3}}{\left(1-X_{2\gamma_1+2\gamma_2+2\gamma_3}\right)^2}$$

$$F'_{p_1} = \frac{1+X_{\gamma_1}+\left(y+y^{-1}\right)X_{\gamma_1+\gamma_2}+X_{\gamma_1+2\gamma_2}+\left(y+y^{-1}\right)X_{\gamma_1+2\gamma_2+\gamma_3}+X_{\gamma_1+2\gamma_2+2\gamma_3}+X_{2\gamma_1+2\gamma_2+2\gamma_3}}{\left(1-X_{2\gamma_1+2\gamma_2+2\gamma_3}\right)^2}$$

$F_{p_{2,3}}$ & $F'_{p_{2,3}}$ are obtained by cyclic shifts of $\gamma_1, \gamma_2, \gamma_3$.

The solution:

$$\mathbb{U} = \left(\prod^{\nearrow}_{n\geq 0} \Phi\left(X_{\gamma_1+n(\gamma_1+\gamma_2)}\right)\right)$$

$$\times \Phi\left(X_{\gamma_3}\right)\Phi\left((-y)X_{\gamma_1+\gamma_2}\right)^{-1}\Phi\left((-y)^{-1}X_{\gamma_1+\gamma_2}\right)^{-1}\Phi\left(X_{2\gamma_1+2\gamma_2+\gamma_3}\right)$$

$$\times \left(\prod^{\searrow}_{n\geq 0} \Phi(X_{\gamma_2+n(\gamma_1+\gamma_2)})\right)$$

Remark 1

The BPS graph can have some **symmetries**. They are inherited by $\mathbb{U}$. Hidden by the factorization $\mathbb{U} = \prod \Phi(X)$, but **manifest on the BPS graph** (Ex. $\mathbb{Z}_3$ symmetry in $\mathcal{N} = 2^*$).

Reflect basic properties of the Schur index **[Cecotti-Neitzke-vafa, Iqbal-vafa, Cordova-Gaiotto-Shao]**, computed as the correlator of a TQFT on C **[Gadde-Pomoni-Rastelli-Razamat]**: it is a symmetric function of the flavor fugacities.

Remark 2

How to make sense of $\mathbb{U}$ physically at $\mathcal{B}_c$? BPS spectrum is ill-defined! Rich physics in the background. Key idea is to use **surface defects**.

- Induce a new sector of “2d-4d” BPS states.
- **(framed) 2d-4d wall-crossing**: creation/decay of 2d-4d states is controlled by 4d BPS spectrum. Unification of Cecotti-Vafa and Kontsevich-Soibelman wall-crossing. **[Gaiotto-Moore-Neitzke]**
- Key to computing $\mathbb{U}$ via 2d-4d wall-crossing: unlike 4d BPS states, **stability** of 2d-4d spectrum is **well-defined at $\mathcal{B}_c$**.

Conclusions

1. We introduce a new object: the **BPS graph** $\mathcal{G}$ of a theory of class $\mathcal{S}$. $\mathcal{G}$ lives on the on the UV curve, and emerges from degenerate spectral networks at $\mathcal{B}_c$.

2. Link between spectral networks and **BPS quivers**. Correctly encodes known quivers. Approach to obtain many new ones, by moduli-deformation of K-lifts.

3. A new construction of Kontsevich-Soibelman invariants based on $\mathcal{G}$. **Manifestly** wall-crossing invariant. Exhibits **symmetries** of $\mathbb{U}$.

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Thank You.