

Killing superalgebras and high supersymmetry

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Based on joint works with J. Figueroa-O'Farrill

Eleven-dimensional supergravity

Let (M, g, F) be Lorentzian mnfd (M, g) , $\dim M = 11$, with closed $F \in \Omega^4(M)$ and endowed with spin bundle $S(M) \rightarrow M$ (the fiber $S(M)_x \simeq S = \mathbb{R}^{32}$). The bosonic **eqs of supergravity** are two coupled PDE [Cremmer-Julia-Scherk '78]:

$$\left. \begin{aligned} \text{Ric}(X, Y) &= -\frac{1}{2}g(i_X F, i_Y F) + \frac{1}{6}g(X, Y)\|F\|^2 \\ d * F &= \frac{1}{2}F \wedge F \end{aligned} \right\} (*)$$

Supersym. transf. $\delta_\epsilon \Psi = D\epsilon + O(\Psi)$ of gravitino Ψ gives **superconnection** on $S(M)$:

$$D_X \epsilon = \nabla_X \epsilon - \frac{1}{24}[X \cdot F - 3F \cdot X] \cdot \epsilon$$

where $X \in \mathfrak{X}(M)$ and $\epsilon \in \Gamma(S(M))$.

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Def. A **symmetry** of a solution of $(*)$ is pair (ξ, ε) given by

- (i) a Killing vector field for (g, F) , i.e., a v.f. ξ s.t. $\mathcal{L}_\xi g = \mathcal{L}_\xi F = 0$;
- (ii) a Killing spinor, i.e., a section ε of $S(M)$ s.t. $D\varepsilon = 0$.

Killing superalgebras

Thm[Figueroa-O'Farrill, Meessen, Philip '05] The v.s. $\mathfrak{k} = \mathfrak{k}_{\bar{0}} \oplus \mathfrak{k}_{\bar{1}}$ of symmetries of (M, g, F) has natural structure of Lie superalgebra, called **Killing superalgebra**.

The Flat Model. (M, g) Minkowski, $F = 0$. In this case $D = \nabla$, $\mathfrak{k}_{\bar{1}} \simeq S$, $\mathfrak{k}_{\bar{0}} \simeq \mathfrak{so}(V) \oplus V$ and \mathfrak{k} is the Poincaré superalgebra $\mathfrak{p} = (\mathfrak{so}(V) \oplus V) \oplus S$.

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The Killing superalgebra is useful invariant of a supergravity bkgd:

- late '90s: first general check of AdS/CFT correspondence;
- early 2000s: it contracts under Penrose limit;
- mid 2000s: **homogeneity conjecture** by Meessen, i.e., if

$$\dim(\mathfrak{k}_1) > \frac{1}{2} \dim S = 16$$

then bkgd is locally homogeneous;

- it is useful to constructing bkgds with prescribed automorphism group.

Supergravity solutions

- Local expressions for metric and 4-form of **low supersymmetric bkgds** have been derived solving the Killing spinor eqs: the G -structure method [Gauntlett, Gutowski, Pakis '03] and the spinorial geometry method [Gillard, Gran, Papadopoulos '05].
- There are (M, g) with parallel spinors that are not Ricci flat [Bryant '00].
- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].

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- Other approaches like exceptional generalized geometry that apply for special compactifications [Coimbra, Strickland-Constable, Waldram '14].
- Classification of **highly supersymmetric bkgds** is largely open. Maximally supersymmetric bkgds are classified [Figuroa-O'Farrill, Papadopoulos '03] and there are non-existence results for 31 and 30 Killing spinors [Gran, Gutowski, Papadopoulos, Roest '07 & '10].
- There is one bkgd with 26 Killing spinors [Michelson '02] and also bkgds with 24, 22, 20, 18 [Gauntlett, Hull '02].

Structural results for highly supersymmetric solutions

The homogeneity thm[Figuroa-O'Farrill, Hustler '12] If $\dim(\mathfrak{k}_{\bar{1}}) > 16$ then bkgd is **locally homogeneous**.

