Vertex Algebras at the Corner

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Based on work with **Davide Gaiotto [1703.00982]** and **Tomáš Procházka [to appear]**.

1. Introduction

Vertex Operator Algebras (VOA) are algebras of chiral operators in 2d CFT with algebra structure given by OPE.

Some **examples** are:

- Stress-energy tensor: T

$$T(z)T(w) \sim \frac{c/2}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

- Kac-Moody algebra: J^a

$$J^a(z)J^b(w) \sim \frac{kg^{ab}}{(z-w)^2} + \sum_c \frac{if_c^{ab} J^c(w)}{z-w}$$

- W_N algebras: $J, T, W_3, W_4, \ldots W_N$

1. Introduction

Main goal:

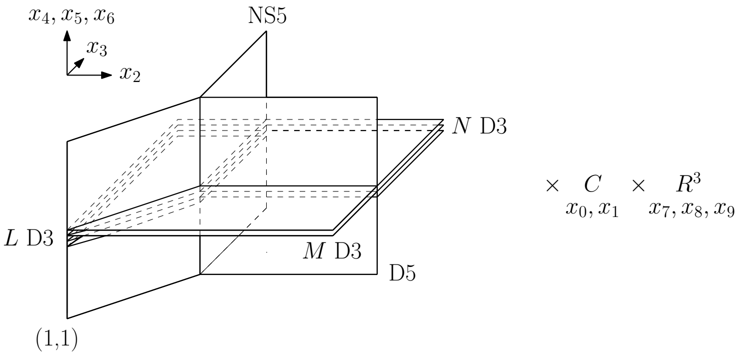
- **Identify VOAs appearing at trivalent junction of interfaces in the Kapustin-Witten twist of $\mathcal{N} = 4$ SYM.**

Applications:

- **Finding new VOA dualities and shedding light on already known ones.**
- **Explicit construction of truncations of $\mathcal{W}_\infty$.**
- **Constructing more complicated VOAs via topological vertex like construction.**

2. Y-algebras

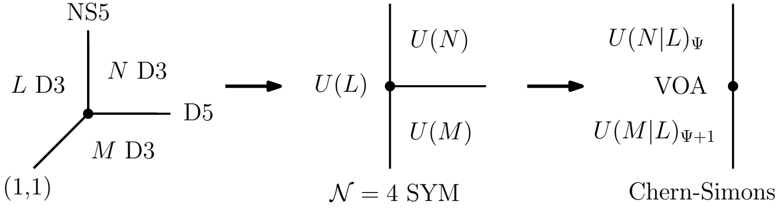
Consider D3-branes attached to the junction of D5, NS5 and (1,1) branes in type IIB string theory.



Low energy theory of N D3 branes can be identified with $\mathcal{N} = 4$ super Yang-Mills theory with $U(N)$ gauge group. From the point of view of the 4d theory, fivebranes play the role of domain-walls between $U(L)$, $U(M)$, and $U(N)$ gauge theories.

2. Y-algebras

In the Kapustin-Witten twist of the theory, local operators living at the corner can be shown to give rise to VOAs:



1. By a variation of arguments of [Witten,10;Mikhaylov,Witten,14], one can argue that path integral localizes to the path integral of $U(N|L)_\Psi$ Chern-Simons (CS) theory and $U(M|L)_{\Psi+1}$ CS theory glued together by a boundary condition that follows from [Gaiotto,Witten,08].
2. Boundaries of CS theory are known to produce various VOAs depending on the choice of boundary condition. [Witten,89;Verlinde,90;...]

2. Y-algebras

Following the outlined construction, one can associate algebra

$$
\begin{aligned}
Y_{L,M,N}[\Psi] &= \frac{W_{N-M}[U(N|L)_\Psi]}{U(M|L)_{\Psi-1}} && N > M \\
Y_{L,N,N}[\Psi] &= \frac{U(N|L)_\Psi \times \mathrm{Sb}^{U(N|L)}}{U(N|L)_{\Psi-1}} && N = M
\end{aligned}
$$

to our configuration of interfaces for each numbers L, M, N of D3-branes. In particular, we need to (roughly):

1. **Start with algebra:**

   $$U(N|L)_\Psi \times U(M|L)_{1-\Psi} \times \text{ghosts}.$$

2. **Perform Drinfeld-Sokolov reduction (W[G]):** Fix upper-triangular part of $(N-M)\times(N-M)$ diagonal block inside $U(N|L)_\Psi$ by looking at cohomology of BRST operator that fixes these components.
3. **Perform Coset (G/H):** Glue $U(M|L)_{1-\Psi}$ with the other diagonal block of $U(N|L)_\Psi$ again by BRST procedure.

