Exact results for class S_k

Elli Pomoni

String Math 2017

[1512.06079 Coman,EP,Taki,Yagi] [1703.00736 Mitev,EP] [in progress]

Motivation: N=2 exact results

- *Seiberg-Witten theory: effective theory in the IR*
- *Nekrasov: instanton partition function*
- *Pestun: observables in the UV (path integral on the sphere localizes)*

▷ String/M-/F-theory realizations

- *Gaiotto: 4D N=2* ***class S****: 6D (2,0) on Riemann surface* $C_{g,n}$
- *AGT: 4D partition functions = 2D CFT correlators*
- *4D SC Index = 2D correlation function of a TFT*

2D/4D relations

What can we do for N=1 theories?

- [x] *Superconformal Index*

- [x] *Intriligator and Seiberg: generalized SW technology*

- [ ] *No Localization on S^4.*

- [ ] *An S^4 partition function plagued with scheme ambiguities.*
  [Gerchkovitz, Gomis, Komargodski 2014]

- [x] *Derivatives of the free energy scheme independent.*
  [Bobev, Elvang, Kol, Olson, Pufu 2014]

Class S_k (S_Γ):

[Gaiotto,Razamat 2015]

* *N=1 SuperConformal*
* *Obtained by* ***orbifolding*** *N=2 (inheritance)*
* *Labeled by punctured Riemann Surface*
* *Index = 2D correlation function of a TFT*

Plan

Is there AGT_k?

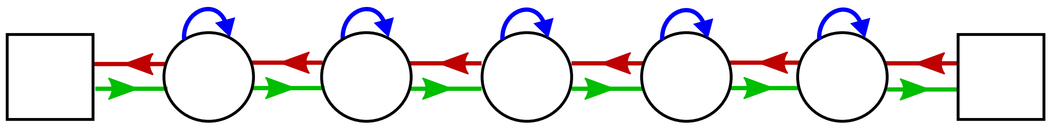
4D partition functions = 2D CFT correlators

- *Spectral curves for N=1 theories in class* S_k
- *From the curves: 2D symmetry algebra and representations*
- *Conformal Blocks* ⟶ *Instanton partition function*
- *Free trinion partition functions on* S^4 ⟶ *3pt functions*

Class S_k

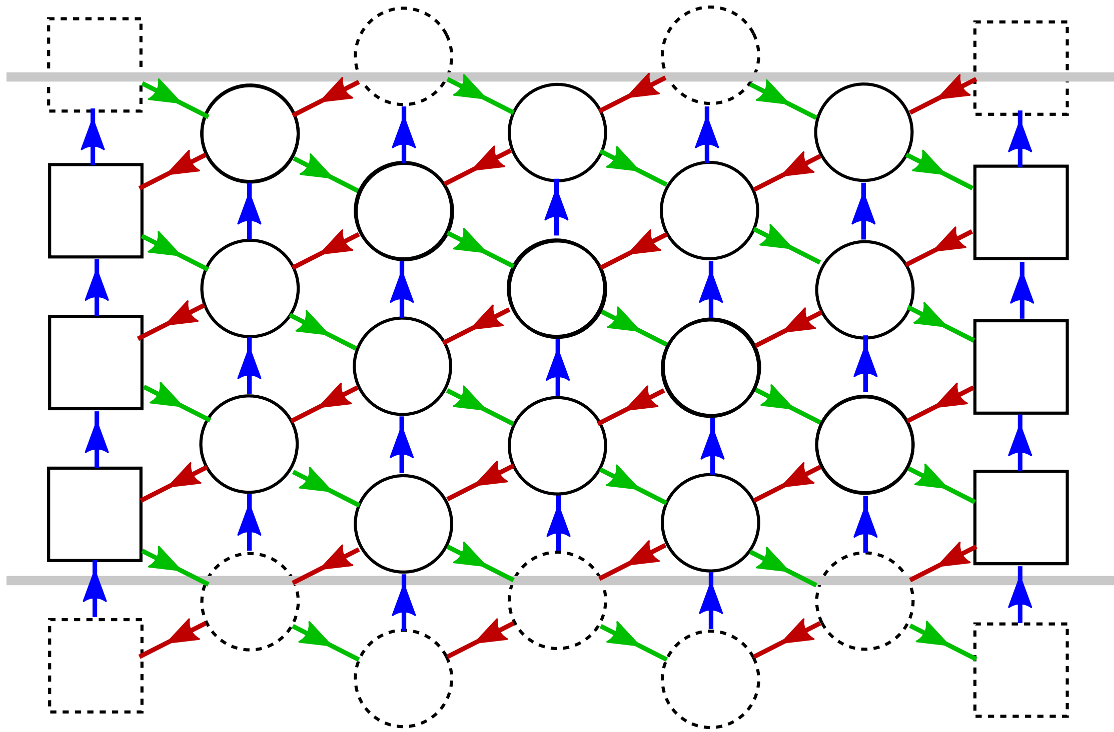
[Gaiotto,Razamat 2015]

