BPS-states and automorphic representations of exceptional groups

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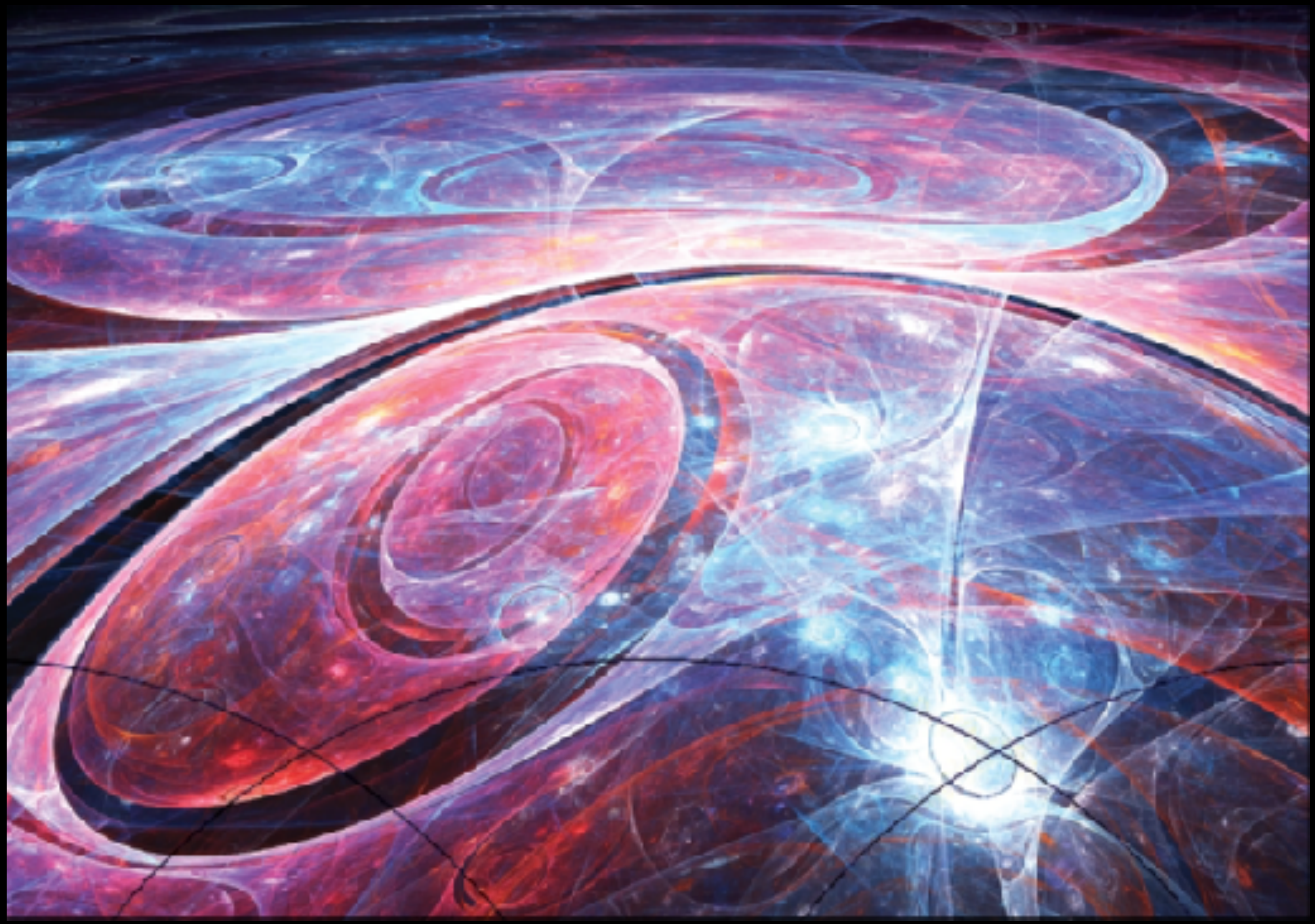
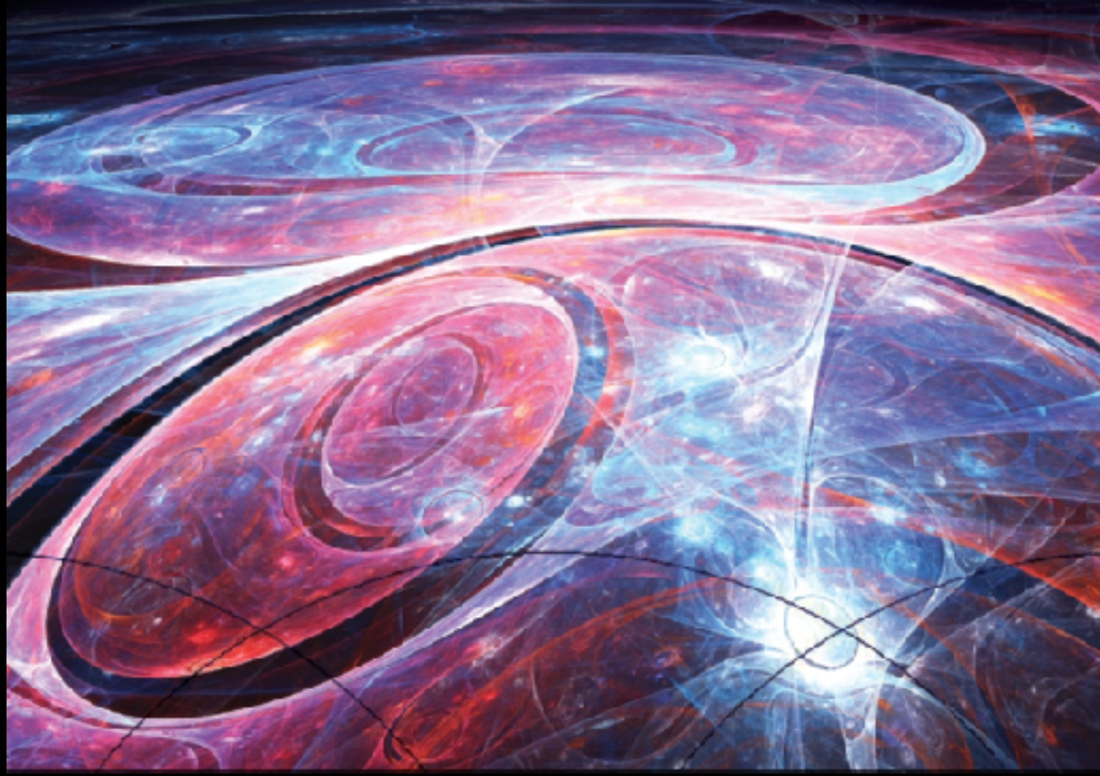