2. Y-algebras

E.g. algebra associated to $N = 5, M = 2, L = 2$ is given by cohomology implementing:

$$J = \left(\begin{array}{ccc|cc|cc} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & * & * & * & * \\ * & * & * & * & * & * & * \\ \hline * & * & 0 & & & & \\ * & * & 0 & & & & \\ \hline * & * & 0 & & & & \\ * & * & 0 & & & & \end{array}\right) \begin{array}{l} \\ N-M \\ \\ \\ M \\ \\ L \\ \end{array} \qquad \tilde{J} = \left(\begin{array}{c|c} & \\ \hline & \end{array}\right) \begin{array}{l} M \\ L \end{array}$$

2. Y-algebras: Virasoro example $Y_{0,0,2}[\Psi]$

Consider $U(2)_\Psi = SU(2)_{\Psi-2} \times U(1)_{2\Psi}$ Kac-Moody algebra generated by fields J, J^0, J^+, J^- with OPEs:

$$J(z)J(w) \sim \frac{2\Psi}{(z-w)^2}, \qquad J^0(z)J^0(w) \sim \frac{2(\Psi-2)}{(z-w)^2},$$
$$J^0(z)J^\pm(w) \sim \frac{\pm 2J^\pm(w)}{z-w}, \qquad J^+(z)J^-(w) \sim \frac{\Psi-2}{(z-w)^2} + \frac{J^0(w)}{z-w}.$$

Fixing upper-triangular part of $U(2)_\Psi$ as $J^+ = 1$ is implemented by

$$Q = \oint dz (J^+ - 1)c$$

where we had to introduce b, c ghost system with OPE

$$b(z)c(w) \sim \frac{1}{z-w}.$$

2. Y-algebras: Virasoro example $Y_{0,0,2}[\Psi]$

Calculation shows that the cohomology is generated by J and

$$T = \frac{1}{\Psi}\left(J^{-} + \frac{1}{4} : J^0 J^0 : + \frac{1}{2} : bc :\right) + \left(1 - \frac{1}{\Psi}\right)\left(\frac{1}{2}\partial J^0 + \partial : bc :\right).$$

The resulting VOA turns out to be $U(1)\times$**Virasoro** algebra of central charge

$$c = 13 - \frac{6}{\Psi} - 6\Psi.$$

Comments:

- Note that this is invariant under Feigin-Frenkel duality:

$$\Psi \leftrightarrow \frac{1}{\Psi}.$$

2. Y-algebras: Virasoro example $Y_{0,0,2}[\Psi]$

- There also exists construction in terms of cohomology of

$$U(2)_{\frac{1}{\Psi-1}} \times U(2)_{\frac{1}{\Psi-1}-1} \times Ff^{U(2)}$$

with BRST charge

$$Q = \oint dz \left[(J^a_{1b} - J^a_{2b} - \psi^a \chi_b) c^b_a + \frac{1}{2} : b[c,c] : \right]$$

where J_1 and J_2 label currents of the two factors and $Ff^{U(2)}$ is a pair of free fermions $\psi^1, \chi_1, \psi^2, \chi_2$.

- The only difference for the simpler G/H case would be absence of the free fermions and J_1 cbeing the components in corresponding diagonal block.