On S there is $\mathfrak{so}(V)$ -invariant symplectic form $\langle -, - \rangle$ and transpose of Clifford multiplication $V \otimes S \rightarrow S$ gives a way to square spinors: the **Dirac current**

$$k : \odot^2 S \rightarrow V ,$$
$$\eta(k(s, s), v) = \langle s, v \cdot s \rangle \quad v \in V, s \in S .$$

It turns out that $k|_{\odot^2 S'} : \odot^2 S' \rightarrow V$ is surjective for all subspaces $S' \subset S$ with $\dim S' > 16$ so v.f. $\xi \in \mathfrak{k}_{\bar{0}}$ already span $T_x M$ at all $x \in M$.

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Rem. Supergravity eqs for homogeneous bkgds are algebraic and simpler than PDEs. However checking **supersymmetry is additional problem** — there exist many homog. bkgds which are not (highly) supersymmetric.

Killing superalgebras are filtered deformations

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The nonzero Lie brackets of $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}} = (\mathfrak{so}(V) \oplus V) \oplus S$ are

$$[A, B] = AB - BA, \quad [A, s] = As, \quad [A, v] = Av, \quad [s, s] = k(s, s),$$

for all $A, B \in \mathfrak{so}(V)$, $s \in S$, $v \in V$. There exists a compatible **\mathbb{Z} -grading**

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0$$

where $\mathfrak{p}_{-2} = V$, $\mathfrak{p}_{-1} = S$ and $\mathfrak{p}_0 = \mathfrak{so}(V)$. Compatibility means

- (i) $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$ for all $i, j \in \mathbb{Z}$;
- (ii) $\mathfrak{p}_{\bar{0}} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_0$ and $\mathfrak{p}_{\bar{1}} = \mathfrak{p}_{-1}$.

Killing superalgebras are filtered deformations

We shall be interested in graded subalgebras $\mathfrak{a} \subset \mathfrak{p}$, i.e.

$$\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 = V' \oplus S' \oplus \mathfrak{h},$$

where $V' \subset V$, $S' \subset S$ and $\mathfrak{h} \subset \mathfrak{so}(V)$. If $\dim S' > 16$ then $V' = V$ (this is the algebraic fact underlying homogeneity thm). The Lie brackets of \mathfrak{a} are:

$$[A, B] = AB - BA$$

$$[A, v] = Av$$

$$[A, s] = As$$

$$[s, s] = \kappa(s, s)$$

$$[v, s] = 0$$

$$[v, w] = 0$$

$A, B \in \mathfrak{h}$, $s \in S'$, $v, w \in V'$.

Killing superalgebras are filtered deformations

There is natural filtration \mathfrak{a}^\bullet on \mathfrak{a} , i.e.

$$\mathfrak{a} = \mathfrak{a}^{-2} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^{-1} = \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \supset \mathfrak{a}^0 = \mathfrak{a}_0 \supset \mathfrak{a}^1 = 0 .$$

Def. A **filtered deformation** of \mathfrak{a} is a Lie superalgebra \mathfrak{g} with same underlying vector space as \mathfrak{a} and a new Lie bracket $[-, -]$ which satisfies:

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- (i) $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j} \oplus \mathfrak{a}_{i+j+1} \oplus \cdots$,
- (ii) components of $[-, -]$ of zero degree coincide with original bracket.

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Def. A **filtered deformation** of \mathfrak{a} is a Lie superalgebra \mathfrak{g} with same underlying vector space as \mathfrak{a} and a new Lie bracket $[-, -]$ which satisfies:

$$[A, B] = AB - BA$$

$$[A, v] = Av + t\delta(A, v)$$

$$[A, s] = As$$

$$[s, s] = \kappa(s, s) + t\gamma(s, s)$$

$$[v, s] = t\beta(v, s)$$

$$[v, w] = t\alpha(v, w) + t^2\rho(v, w)$$

for some maps $\delta : \mathfrak{h} \otimes V' \rightarrow \mathfrak{h}$, $\gamma : \odot^2 S' \rightarrow \mathfrak{h}$, $\beta : V' \otimes S' \rightarrow S'$, $\alpha : \Lambda^2 V' \rightarrow V'$ and $\rho : \Lambda^2 V' \rightarrow \mathfrak{h}$ subject to the Jacobi identities for all values of parameter t .

