# New perspective on Sine-Gordon model and perturbative QFT

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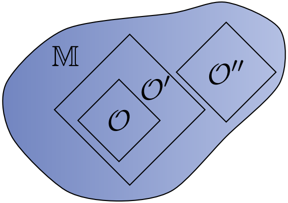
[1] Joint work with Dorothea Bahns.

# Outline of the talk

# Algebraic quantum field theory

- A convenient framework to investigate conceptual problems in QFT is the Algebraic Quantum Field Theory.



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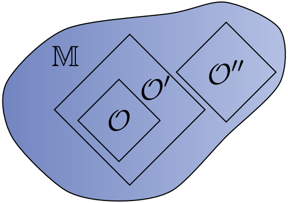


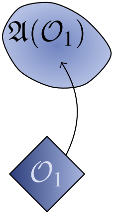

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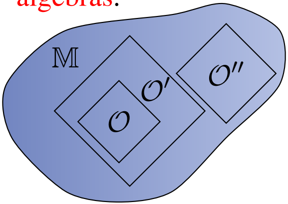
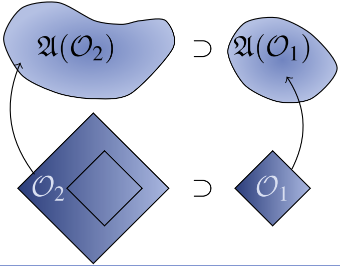
- Haag-Kastler idea: a model is defined by associating to each bounded contractible region $\mathcal{O}$ of Minkowski spacetime $\mathbb{M}$ the algebra $\mathfrak{A}(\mathcal{O})$ of observables that can be measured in $\mathcal{O}$.
- The physical notion of subsystems is realized by the condition of isotony, i.e.: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. We obtain a net of algebras.



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where $[.,.]$ is the commutator in the sense of $\mathfrak{A}(\mathcal{O}_3)$, where $\mathcal{O}_3$ contains both $\mathcal{O}_1$ and $\mathcal{O}_2$.

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- **Time-slice axiom:** If $\mathcal{N}$ is a neighborhood of a Cauchy-surface in $\mathcal{O}$, then $\mathfrak{A}(\mathcal{N})$ is isomorphic to $\mathfrak{A}(\mathcal{O})$.
- This is a QFT version of the initial value problem (or local constancy in the time direction).

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- For a review see the book: *Perturbative algebraic quantum field theory. An introduction for mathematicians*, KR, Springer 2016.

# Recent results

- Treatment of gauge theories using the BV formalism
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- Construction of the local net of observables in Sine-Gordon model: D. Bahns, KR, **[arXiv:math-ph/1609.08530]**.

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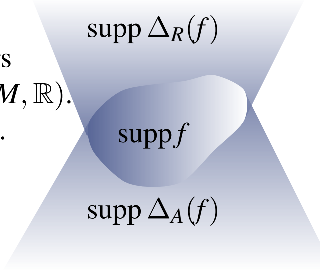
- $P$ has retarded and advanced Green operators $\Delta_R,\ \Delta_A : \mathcal{D}(M) \to \mathcal{E}(M)$, where $\mathcal{D} \equiv \mathcal{C}^\infty(M, \mathbb{R})$. They satisfy: $P \circ \Delta_{R/A} = \mathrm{id}$, $\Delta_{R/A} \circ P = \mathrm{id}$.



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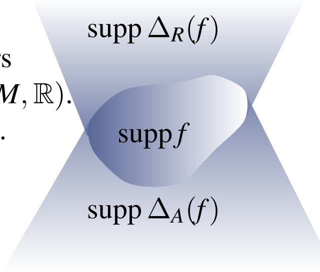
- Their difference is the causal propagator
$$\Delta \doteq \Delta_R - \Delta_A .$$



# Poisson structure and the $\star$-product

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$$(F \star G)(\varphi) \doteq e^{\hbar \left\langle W, \frac{\delta^2}{\delta\varphi_1 \delta\varphi_2} \right\rangle} F(\varphi_1) G(\varphi_2) \Big|_{\varphi_1 = \varphi_2 = \varphi} . ,$$

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- The free QFT is defined as $\mathfrak{A}_0(\mathcal{O}) \doteq (\mathcal{F}(\mathcal{O})[[\hbar]], \star, *)$, where $F^*(\varphi) \doteq \overline{F(\varphi)}$ and $\mathcal{F}(M)$ is an appropriate functional space.

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\text{reg}}(M)[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

# Interaction

- We have $(\mathcal{F}_{\text{reg}}(M)[[\hbar]], \star, \cdot_{\mathcal{T}})$, where $\star$ is non-commutative, $\cdot_{\mathcal{T}}$ is commutative and they satisfy the relation

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- Interaction is a functional $V \in \mathcal{F}_{\mathrm{reg}}(M))$. Using the commutative product $\cdot_{\mathcal{T}}$ we define the S-matrix:

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- Renormalization problem: extend $\cdot_{\mathcal{T}}$ to local non-linear functionals.

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- The 2-point function of the free massless scalar field in 2D coincides with the Hadamard parametrix

$$W(x) = \frac{i}{2}(\Delta_R(x) - \Delta_A(x)) + H(x) = -\frac{1}{4\pi}\ln\left(\frac{-x \cdot x + i\varepsilon t}{\mu^2}\right)$$

where $\mu > 0$ is the scale parameter that we need to fix.

# Vertex operators

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- In particular, for $\boldsymbol{x} = \boldsymbol{y}$ and $t > t'$,

$$V_a(t,\boldsymbol{x}) \star V_{a'}(t',\boldsymbol{x}) = e^{aa'i\hbar/2} V_{a'}(t',\boldsymbol{x}) \star V_a(t,\boldsymbol{x})\,,$$

which is the well-known relation for vertex operators.

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- Define $\star_v$ as the star product induced by $W_v$, so $\star_0$ denotes the star product induced by the parametrix $W$. We have

$$F \star_v G = \alpha_{H_v}(\alpha_{H_v}^{-1} F \star \alpha_{H_v}^{-1} G),$$

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  where $\alpha_{H_v} \doteq e^{\frac{\hbar}{2}\left\langle H_v, \frac{\delta^2}{\delta\varphi^2}\right\rangle}$.
- Hence $\star$, and $\star_v$ are equivalent products, and $\alpha_{H_v}$, is a "gauge transformation".

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$$\omega_\nu(:F:_\nu \star :G:_\nu) \doteq \alpha_\nu(:F:_\nu \star :G:_\nu)(0) = (F \star_\nu G)(0).$$

- Similar for the $S$-matrix:

$$\omega_\nu(\mathcal{S}(\lambda :V:_\nu)) \doteq \alpha_\nu \left( e_{\mathcal{T}}^{i\lambda :V:_\nu /\hbar} \right)(0) = e_{\mathcal{T}_\nu}^{i\lambda V/\hbar}(0).$$

Here $\cdot_{\mathcal{T}_\nu}$ is the time-ordered product corresponding to $\star_\nu$.

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**Theorem (Bahns, KR 2016)**

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- No issues with positivity/IR problems, no Wick rotation.
- The abstract formal $S$-matrix is constructed before a state is chosen.
- Local observables are constructed using the Bogoliubov formula and one obtains an interacting local net $\mathfrak{A}(\mathcal{O})$.

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- Apply the same methods to a larger class of integrable models and other models in 2D.
- Show equivalence with the $\mathcal{O}(3)$ model and the Thirring model.

Thank you very much for your attention!