3. S-duality

We have show an existence of **triality of constructions**:

$$W_2\left[U(2)_\Psi\right] \longleftrightarrow W_2\left[U(2)_{\frac{1}{\Psi}}\right]$$

$$\nwarrow\!\!\searrow \qquad \frac{U(2)_{\frac{1}{\Psi-1}} \times Ff^{U(2)}}{U(2)_{\frac{1}{\Psi-1}+1}} \qquad \swarrow\!\!\nearrow$$

The triality has generalization for $U(N)$ Kac-Moody algebras giving rise to W_N algebras:

$$W_N\left[U(N)_\Psi\right] \longleftrightarrow W_N\left[U(N)_{\frac{1}{\Psi}}\right]$$

$$\nwarrow\!\!\searrow \qquad \frac{U(N)_{\frac{1}{\Psi-1}} \times Ff^{U(N)}}{U(N)_{\frac{1}{\Psi-1}+1}} \qquad \swarrow\!\!\nearrow$$

3. S-duality

S-duality of type IIB gives rise to $SL(2,Z)$ action on (p,q) interfaces and parameter Ψ generated by

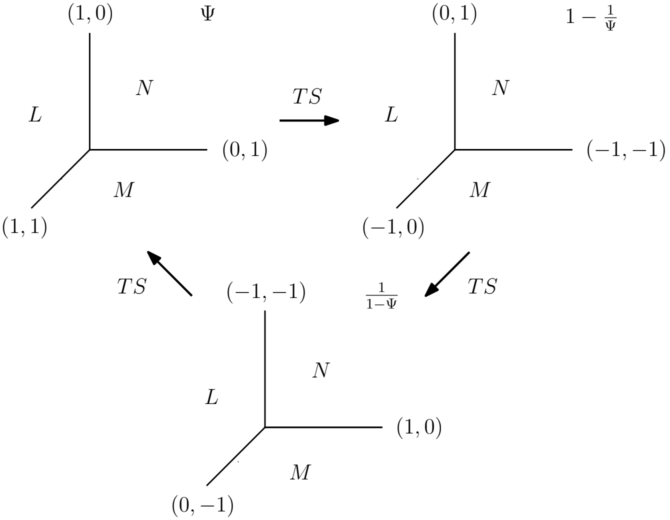
$$
\begin{aligned}
T: &\quad \Psi \to \Psi + 1, \quad (p,q) \to (p, q+p) \\
S: &\quad \Psi \to -\frac{1}{\Psi}, \quad (p,q) \to (-q, p)
\end{aligned}
$$

S_3 **subgroup of the S-duality group** preserves our trivalent junction and acts as a permutation of the branes accompanied by a transformation of Ψ. This induces S_3 **duality action on Y-algebras**:

$$
\boxed{Y_{L,M,N}[\Psi] = Y_{N,L,M}\left[\frac{1}{1-\Psi}\right] = Y_{M,N,L}\left[1-\frac{1}{\Psi}\right]}
$$

3. S-duality

Diagrammatically, the triality action is given by transformations:



3. S-duality: Examples

- Three realizations of W_N ($Y_{0,0,N}[\Psi]$):

$$W_N[U(N)_\Psi] \quad \leftrightarrow \quad W_N[U(N)_{\frac{1}{\Psi}}] \quad \leftrightarrow \quad \frac{U(N)_{\frac{1}{\Psi-1}} \times Ff^{U(N)}}{U(N)_{\frac{1}{\Psi-1}-1}}$$

- Three realizations of parafermions ($Y_{0,1,2}[\Psi]$):

$$\frac{U(2)_\Psi}{U(1)_{\Psi-1}} \quad \leftrightarrow \quad \frac{W_2[U(2|1)_{\frac{1}{\Psi}}]}{U(1)_{1-\frac{1}{\Psi}}} \quad \leftrightarrow \quad \frac{U(2|1)_{\frac{1}{\Psi-1}}}{U(2)_{\frac{1}{\Psi-1}-1}}$$