String Math
Hamburg
July 26, 2017

Outline



1. Motivation from string theory

2. Automorphic forms and representation theory

3. Small representations and BPS-couplings

4. Outlook

Talk based on our papers/book:

[1511.04265] w/ Fleig, Gustafsson, Kleinschmidt

[1412.5625] w/ Gustafsson, Kleinschmidt

[1312.3643] w/ Fleig, Kleinschmidt

to appear on Friday this week

[1707.XXXX] w/ Ahlén, Gustafsson, Kleinschmidt, Liu

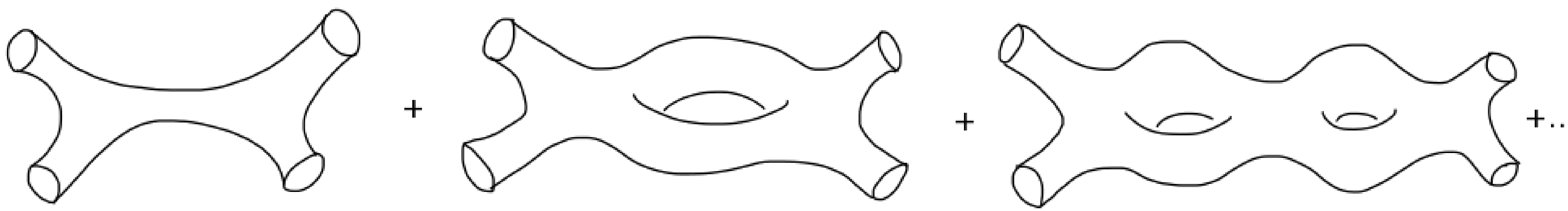
and in progress

[17YY.XXXX] w/ Gourevitch, Gustafsson, Kleinschmidt, Sahi

1. Motivation from string theory

String amplitudes

Understand the structure of **string interactions**



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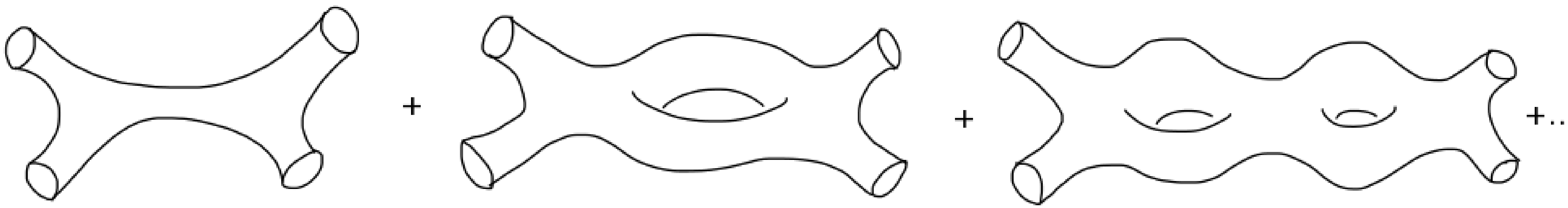

Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

→ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

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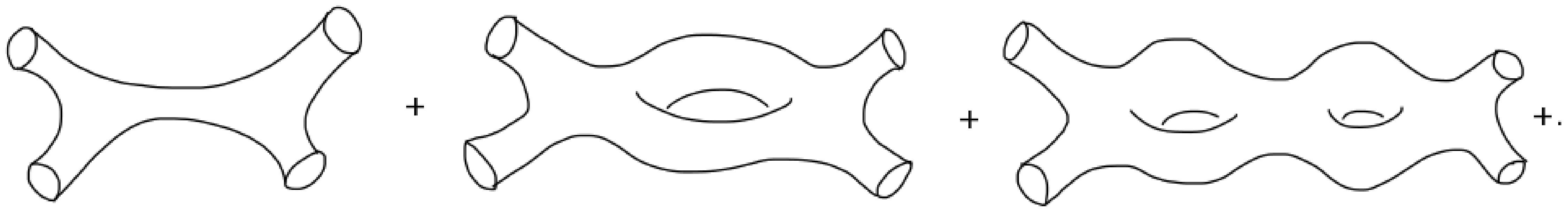


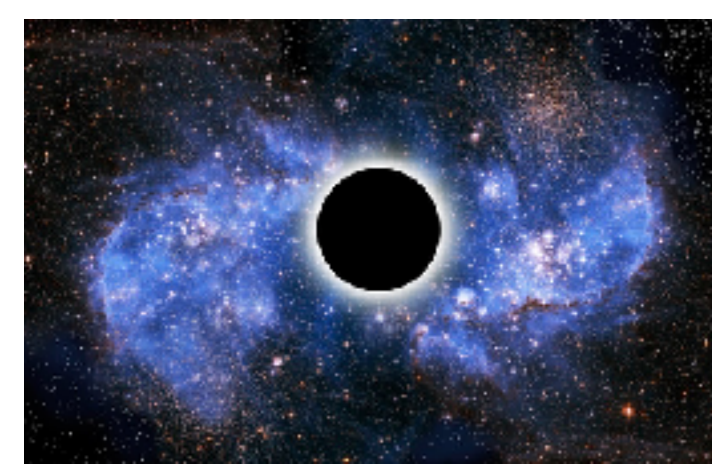
Strongly constrained by **symmetries**!

- supersymmetry
- U-duality

→ amplitudes have intricate **arithmetic structure** $G(\mathbb{Z})$

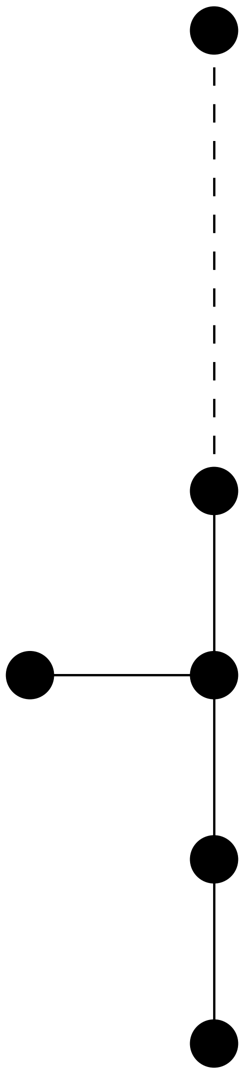
Symmetry constrains interactions, leads to insights about:

- ultraviolet properties of gravity
- non-perturbative effects (black holes, instantons)
- novel mathematical predictions from physics



Toroidal compactifications yield the famous chain of **U-duality groups**

[Cremmer, Julia][Hull, Townsend]



D	G	K	$G(\mathbb{Z})$
10	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
9	$\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$	$\mathrm{SO}(2)$	$\mathrm{SL}(2,\mathbb{Z})$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$	$\mathrm{SL}(5,\mathbb{Z})$
6	$\mathrm{Spin}(5,5,\mathbb{R})$	$(\mathrm{Spin}(5) \times \mathrm{Spin}(5))/\mathbb{Z}_2$	$\mathrm{Spin}(5,5,\mathbb{Z})$
5	$\mathrm{E}_6(\mathbb{R})$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathrm{E}_6(\mathbb{Z})$
4	$\mathrm{E}_7(\mathbb{R})$	$\mathrm{SU}(8)/\mathbb{Z}_2$	$\mathrm{E}_7(\mathbb{Z})$
3	$\mathrm{E}_8(\mathbb{R})$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	$\mathrm{E}_8(\mathbb{Z})$

Physical couplings are given by **automorphic forms** on

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K$$

Green, Gutperle, Sethi, Vanhove, Kiritsis, Pioline, Obers, Kazhdan, Waldron, Basu, Russo, Cederwall, Bao, Nilsson, D.P., Lambert, West, Gubay, Miller, Bossard, Fleig, Kleinschmidt, Gustafsson, Cosnier-Horeau…

Higher-derivative action in type II string theory on tori

$$\int d^{10-n}x\sqrt{G}\,\left[(\alpha')^3 f_0(g)\mathcal{R}^4 + (\alpha')^5 f_4(g)\partial^4\mathcal{R}^4 + \cdots\right]$$

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contraction of four Riemann tensors

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- → $f_0(g), f_4(g)$ are functions of $g \in E_{n+1}(\mathbb{R})/K$
- → must be **invariant** under U-duality $E_{n+1}(\mathbb{Z})$
- → supersymmetry requires that they are **Laplacian eigenfunctions**
- → well-defined **weak-coupling expansions** as $g_s \to 0$

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defining properties of an **automorphic form!**

2. Automorphic forms and representation theory

Data:

- $G(\mathbb{R})$ real semi-simple Lie group (e.g. $SL(n,\mathbb{R})$)
- $G(\mathbb{Z}) \subset G$ arithmetic subgroup (e.g. $SL(n,\mathbb{Z})$)

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Definition:

An **automorphic form** is a smooth function $\varphi : G \longrightarrow \mathbb{C}$ satisfying

1. Automorphy: $\forall \gamma \in G(\mathbb{Z}),\ \ \varphi(\gamma g) = \varphi(g)$
2. φ is an eigenfunction of the ring of inv. diff. operators on G
3. φ has well-behaved growth conditions

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Example: Non-holomorphic Eisenstein series on $G(\mathbb{R}) = SL(2,\mathbb{R})$

$$E(s,\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|m\tau+n|^{2s}} \qquad s \in \mathbb{C}$$

$$\tau = x + iy \in \mathbb{H} \cong SL(2,\mathbb{R})/U(1)$$

Automorphic representations

$$\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R})) = \{\text{space of automorphic forms on } G(\mathbb{R})\}$$

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$$\left(\rho(h)\varphi\right)(g) = \varphi(gh)$$

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for $\varphi \in \mathcal{A}$ and $h, g \in G$

Definition: *An* ***automorphic representation*** π *of* G *is an irreducible constituent in the decomposition of* $\mathcal{A}$ *under the right-regular action.*

[Gelfand, Graev, Piatetski-Shapiro][Langlands]…

Automorphic representations

$$\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R})) = \mathcal{A}_{discrete} \oplus \mathcal{A}_{continuous}$$

- $\mathcal{A}_{discrete}$: generated by cusp forms

  (and residues of Eisenstein series)

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)du = 0$$

all unipotents

$U \subset G$

- $\mathcal{A}_{continuous}$: generated by Eisenstein series

Example: $G = SL(2,\mathbb{R})$

$$\int d^{10}x\sqrt{G}\, f_0(\tau)\mathcal{R}^4$$

$$y = g_s^{-1}$$

$$f_0(\tau) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+ 2\pi \sum_{m\neq 0} \sqrt{|m|}\,\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1 + \mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}$$

amplitudes in the presence of instantons

unique solution!

