BPS-states and automorphic representations of exceptional groups

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String Math
Hamburg
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Outline

1. Motivation from string theory

2. Automorphic forms and representation theory

3. Small representations and BPS-couplings

4. Outlook
Talk based on our papers/book:

[1511.04265] w/ Fleig, Gustafsson, Kleinschmidt
[1412.5625] w/ Gustafsson, Kleinschmidt
[1312.3643] w/ Fleig, Kleinschmidt

to appear on Friday this week

[1707.XXXX] w/ Ahlén, Gustafsson, Kleinschmidt, Liu

and in progress

[17YY.XXXX] w/ Gourevitch, Gustafsson, Kleinschmidt, Sahi
I. Motivation from string theory
String amplitudes

Understand the structure of string interactions
String amplitudes

Understand the structure of **string interactions**

![String diagrams](image)

Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

Amplitudes have intricate arithmetic structure $G(\mathbb{Z})$
String amplitudes

Understand the structure of **string interactions**

- supersymmetry
- U-duality

Strongly constrained by **symmetries**!

Symmetry constrains interactions, leads to insights about:

- ultraviolet properties of gravity
- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics
Toroidal compactifications yield the famous chain of \textbf{U-duality groups} \cite{Cremmer, Julia} \cite{Hull, Townsend}

Physical couplings are given by \textbf{automorphic forms} on

$$G(\mathbb{Z}) \backslash G(\mathbb{R}) / K$$

<table>
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<th>$D$</th>
<th>$G$</th>
<th>$K$</th>
<th>$G(\mathbb{Z})$</th>
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<td>$\text{SL}(2, \mathbb{R})$</td>
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<td>$\text{SL}(5, \mathbb{R})$</td>
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<td>$\text{Spin}(5, 5, \mathbb{R})$</td>
<td>$(\text{Spin}(5) \times \text{Spin}(5)) / \mathbb{Z}_2$</td>
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<td>$\text{E}_6(\mathbb{R})$</td>
<td>$\text{USp}(8) / \mathbb{Z}_2$</td>
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<td>$\text{E}_8(\mathbb{R})$</td>
<td>$\text{Spin}(16) / \mathbb{Z}_2$</td>
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Higher-derivative action in type II string theory on tori

$$\int d^{10-n} x \sqrt{G} \left[ (\alpha')^3 f_0(g) \mathcal{R}^4 + (\alpha')^5 f_4(g) \partial^4 \mathcal{R}^4 + \cdots \right]$$
Higher-derivative action in type II string theory on tori

\[ \int d^{10-n}x \sqrt{G} \left[ (\alpha')^3 f_0(g) R^4 + (\alpha')^5 f_4(g) \partial^4 R^4 + \cdots \right] \]

contraction of four Riemann tensors
Higher-derivative action in type II string theory on tori

\[ \int d^{10-n}x \sqrt{G} \left[ (\alpha')^3 f_0(g) R^4 + (\alpha')^5 f_4(g) \partial^4 R^4 + \cdots \right] \]

\[ \rightarrow f_0(g), f_4(g) \text{ are functions of } g \in E_{n+1}(\mathbb{R})/K \]

\[ \rightarrow \text{must be invariant under } U\text{-duality } E_{n+1}(\mathbb{Z}) \]

\[ \rightarrow \text{supersymmetry requires that they are } \text{Laplacian eigenfunctions} \]

\[ \rightarrow \text{well-defined weak-coupling expansions as } g_s \rightarrow 0 \]
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defining properties of an automorphic form!
2. Automorphic forms and representation theory
Data:

- \( G(\mathbb{R}) \) real semi-simple Lie group (e.g. \( SL(n, \mathbb{R}) \))
- \( G(\mathbb{Z}) \subset G \) arithmetic subgroup (e.g. \( SL(n, \mathbb{Z}) \))
An automorphic form is a smooth function $\varphi : G \rightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}), \varphi(\gamma g) = \varphi(g)$