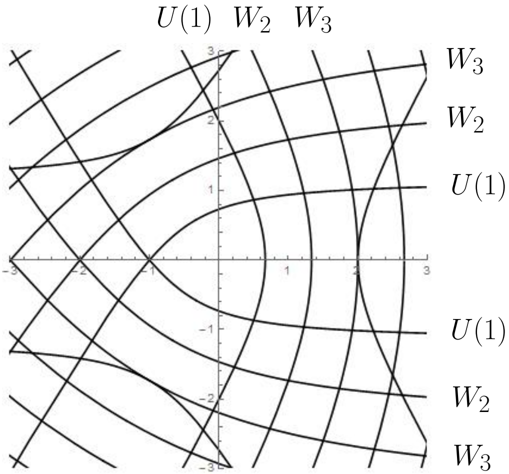
- And many new dualities, such as

$$\frac{W_2[U(3)_\Psi]}{U(1)_{\Psi-1}} \quad \leftrightarrow \quad \frac{U(1|3)_{\frac{1}{1-\Psi}}}{U(3)_{\frac{1}{1-\Psi}-1}} \quad \leftrightarrow \quad \frac{W_3[U(1|3)_{\frac{1}{\Psi}}]}{U(1)_{\frac{1}{\Psi}-1}}$$

Central charges and vacuum characters match for all values of L, M, N!!!

4. Truncations of $\mathcal{W}_\infty$ = Y-algebras

Bootstraping VOA with field content $J, T, W_3, W_4, \ldots$ leads to **two parameter family of VOAs** called $\mathcal{W}_\infty$. At lines in the parameter space **truncations** $\mathcal{W}_\infty/\mathcal{I}$ by ideal $\mathcal{I}$ are possible.



4. Truncations of $\mathcal{W}_\infty$ = Y-algebras

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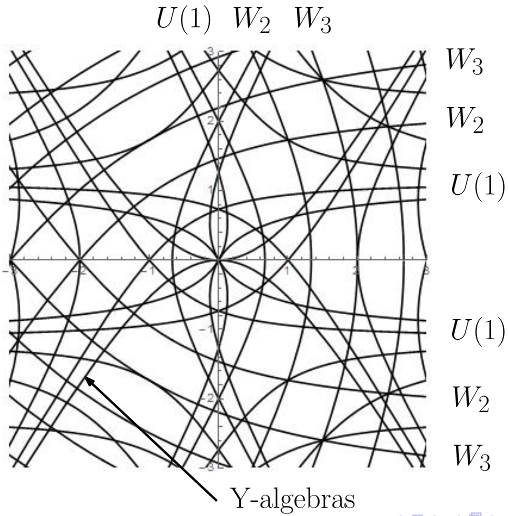
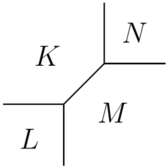


5. Gluing Y-algebras

Y-algebras associated to trivalent junctions can be thought of as atomic elements for VOAs associated to more complicated webs of defects. Consider resolved conifold diagram as an example.



Local operators at the two corners give rise to a pair of Y-algebras. One can extend this VOA by bi-modules associated to line defects stretched between the two corners. The resulting VOA is a conformal extensions

$$Y_{K,M,N}[\Psi] \boxtimes Y_{M,K,L}[\Psi].$$

5. Gluing Y-algebras

Even the simplest diagram of resolved conifold describe large family of VOAs with $Z_2 \times Z_2$ duality action that are truncations of a variation of $\mathcal{W}_\infty$ algebra with vacuum character given by

$$\chi = \prod_{n=1}^{\infty} \frac{(1+q^{n+\rho})^{2n}}{(1-q^n)^{2n}}$$

for ρ (half-) integral. Note two special values:

- $\rho = 0 \quad \rightarrow \quad \mathcal{W}_\infty^{1|1}$,
- $\rho = \frac{1}{2} \quad \rightarrow \quad \mathcal{W}_\infty^{\mathcal{N}=2}$.

Considering also more complicated diagrams, the construction seems to contain many/most/all of the known VOAs.

6. Summary

Brane realization of VOAs leads to:

- Explanation of triality for W_N as a consequence of S-duality.
- Generalization of the triality to $Y_{L,M,N}$.
- Construction of truncations of $\mathcal{W}_\infty$ algebras.
- Y-algebras serve as building blocks for more complicated algebras via topological vertex like construction.

Further developments:

- Further exploration of more complicated webs of defects.
- O3-planes and ortho-symplectic counterpart of our story.
- Exploration of relations to other problems of physics and mathematics such as AGT correspondence, counting of D0-D2-D4 bound stated, spiked instantons, geometric Langlands program, $AdS_3/CFT_2 \ldots$