Data collection theories (examples) with a Lagrangian description:



N=2 class S mother theory

Begin with N=2 class S with SU(kN) factors and Orbifold: [Douglas,Moore 1996]

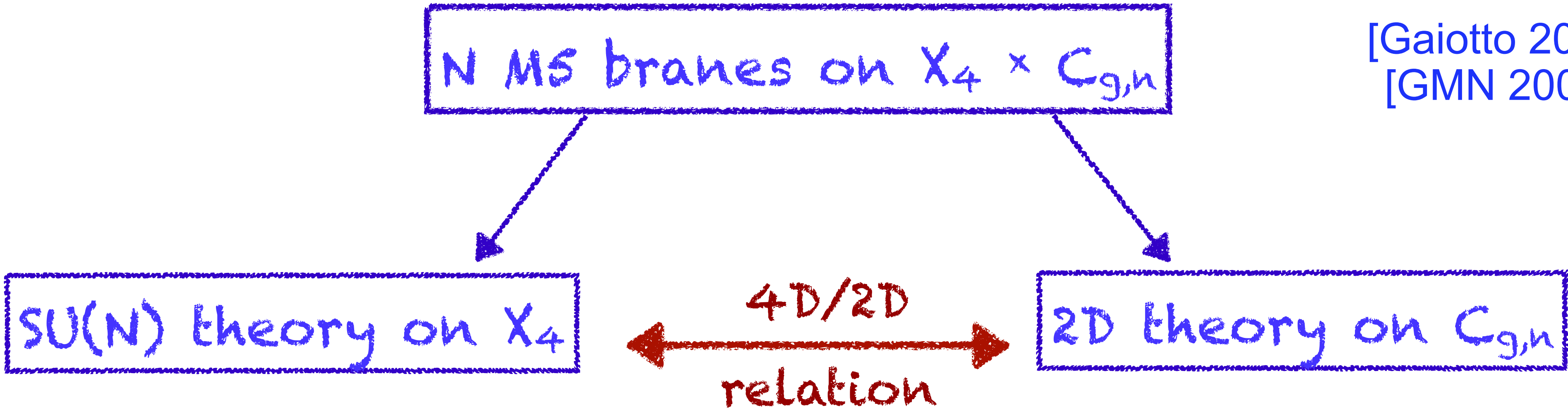


N=1 class S_k orbifold daughter

$$\underset{kN\times kN}{\Phi} = \begin{pmatrix} & \Phi_{(1)} & & & \\ & & \Phi_{(2)} & & \\ & & & \ddots & \\ & & & & \Phi_{(k-1)} \\ \underset{N\times N}{\Phi_{(k)}} & & & & \end{pmatrix}$$

Class S and S_k

[Gaiotto 2009]
[GMN 2009]