$$f_0(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^{3/2}}{|m+n\tau|^3}$$

[Green, Gutperle]
[Green, Sethi]
[Pioline]

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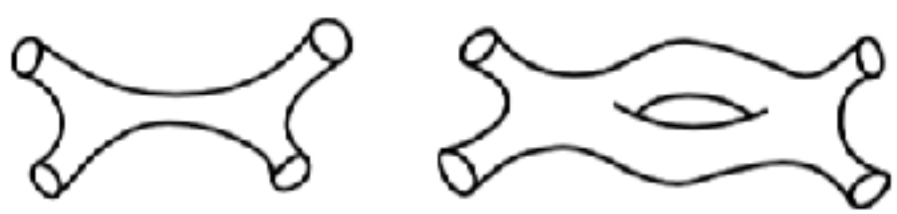

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instanton action

$$S_{\text{inst}}(z) := 2\pi\,|m|\,y - 2\pi i m x$$

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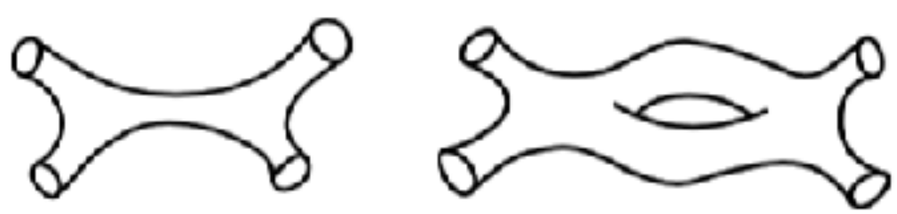

amplitudes in the presence of instantons

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$$S_{\text{inst}}(z) := 2\pi\,|m|\,y - 2\pi i m x$$

instanton measure

$$\sigma_{-2}(m) = \sum_{d|m} d^{-2}$$

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*
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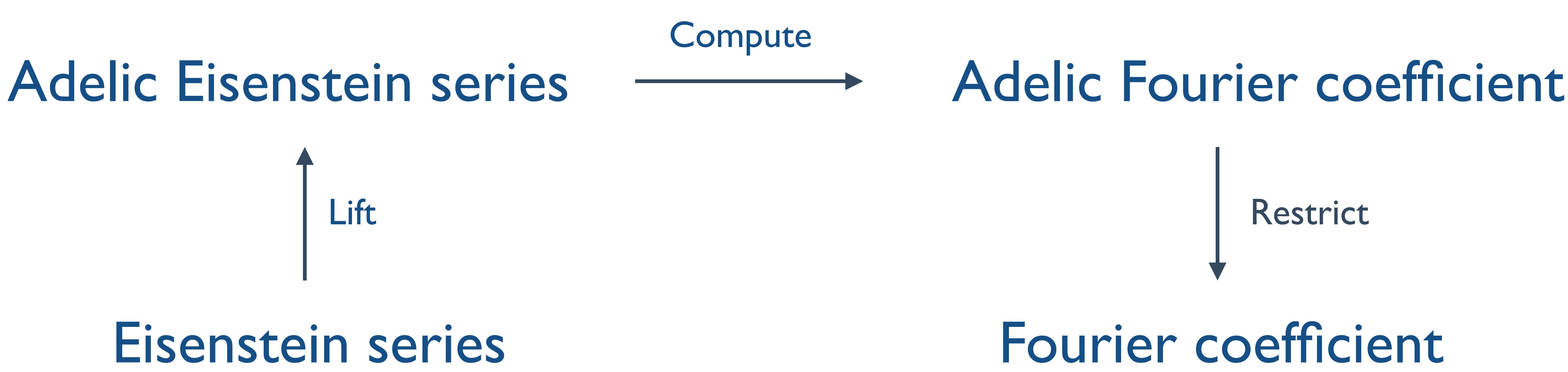