2. $\varphi$ is an eigenfunction of the ring of inv. diff. operators on $G$

3. $\varphi$ has well-behaved growth conditions

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Example: Non-holomorphic Eisenstein series on $G(\mathbb{R}) = SL(2, \mathbb{R})$

$$E(s, \tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{y^s}{|m\tau + n|^{2s}}$$

$s \in \mathbb{C}$

$$\tau = x + iy \in \mathbb{H} \cong SL(2, \mathbb{R})/U(1)$$
Automorphic representations

\[ A(\mathbb{G}(\mathbb{Z}) \backslash \mathbb{G}(\mathbb{R})) = \{ \text{space of automorphic forms on } \mathbb{G}(\mathbb{R}) \} \]
Automorphic representations

\[ \mathcal{A}(G(\mathbb{Z}) \backslash G(\mathbb{R})) = \{ \text{space of automorphic forms on } G(\mathbb{R}) \} \]

The group \( G \) acts on this space via the right-regular representation:

\[(\rho(h)\varphi)(g) = \varphi(gh)\]

for \( \varphi \in \mathcal{A} \) and \( h, g \in G \)
Automorphic representations

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The group \( G \) acts on this space via the \textbf{right-regular representation}:

\[
(\rho(h)\varphi)(g) = \varphi(gh)
\]

for \( \varphi \in \mathcal{A} \) and \( h, g \in G \)

\[\text{Definition: An \textbf{automorphic representation} } \pi \text{ of } G \]

\[\text{is an irreducible constituent in the decomposition of } \mathcal{A} \text{ under the right-regular action.}\]
Automorphic representations

\[ \mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R})) = \mathcal{A}_{\text{discrete}} \oplus \mathcal{A}_{\text{continuous}} \]

- \( \mathcal{A}_{\text{discrete}} \): generated by cusp forms
  (and residues of Eisenstein series)

\[ \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)\,du = 0 \quad \text{all unipotents} \quad U \subset G \]

- \( \mathcal{A}_{\text{continuous}} \): generated by Eisenstein series
Example: $G = SL(2, \mathbb{R})$

$$f_0(\tau) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2} + 2\pi \sum_{m \neq 0} \sqrt{|m|} \sigma_2(m) e^{-S_{\text{inst}}(z)} \left[ 1 + O(y^{-1}) \right]$$

perturbative terms

$$y = g_s^{-1}$$

tree-level

non-perturbative terms

one-loop

amplitudes in the presence of instantons

unique solution!

$$f_0(\tau) = \sum_{(m,n) \neq (0,0)} {y^{3/2} \over |m + n\tau|^3}$$

[Green, Gutperle]
[Green, Sethi]
[Pioline]
Example: \( G = SL(2, \mathbb{R}) \)

\[
f_0(\tau) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2}
\]

\[
y = g_s^{-1}
\]

perturbative terms

\[
f_0(\tau) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2} + 2\pi \sum_{m \neq 0} \sqrt{|m|} \sigma_2(m) e^{-S_{\text{inst}}(z)} [1 + \mathcal{O}(y^{-1})]
\]

non-perturbative terms

\[
\mathcal{A}_{\text{amplitudes in the presence of instantons}}
\]

\[
S_{\text{inst}}(z) := 2\pi |m| y - 2\pi im x
\]
Example: \( G = SL(2, \mathbb{R}) \)

\[
y = g_s^{-1}
\]

\[
f_0(\tau) = 2\zeta(3)y^{3/2} + 4\zeta(2)y^{-1/2}
\]

\[
+ 2\pi \sum_{m \neq 0} \sqrt{|m|} \sigma_{-2}(m) e^{-S_{\text{inst}}(z)} \left[ 1 + \mathcal{O}(y^{-1}) \right]
\]

\[
S_{\text{inst}}(z) := 2\pi |m| y - 2\pi i m x
\]

\[
\sigma_{-2}(m) = \sum_{d|m} d^{-2}
\]
An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.

— Robert P. Langlands
An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.