6D **(2,0)** SCFT on Riemann surface: 4D N=2 theories of **class S**

Transverse C^2/Z_k Orbifold the 6D (2,0) SCFT to 6D (1,0) SCFT

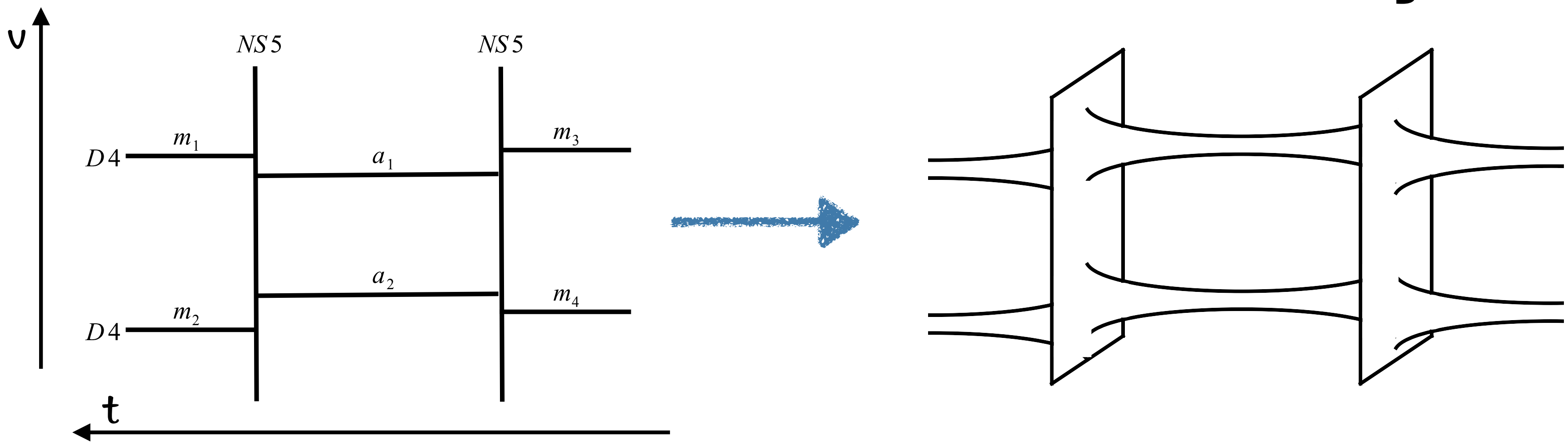
6D **(1,0)** SCFT on Riemann surface: 4D N=1 theories of **class S_k**

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
N M5-branes	—	—	—	—	.	.	—	.	.	.	—
A_{k-1} orbifold	.	.	.	.	*	*	.	*	*	.	.

[Gaiotto,Razamat 2015]

Curves from M-theory

[Witten 1997]



$$t = e^{-\frac{x^6 + i x^{10}}{R_{10}}}$$

$$v = x^4 + i x^5$$

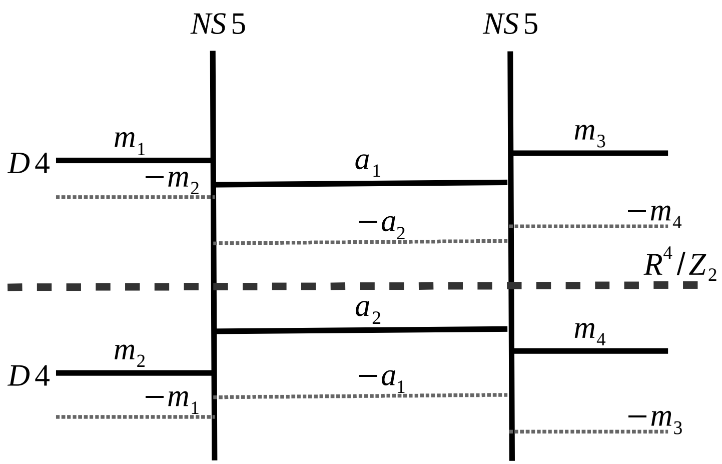
The **Coulomb branch** physics encoded in v,t space.

The NS5/D4 is the classical configuration.

2D surface F(t,v)=0 in the 4D space $\{x^4, x^5, x^6, x^{10}\}=\{v,t\}$.

Take in account tension of the branes: include quantum effects.

single M5 brane with non trivial topology: SW curve



$U(1)_r$ (over x^4, x^5) ~~SU(2)~~$_R$ of N=2 (over x^7, x^8, x^9)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5 branes	—	—	—	—	—	—	.	.	.	.	.
N D4-branes	—	—	—	—	.	.	—	.	.	.	—
A_{k-1} orbifold	.	.	.	.	*	*	.	*	*	.	.