Adelic framework

For each **prime number** *p*

Euclidean norm: $\mathbb{Q} \to \mathbb{R}$ — *real numbers*

p-adic norm: $\mathbb{Q} \to \mathbb{Q}_p$ — *p*-adic numbers

$$\mathbb{Q}_\infty = \mathbb{R}$$

completions of $\mathbb{Q}$

Adelic framework

For each prime number p

Euclidean norm

$\mathbb{Q} \to \mathbb{R}$ *real numbers*

$\mathbb{Q} \to \mathbb{Q}_p$ *p*-adic numbers

p-adic norm

$$\mathbb{Q}_\infty = \mathbb{R}$$

The adeles are then defined as

$$\mathbb{A} = \mathbb{R} \times \prod_{p \, \mathrm{prime} < \infty}' \mathbb{Q}_p$$

global — local

$$x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

$$\mathbb{Q} \hookrightarrow \mathbb{A}$$

$\mathbb{Q} \subset \mathbb{A}$ **discrete embedding**

$$q \mapsto (q; q, q, q, \ldots)$$

analogous to: $\mathbb{Z} \subset \mathbb{R}$

Adelic framework

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$$\mathbb{Q} \subset \mathbb{A}$$

much easier to work with since $\mathbb{Q}$ is a **field**.

analogous to: $\mathbb{Z} \subset \mathbb{R}$

Adelic framework

(completed) **Riemann zeta function**: $\xi(s) = \xi(1-s)$

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-s/2}\Gamma(s/2) \prod_{p \text{ prime} < \infty} \frac{1}{1-p^{-s}}$$

$$= \int_{\mathbb{R}} e^{-\pi x^2} |x|^s dx \prod_{p \text{ prime} < \infty} \int_{\mathbb{Q}_p} \gamma_p(x) |x|_p^s dx$$

$$= \int_{\mathbb{A}} \gamma_{\mathbb{A}}(x) |x|_{\mathbb{A}}^s dx$$

In his famous thesis, Tate gave elegant new proofs of the **functional equation and analytic continution** of $\xi(s)$ using these techniques

Adelic framework

arithmetic groups	$G(\mathbb{Z}) \subset G(\mathbb{R})$	$\longrightarrow$	$G(\mathbb{Q}) \subset G(\mathbb{A})$
space of automorphic forms	$\mathcal{A}\big(G(\mathbb{Z})\backslash G(\mathbb{R})\big)$	$\longrightarrow$	$\mathcal{A}\big(G(\mathbb{Q})\backslash G(\mathbb{A})\big)$

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	$\bigcup$		$\bigcup$
Eisenstein series	$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho \vert H(\gamma g)\rangle}$	$\longrightarrow$	$\sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho \vert H(\gamma g)\rangle}$

$\lambda \in \mathfrak{h}^\star \otimes \mathbb{C}$

$H \ : G \to \mathfrak{h}$

3. Small representations and BPS-couplings

Minimal automorphic representations

Definition: *An automorphic representation*

$$\pi = \bigotimes_{p \leq \infty} \pi_p$$

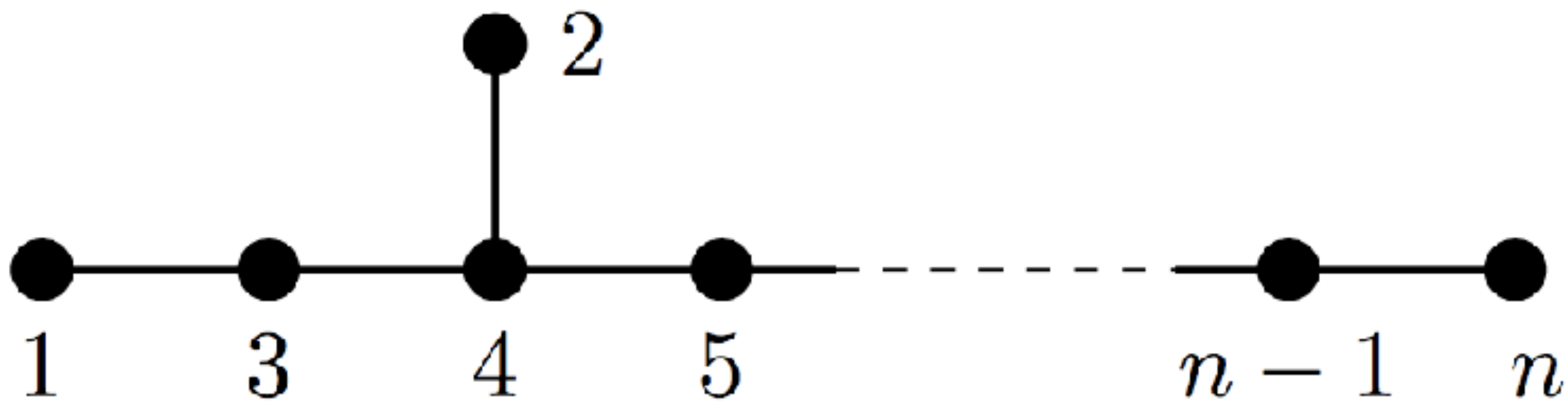
is minimal if each factor π_p *has smallest non-trivial Gelfand-Kirillov dimension.*

[Joseph][Brylinski, Kostant][Ginzburg, Rallis, Soudry][Kazhdan, Savin]. . . .