— Robert P. Langlands
Adelic framework

For each prime number $p$

$\mathbb{Q} \rightarrow \mathbb{R} \rightarrow \mathbb{Q}_p$

Euclidean norm

$p$-adic norm

real numbers

$p$-adic numbers

completions of $\mathbb{Q}$

$\mathbb{Q}_\infty = \mathbb{R}$
Adelic framework

For each prime number $p$

Euclidean norm

$\mathbb{Q} \overset{\text{Euclidean norm}}{\longrightarrow} \mathbb{R}$ \hspace{1cm} \text{real numbers} \hspace{1cm} \mathbb{Q}_\infty = \mathbb{R}$

$p$-adic norm

$\mathbb{Q} \overset{\text{$p$-adic norm}}{\longrightarrow} \mathbb{Q}_p$ \hspace{1cm} \text{$p$-adic numbers}$

The adeles are then defined as

$A = \mathbb{R} \times \prod_{p \text{ prime} < \infty} \mathbb{Q}_p$

$x = (x_\infty; x_2, x_3, x_5, \ldots) \in A$

Q $\hookrightarrow A$

$q \mapsto (q; q, q, q, \ldots)$

$\mathbb{Q} \subset A$ \hspace{1cm} \text{discrete embedding}$

analogous to: \hspace{1cm} \mathbb{Z} \subset \mathbb{R}$

$\text{global}$

$\text{local}$
Adelic framework

For each prime number $p$

Euclidean norm

\[
\begin{array}{ccc}
\mathbb{Q} & \rightarrow & \mathbb{R} \\
& & \text{real numbers} \\
& \rightarrow & \mathbb{Q}_p \\
& & \text{$p$-adic numbers}
\end{array}
\]

\[Q_\infty = \mathbb{R}\]

The adeles are then defined as

\[A = \mathbb{R} \times \prod_{p \text{ prime } < \infty} \mathbb{Q}_p\]

The adeles $x = (x_\infty; x_2, x_3, x_5, \ldots) \in A$

\[Q \hookrightarrow A\]

\[q \mapsto (q; q, q, q, \ldots)\]

much easier to work with since $\mathbb{Q}$ is a field.

analogous to: $\mathbb{Z} \subset \mathbb{R}$
Adelic framework

(completed) **Riemann zeta function:** \( \xi(s) = \xi(1 - s) \)

\[
\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-s/2} \Gamma(s/2) \prod_{p \text{ prime} < \infty} \frac{1}{1 - p^{-s}}
\]

\[
= \int_{\mathbb{R}} e^{-\pi x^2} |x|^s \, dx \prod_{p \text{ prime} < \infty} \int_{\mathbb{Q}_p} \gamma_p(x) |x|_p^s \, dx
\]

\[
= \int_{\mathbb{A}} \gamma_\mathbb{A}(x) |x|_\mathbb{A}^s \, dx
\]

In his famous thesis, Tate gave elegant new proofs of the **functional equation and analytic continuation** of \( \xi(s) \) using these techniques.
Adelic framework

arithmetic groups \[ G(\mathbb{Z}) \subset G(\mathbb{R}) \quad \rightarrow \quad G(\mathbb{Q}) \subset G(\mathbb{A}) \]

space of automorphic forms \[ \mathcal{A}(G(\mathbb{Z}) \backslash G(\mathbb{R})) \quad \rightarrow \quad \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \]
Adelic framework

arithmetic groups $G(\mathbb{Z}) \subset G(\mathbb{R}) \quad \rightarrow \quad G(\mathbb{Q}) \subset G(\mathbb{A})$

space of automorphic forms $\mathcal{A}(G(\mathbb{Z}) \backslash G(\mathbb{R})) \quad \rightarrow \quad \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$

Eisenstein series

$$\sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} e^{\langle \lambda + \rho | H(\gamma g) \rangle} \quad \rightarrow \quad \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

$\lambda \in \mathfrak{h}^* \otimes \mathbb{C}$

$H : G \rightarrow \mathfrak{h}$
3. Small representations and BPS-couplings
Minimal automorphic representations

**Definition:** An automorphic representation 

\[ \pi = \bigotimes_{p \leq \infty} \pi_p \]

is minimal if each factor \( \pi_p \) has smallest non-trivial Gelfand-Kirillov dimension.