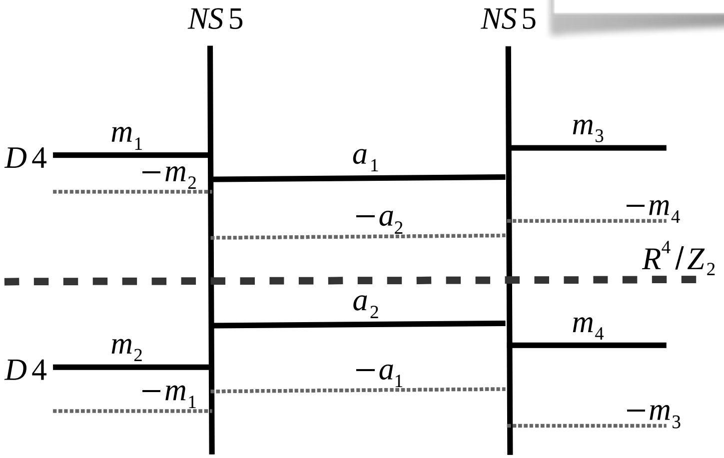
$U(1)_R$ (under x^7, x^8)

$$v \sim e^{\frac{2\pi i}{k}} v$$

S_k curves

[1512.06079 Coman,EP,Taki,Yagi]

$$(v^k - m_1^k)(v^k - m_2^k)t^2 + P(v)t + q(v^k - m_3^k)(v^k - m_4^k) = 0$$



$$P(v) = -(1+q)v^{2k} + u_k v^k + u_{2k}$$

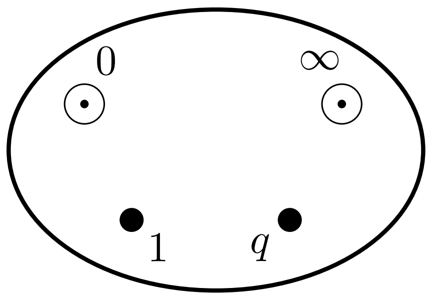
vevs of gauge invariant operators: parameterize the Coulomb branch

$$\langle \mathrm{tr}\,(\Phi_{(1)} \cdots \Phi_{(k)}) \rangle \sim u_k \qquad \langle \mathrm{tr}\,(\Phi_{(1)} \cdots \Phi_{(k)})^2 \rangle \sim u_{2k}$$

SW or IR curve Σ of g=kN-1 → Gaiotto or UV curve $C_{0,n}$ a sphere with n=4 punctures

$$x^{kN} = -\sum_{\ell=1}^{N} \phi_{k\ell}^{(4)}(t)\, x^{k(N-\ell)}$$

$$\phi_{k\ell}^{(4)}(t) = \frac{(-1)^\ell\, \mathfrak{c}_L^{(\ell,k)}\, t^2 + u_{k\ell} t + (-1)^\ell\, \mathfrak{c}_R^{(\ell,k)}\, q}{t^{k\ell}(t-1)(t-q)}$$



$\mathcal{N}=1$ SU(N) SCQCD$_k$

$$\mathfrak{c}^{(s,k)} = \sum_{i_1<\cdots<i_s=1}^{N} m_{i_1}^k \cdots m_{i_s}^k$$

The AGT correspondence

[Alday,Gaiotto,Tachikawa] [Wyllard]

A relation between:

- 4D N=2 theories of class S with SU(2)/SU(N) factors
- 2D Liouville/Toda CFT

$$\mathcal{Z}_{\mathbb{S}^4}\left[\mathcal{T}_{g,n}\right] = \int da\, \mathcal{Z}_{pert}\, |\mathcal{Z}_{inst}|^2 = \int d\alpha\, C \ldots C\, |\mathcal{B}_{\alpha}^{\alpha_i}|^2 = \langle \prod_{i=1}^{n} V_{\alpha_i} \rangle_{\mathcal{C}_{g,n}}$$

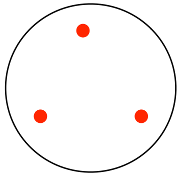
4D gauge theory	2D CFT
instanton partition function	conformal block
perturbative part	3-point function
coupling constants	cross ratios
masses	external momenta
Coulomb moduli	internal momenta
generalized S-duality	crossing symmetry
Omega background	Coupling constant/central charge

$$Q = b + \frac{1}{b}$$

$$b^2 = \frac{\epsilon_1}{\epsilon_2}$$

The AGT relation from the curve

Example:
N=2 SU(2) Free trinion



Close to the punctures:

$$\phi_2^{(3)}(z) \sim \frac{m_j^2}{(z-z_j)^2}$$

Recall 2D Ward Identities:

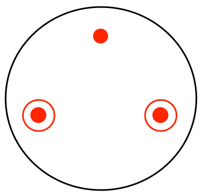
$$\langle T(z)\mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\rangle = \sum_{j=1}^{3}\left[\frac{h_j}{(z-z_j)^2}+\frac{\partial_j}{z-z_j}\right]\langle \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\rangle$$

$$h_i = -m_i^2$$

$$\phi_2^{(3)}(z) = \frac{\langle T(z)\mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\rangle}{\langle \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\rangle}$$

Free trinions the curves are equivalent to Ward identities!