Automorphic forms $\varphi \in \pi_{min}$ are characterised by having **very few non-vanishing Fourier coefficients**.

[Ginzburg, Rallis, Soudry]

Exceptional groups



Functional dimension of minimal representations:

$$\mathrm{GKdim}\,\pi_{min} = \begin{cases} 11, & E_6 \\ 17, & E_7 \\ 29, & E_8 \end{cases}$$

Automorphic realization

Consider the Borel-Eisenstein series on $G(\mathbb{A})$

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}$$

Now fix the weight to

$$\lambda = 2s\Lambda_1 - \rho$$

where Λ_1 is the fundamental weight associated to node 1.

Theorem [Ginzburg,Rallis,Soudry][Green,Miller,Vanhove]

For $G = E_6, E_7, E_8$ *the Eisenstein series* $E(2s\Lambda - \rho, g)$ *evaluated at* $s = 3/2$ *is attached to the representation* π_{min}

This theorem yields an explicit automorphic realisation of the minimal representation.

Our aim is to use this to calculate Fourier coefficients associated with maximal parabolic subgroups.

BPS-couplings

$g \in E_n(\mathbb{R})$

$$\int d^{11-n}x\sqrt{G}f_0(g)\mathcal{R}^4 \qquad f_0(g) = E(3/2, g) \qquad s = 3/2$$

$$\int d^{11-n}\sqrt{G}f_4(g)\partial^4\mathcal{R}^4 \qquad f_4(g) = E(5/2, g) \qquad s = 5/2$$

These partition functions are Eisenstein series attached to **small automorphic representations** of G.

[Green, Miller, Vanhove][Pioline]

minimal automorphic representation π_{min}	**next-to-minimal automorphic representation** π_{ntm}
1/2 - BPS	1/4 - BPS

Perturbative limit - choices of unipotent subgroups

Decompactification limit

- perturbative parameter: radius of decompactified circle
- non-perturbative effects: KK-instantons, BPS-instantons

String perturbation limit

- perturbative parameter: string coupling
- non-perturbative effects: D-instantons, NS5-instantons

M-theory limit

- perturbative parameter: volume of M-theory torus
- non-perturbative effects: M2- & M5-instantons

general Fourier coefficients

- $P = LU$ **standard parabolic** of G
- **unitary character** $\psi_U : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$
- We then have the U -**Fourier coefficient:**

$$F_{\psi_U}(f_\chi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(f_\chi, ug)\overline{\psi_U(u)}du$$

very little is known in general in this case…

Theorem [Miller-Sahi]: *Let* G *be a split group of type* E_6 *or* E_7

Then any Fourier coefficient of $\varphi \in \pi_{min}$ *of* G *is completely determined by the maximally degenerate Whittaker coefficient*

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

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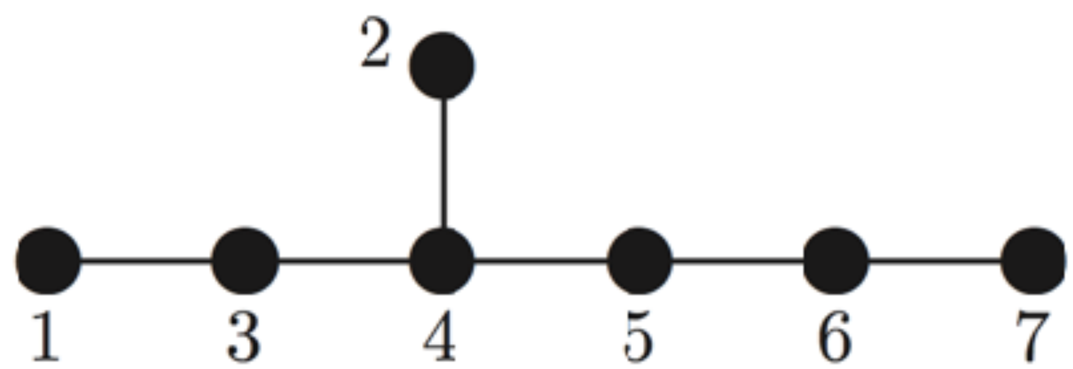
Can one use this to calculate

$$F_{\psi_U}(\varphi, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\varphi, ug)\overline{\psi_U(u)}du$$

in terms of W_{ψ_α} **?**

Example: $G = E_7$



Consider the **3-grading** of the Lie algebra

$$\mathfrak{e}_7 \;=\; \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \;=\; \mathbf{27} \oplus (\mathfrak{e}_6 \oplus \mathbf{1}) \oplus \mathbf{27}$$