Main Motivations and Questions

Motivations.

- **Idea**: instead of studying directly bkgds, we set to study filt. def.
- The problem of classifying filt. def. of \mathbb{Z} -graded Lie (super)algebras is well-defined mathematically [Sternberg, Guillemin '70s] and subject of recent investigations [Kac, Cantarini, Cheng '00s] and [Kruglikov, The '14].
- According to Klein's Erlangen program, any geometry should be described by its transformation group. We shall set up a "supergravity Erlangen program", systematising search for supergravity bkgds.

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Questions.

- Is every filt. def. really a Killing superalgebra?
- Any 11-dimensional (M, g) with **closed** $F \in \Omega^4(M)$ has associated Killing superalgebra. Should filt. def. be further constrained by eqs of supergravity?

Spencer cohomology

Filtered deformations are governed by **Spencer cohomology**, a bi-graded refinement of the usual Chevalley-Eilenberg cohomology of a Lie (super)algebra and its adjoint representation to the case of \mathbb{Z} -graded Lie (super)algebras. The **space of q -cochains** for *Poincaré superalgebra* is $C^q(\mathfrak{p}_-, \mathfrak{p}) = \mathfrak{p} \otimes \Lambda^q \mathfrak{p}_-^*$, where $\mathfrak{p}_- = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} = V \oplus S$. It decomposes $C^q(\mathfrak{p}_-, \mathfrak{p}) = \bigoplus C^{p,q}(\mathfrak{p}_-, \mathfrak{p})$ in components of different $\text{deg} = p$.

	q				
p	0	1	2	3	4
0	$\mathfrak{so}(V)$	$S \rightarrow S$ $V \rightarrow V$	$\odot^2 S \rightarrow V$		
2		$V \rightarrow \mathfrak{so}(V)$	$\Lambda^2 V \rightarrow V$ $V \otimes S \rightarrow S$ $\odot^2 S \rightarrow \mathfrak{so}(V)$	$\odot^3 S \rightarrow S$ $\odot^2 S \otimes V \rightarrow V$	$\odot^4 S \rightarrow V$
4			$\Lambda^2 V \rightarrow \mathfrak{so}(V)$	$\odot^2 S \otimes V \rightarrow \mathfrak{so}(V)$ $\Lambda^2 V \otimes S \rightarrow S$ $\Lambda^3 V \rightarrow V$	$\odot^4 S \rightarrow \mathfrak{so}(V)$ $\odot^3 S \otimes V \rightarrow S$

Spencer cohomology and Killing spinors

Thm[Figueroa-O'Farrill, A.S.] $H^{4,2}(\mathfrak{p}_-, \mathfrak{p}) = 0$, $H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^4 V$.

Spencer cohomology recovers the 4-form of supergravity! But there is more: the β -component (remember $\beta : V \otimes S \rightarrow S$) of the Spencer cocycle is exactly

$$\beta(v, s) = v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s,$$

where $\varphi \in \Lambda^4 V$. In other words, it indicates what relevant Killing spinor eqs are.

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where $\varphi \in \Lambda^4 V$. In other words, it indicates what relevant Killing spinor eqs are.

Rem. Filtr. def. are useful also in relation with the problem of determining the geometries admitting rigid supersymmetric field theories. We obtained:

$$\underline{d=4} \quad H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^4 V;$$

$$\underline{d=6} \quad H^{2,2}(\mathfrak{p}_-, \mathfrak{p}) \simeq \Lambda_+^3 V \oplus (V \otimes \odot^2 \mathbb{C}^2).$$

Joint works with P. de Medeiros and J. Figueroa-O'Farrill.