Automorphic forms \( \varphi \in \pi_{min} \) are characterised by having very few non-vanishing Fourier coefficients.
Exceptional groups

Functional dimension of minimal representations:

\[
\text{GKdim } \pi_{\text{min}} = \begin{cases} 
11, & E_6 \\
17, & E_7 \\
29, & E_8 
\end{cases}
\]
Automorphic realization

Consider the Borel-Eisenstein series on $G(\mathbb{A})$

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Now fix the weight to

$$\lambda = 2s \Lambda_1 - \rho$$

where $\Lambda_1$ is the fundamental weight associated to node 1.
Theorem [Ginzburg,Rallis,Soudry][Green,Miller,Vanhove]

For $G = E_6, E_7, E_8$ the Eisenstein series $E(2s\Lambda - \rho, g)$ evaluated at $s = 3/2$ is attached to the representation $\pi_{min}$

This theorem yields an explicit automorphic realisation of the minimal representation.

Our aim is to use this to calculate Fourier coefficients associated with maximal parabolic subgroups.
BPS-couplings

\[
\int d^{11-n} x \sqrt{G} f_0(g) \mathcal{R}^4 \quad f_0(g) = E(3/2, g) \quad s = 3/2
\]

\[
\int d^{11-n} \sqrt{G} f_4(g) \partial^4 \mathcal{R}^4 \quad f_4(g) = E(5/2, g) \quad s = 5/2
\]

These partition functions are Eisenstein series attached to small automorphic representations of \( G \).

[minimal automorphic representation] \( \pi_{\text{min}} \)

[next-to-minimal automorphic representation] \( \pi_{\text{ntm}} \)

\( g \in E_n(\mathbb{R}) \)

I/2 - BPS

I/4 - BPS

[Green, Miller, Vanhove][Pioline]
Perturbative limit - choices of unipotent subgroups

→ **Decompactification limit**
- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

→ **String perturbation limit**
- perturbative parameter: string coupling
- non-perturbative effects: D-instantons, NS5-instantons

→ **M-theory limit**
- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons
general Fourier coefficients

$P = LU \text{ standard parabolic of } G$

unitary character $\psi_U : U(\mathbb{Q}) \backslash U(\mathbb{A}) \rightarrow U(1)$

We then have the $U$-Fourier coefficient:

$$F_{\psi_U}(f_{\chi}, g) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(f_{\chi}, u g) \overline{\psi_U(u)} du$$

very little is known in general in this case…
Theorem [Miller-Sahi]: Let $G$ be a split group of type $E_6$ or $E_7$.

Then any Fourier coefficient of $\varphi \in \pi_{min}$ of $G$ is completely determined by the maximally degenerate Whittaker coefficient

$$ W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(ng) \overline{\psi_\alpha(n)} \, dn $$
**Theorem** [Miller-Sahi]: Let $G$ be a split group of type $E_6$ or $E_7$. Then any Fourier coefficient of $\varphi \in \pi_{\min} \text{ of } G$ is completely determined by the maximally degenerate Whittaker coefficient

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(ng) \overline{\psi_\alpha(n)} \, dn$$

Can one use this to calculate

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(\varphi, ug) \overline{\psi_U(u)} \, du$$

in terms of $W_{\psi_\alpha}$?
Example: \( G = E_7 \)

Consider the **3-grading** of the Lie algebra

\[
e_7 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = 27 \oplus (e_6 \oplus 1) \oplus 27.
\]

The space \( \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is the Lie algebra of a maximal parabolic \( P = LU \) with 27-dim unipotent \( U \) and Levi \( L = E_6 \times GL(1) \)
The degenerate Whittaker vector associated with $\alpha_7$ is given by:

$$W_{\psi_k}(3/2, a) = |k|^{3/2} \sigma_{-3}(k) K_{3/2}(2\pi |k| a)$$

where $a \in A \subset E_7$ and

$$\sigma_s(k) = \sum_{d|k} d^s$$
The degenerate Whittaker vector associated with $\alpha_7$ is given by:

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$$\sigma_s(k) = \sum_{d|k} d^s$$

We now want to relate this to the $U$ - Fourier coefficient

$$F_{\psi_U}(3/2, g) = \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} E(3/2, ug) \overline{\psi_U(u)} du$$

This captures **instantons in the decompactification limit** of $\mathbb{II}/T^6$!
Claim: [Pioline][Gustafsson, Kleinschmidt, D.P.][Bossard, Verschinin]

\[ F_{\psi_U} \left( \frac{3}{2}; h, r \right) = \left| k \right|^{3/2} \sigma_3(k) K_{3/2} \left( 2\pi r \left| k \right| \times \| h^{-1} x \| \right) \]

where \( h \in E_6, r \in GL(1) \) and \( \vec{x} \in \mathbb{Z}^{27} \)
Claim: [Pioline][Gustafsson, Kleinschmidt, D.P.][Bossard, Verschinin]

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where \( h \in E_6, r \in GL(1) \) and \( \vec{x} \in \mathbb{Z}^{27} \)

Proof: To appear by [Gourevitch, Gustafsson, Kleinschmidt, D.P., Sahi]

This gives the complete abelian Fourier expansion of the minimal representation

Physically the vector \( \vec{x} \) corresponds to the instanton charges of the 27 vector fields in D=5.
Next-to-minimal representations

Relevant for $\partial^4 \mathcal{R}^4$-couplings.

**Theorem** [Green, Miller, Vanhove]: Let $G = E_6, E_7, E_8$

The Eisenstein series

$$E(s, g) = \sum_{\gamma \in \mathcal{B}(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle 2s\Lambda_1 | H(\gamma g) \rangle}$$

evaluated at $s = 5/2$ is a spherical vector in $\pi_{ntm}$. 
Conjecture [Gustafsson, Kleinschmidt, D.P.]:

Let $G$ be a semisimple, simply laced Lie group.

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_\alpha} (\varphi, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(n g) \overline{\psi_\alpha(n)} dn$$

$$W_{\psi_\alpha, \beta} (\varphi, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(n g) \overline{\psi_{\alpha, \beta}(n)} dn$$

where $\alpha, \beta$ are commuting simple roots.

*Proof.* For $SL(n)$ to appear by [Ahlén, Gustafsson, Kleinschmidt, Liu, D.P.]

For exceptionals, in progress by [Gustafsson, Gourevitch, Kleinschmidt, D.P., Sahi]
Conjecture [Gustafsson, Kleinschmidt, D.P.]:

Let $G$ be a semisimple, simply laced Lie group.

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ are completely determined by degenerate Whittaker vectors of the form

$$W_{\psi_{\alpha}}(\varphi, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(n g) \overline{\psi_{\alpha}(n)} dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \varphi(n g) \overline{\psi_{\alpha,\beta}(n)} dn$$

where $(\alpha, \beta)$ are commuting simple roots.

This will allow us to extract instanton effects from $\partial^4 R^4$ couplings!

See also [Bossard, Pioline] [Bossard, Cosnier-Horeau, Pioline]
4. Outlook
So what happens at the next order?

1/8-BPS and non-BPS couplings seem to require more general automorphic objects.

[Green, Miller, Vanhove][Pioline][Bossard, Verschinin][Bossard, Kleinschmidt]

This is uncharted mathematical territory.
So what happens at the next order?

1/8-BPS and non-BPS couplings seem to require more general automorphic objects.

[Green, Miller, Vanhove][Pioline][Bossard, Verschinin][Bossard, Kleinschmidt]

This is uncharted mathematical territory.

Below D=3 we also enter the realm of Kac-Moody groups!

Connections with double affine Hecke algebras (DAHAs)?

Nicolai, Damour, Henneaux, West, Kleinschmidt, Fleig, D.P., Garland, Lee, Patnaik, Braverman, Kazhdan, Miller, Carbone…