The space $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of a maximal parabolic $P = LU$ with 27-dim unipotent U and Levi $L = E_6 \times GL(1)$

The degenerate Whittaker vector associated with α_7 is given by:

[Fleig, Kleinschmidt, D.P.]

$$W_{\psi_k}(3/2, a) = |k|^{3/2}\sigma_{-3}(k)K_{3/2}(2\pi|k|a)$$

where $a \in A \subset E_7$ and

$$\sigma_s(k) = \sum_{d|k} d^s$$

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We now want to relate this to the U - Fourier coefficient

$$F_{\psi_U}(3/2, g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(3/2, ug)\overline{\psi_U(u)}du$$

This captures **instantons in the decompactification limit** of II/T^6!

Claim: [Pioline][Gustafsson, Kleinschmidt, D.P.][Bossard, Verschinin]

$$F_{\psi_U}(3/2; h, r) = |k|^{3/2}\sigma_{-3}(k)K_{3/2}(2\pi r|k| \times ||h^{-1}\vec{x}||)$$

where $h \in E_6$, $r \in GL(1)$ and $\vec{x} \in \mathbb{Z}^{27}$

Claim: [Pioline][Gustafsson, Kleinschmidt, D.P.][Bossard, Verschinin]

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Proof: ***To appear by*** *[Gourevitch, Gustafsson, Kleinschmidt, D.P., Sahi]*

This gives the **complete abelian Fourier expansion** of the minimal representation

Physically the vector $\vec{x}$ corresponds to the **instanton charges** of the 27 vector fields in D=5.

Next-to-minimal representations

Relevant for $\partial^4\mathcal{R}^4$-couplings.

Theorem [Green, Miller, Vanhove]: *Let* $G = E_6, E_7, E_8$
The Eisenstein series

$$E(s,g) = \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle 2s\Lambda_1 | H(\gamma g)\rangle}$$

evaluated at $s = 5/2$ *is a spherical vector in* π_{ntm}.

Conjecture [Gustafsson, Kleinschmidt, D.P.]:

Let G *be a semisimple, simply laced Lie group.*

Then all Fourier coefficients of $\varphi \in \pi_{ntm}$ *are completely determined by degenerate Whittaker vectors of the form*

$$W_{\psi_\alpha}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_\alpha(n)}dn$$

$$W_{\psi_{\alpha,\beta}}(\varphi, g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi_{\alpha,\beta}(n)}dn$$

where (α, β) *are commuting simple roots.*

Proof. For $SL(n)$ to appear by ***[Ahlén, Gustafsson, Kleinschmidt, Liu, D.P.]***
For exceptionals, in progress by ***[Gustafsson, Gourevitch, Kleinschmidt, D.P., Sahi]***

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where (α, β) *are commuting simple roots.*

This will allow us to **extract instanton effects** from $\partial^4\mathcal{R}^4$ couplings!

See also ***[Bossard, Pioline][Bossard, Cosnier-Horeau, Pioline]***

4. Outlook

So what happens at the next order?

1/8-BPS and non-BPS couplings seem to require more general automorphic objects.

[Green, Miller, Vanhove][Pioline][Bossard, Verschinin][Bossard, Kleinschmidt]

This is uncharted mathematical territory.

So what happens at the next order?

1/8-BPS and non-BPS couplings seem to require more general automorphic objects.

[Green, Miller, Vanhove][Pioline][Bossard, Verschinin][Bossard, Kleinschmidt]

This is uncharted mathematical territory.

Below D=3 we also enter the realm of **Kac-Moody groups**!

Connections with **double affine Hecke algebras** (DAHAs)?

Nicolai, Damour, Henneaux, West, Kleinschmidt, Fleig, D.P., Garland, Lee, Patnaik, Braverman, Kazhdan, Miller, Carbone…