N=2 SU(N)
Free trinion



$$\phi_\ell^{(3)}(z) = \frac{\langle W_\ell(z)\mathsf{V}_\odot(z_1)\mathsf{V}_\bullet(z_2)\mathsf{V}_\odot(z_3)\rangle}{\langle \mathsf{V}_\odot(z_1)\mathsf{V}_\bullet(z_2)\mathsf{V}_\odot(z_3)\rangle}$$

From the curves to the 2D CFT

$$\lim_{\epsilon_{1,2}\to 0} \langle\langle\, J_\ell(t)\,\rangle\rangle_n = \phi_\ell^{(n)}(t)$$

$$\langle\langle\, J(t)\,\rangle\rangle_n \stackrel{\text{def}}{=} \frac{n\text{-point W-block with insertion of } J(t)}{n\text{-point W-block}}$$

- *The symmetry algebra that underlies the 2D CFT = W_{kN} algebra*

- *The reps are **very special** reps of the W_{kN} algebra*

- *Obtain them from the N=2 SU(kN) after replacing:*

$$m^{\mathrm{SU}(Nk)}_{j+Ns} \longmapsto m_j\, \mathrm{e}^{\frac{2\pi i}{k} s} \qquad a^{\mathrm{SU}(Nk)}_{j+Ns} \longmapsto a_j\, \mathrm{e}^{\frac{2\pi i}{k} s}$$

[1703.00736 Mitev,EP]

2D Conformal Blocks = Instanton P.F.

- *We have the reps of the W_{kN} algebra for $\varepsilon_{1,2} = 0$ (from the curve)*

- *Demand: the structure of the multiplet (null states) not change $\varepsilon_{1,2} \neq 0$*

- *The blocks for $\varepsilon_{1,2} \neq 0$: proposal for the instanton partition functions:*

$$\mathcal{Z}_{\text{inst}} = \mathcal{B}_{\mathbf{w}}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 | q)$$

- *If w and c turn on $Q \neq 0$ as in Liouville/Toda,*

  then we obtain them from the N=2 SU(kN) after replacing:

$$m^{\mathrm{SU}(Nk)}_{j+Ns} \longmapsto m_j \, \mathrm{e}^{\frac{2\pi i}{k} s} \qquad a^{\mathrm{SU}(Nk)}_{j+Ns} \longmapsto a_j \, \mathrm{e}^{\frac{2\pi i}{k} s}$$

[1703.00736 Mitev,EP]

Free trinion P.F. = 2D CFT 3pt functions

[in progress Carstensen,EP,Mitev]

$$\mathcal{Z}^{S^4}_{\text{free trinion}} = \left\langle \mathsf{V}_{\odot}(\infty)\mathsf{V}_{\bullet}(1)\mathsf{V}_{\odot}(0) \right\rangle$$

* *For the free trinion theory on S^4 : explicitly do the PI (determinant).*
* *We know the conformal blocks: can write crossing equations.*
* *Is the free trinion P.F. a solution of the crossing equations ??*
* *3pt functions (dynamics) + Blocks = AGT_k*

Summary

Is there AGT_k?

4D partition functions = 2D CFT correlators

- ☑ *We constructed spectral **curves** for N=1 theories in class S_k.*
- ☑ *The curves: 2D symmetry algebra (W_{kN}) and representations.*
- ☑ *Conformal Blocks* ⟶ *Instanton partition function*
- ☐ *Free trinion partition functions on S^4 = **3pt functions***

[in progress Carstensen,EP,Mitev]

Future

* *Compute the instantons with standard Field theory techniques.*

* *Orbifold Nekrasov or use Dp/D(p-4) systems.* [in progress Bourton, EP]

* *Orbifold Pestun, to get the partition function on S^4.*
  [with Carstensen, Hayling, Panerai, Papageorgakis]

* *Go away from the orbifold point (we have the curves and the 2D blocks).*
  [in progress Bourton, EP]

* *Get the AGT_k from (1,0) 6D à la Cordova and Jafferis.*

Thank you!