Maximal supersymmetry

We classified maximally supersymmetric filt. def., i.e., filt. def. \mathfrak{g} of subalgebras $\mathfrak{a} = \mathfrak{a}_{-2} \oplus \mathfrak{a}_{-1} \oplus \mathfrak{a}_0$ of \mathfrak{p} with $\mathfrak{a}_{-2} = V$, $\mathfrak{a}_{-1} = S$ and $\mathfrak{a}_0 = \mathfrak{h}$ subalgebra of $\mathfrak{so}(V)$. The fact that $S' = S$ means we have maximal supersymmetry, whereas $V' = V$ (which is forced) means we are describing (locally) homogeneous geometries. We bootstrapped the computation of $H^{2,2}(\mathfrak{a}_-, \mathfrak{a})$ from $H^{2,2}(\mathfrak{p}_-, \mathfrak{p})$ and obtained:

Thm[Figueroa-O'Farrill, A.S.] The maximally supersymmetric filt. def. are exactly the Killing superalgebras of **maximally supersymmetric bkgds** and nothing else:

- (i) \mathfrak{p} itself for Minkowski spacetime;
- (ii) $\mathfrak{osp}(8|4)$ for $AdS_4 \times S^7$ [Freund, Rubin '80];
- (iii) $\mathfrak{osp}(6, 2|4)$ for $S^4 \times AdS_7$ [Pilch, van Nieuwenhuizen, Townsend '84];
- (iv) the Killing superalgebra of max. susy pp-wave [Kowalski-Glikman '84].

In all cases $\mathfrak{h} = \mathfrak{so}(V) \cap \text{stab}(\varphi)$ where $\varphi \in \Lambda^4 V$.

High supersymmetry

Thm[Figueroa-O'Farrill, A.S.] Let (M, g) be 11-dimensional Lorentzian mnfd with **closed** $F \in \Omega^4(M)$. If space $\mathfrak{k}_{\bar{1}}$ of Killing spinors has **$\dim \mathfrak{k}_{\bar{1}} > 16$** , then (M, g, F) satisfies the Einstein and Maxwell eqs of supergravity. This result is sharp.

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Sketch of the proof. The Jacobi identity in $[S' S' V]$ gives

$$\frac{1}{2}R(v, \kappa(s, s))w = \kappa((X_v \beta)(w, s), s) - \kappa(\beta_v(s), \beta_w(s)) - \kappa(\beta_w \beta_v(s), s) \quad (1)$$

for all $s \in S'$ and $v, w \in V$, for some map $X : V \rightarrow \mathfrak{so}(V)$. As $\kappa(S', S') = V$, this fully determines the curvature R and, by a further contraction, the Ricci tensor:

$$\begin{aligned} \text{Ric}(v, \kappa(s, s)) &= \frac{1}{2}F_{ab}^2 v^a \langle \Gamma^b s, s \rangle - \frac{1}{6} \|F\|^2 \langle v \cdot s, s \rangle \\ &\quad + \frac{1}{6} \langle (v \wedge F \wedge F + 2\iota_v \delta F - v \wedge dF) \cdot s, s \rangle. \end{aligned} \quad (2)$$

We then showed that the terms which depend on forms of different degree in

$$\odot^2 S' \subset \odot^2 S = \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^5 V \quad (3)$$

satisfy the eqs separately (not immediate: embedding (3) is in general diagonal) ■ 🔍 ↻

High supersymmetry

The theorem allows to establish a reconstruction result for highly supersymmetric bkgds and to map their classification to an algebraic problem.

Def. A filtered subdeformation $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of \mathfrak{p} with $\dim \mathfrak{g}_1 > 16$ is **aligned** if it is constructed out of a closed 4-form $\varphi \in \Lambda^4 V$.

Thm[Figuroa-O'Farrill, A.S.] The Killing superalgebra of any highly susy bkgd is aligned. Conversely any aligned filtered subdeformation \mathfrak{g} is (a subalgebra of) the Killing superalgebra \mathfrak{k} of a highly susy bkgd (M, g, F) . In particular highly susy bkgds, up to local equivalence, are in a **one-to-one correspondence** with maximal aligned filtered subdeformations of \mathfrak{p} , up to isomorphism of filtered subdeformations.

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Thanks!