Super $p$-Brane Theory
emerging from
Super Homotopy Theory

Urs Schreiber
(CAS Prague and HCM Bonn)

talk at [String Math 2017](#)

Based on

- [arXiv:1611.06536](http://arxiv.org/abs/1611.06536) with D. Fiorenza and H. Sati

these slides are kept at [ncatlab.org/schreiber/print/StringMath2017](http://ncatlab.org/schreiber/print/StringMath2017)
Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?
Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?

*We still have no fundamental formulation of “M-theory” - Work on formulating the fundamental principles underlying M-theory has noticeably waned. [...] If history is a good guide, then we should expect that anything as profound and far-reaching as a fully satisfactory formulation of M-theory is surely going to lead to new and novel mathematics. Regrettably, it is a problem the community seems to have put aside - temporarily. But, ultimately, Physical Mathematics must return to this grand issue.*

G. Moore, *Physical Mathematics and the Future*, at *Strings 2014*
Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?

What is even its **Principle**?
### Principles

<table>
<thead>
<tr>
<th>physics</th>
<th>mathematics</th>
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<tbody>
<tr>
<td>gauge principle</td>
<td>homotopy theory</td>
</tr>
<tr>
<td>&amp; Pauli exclusion</td>
<td>super-geometry</td>
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</tbody>
</table>

= super-homotopy theory
Homotopy Theory

- Stable
- Parameterized over the point
- Underlying parameter space
- Spaces

Plain

Rational

Super rational

Spectra
Homotopy Theory

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<thead>
<tr>
<th>Plain</th>
<th>Stable</th>
<th>Parameterized Stable</th>
<th>Plain</th>
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<td>Plain Spectra</td>
<td>Parameterized over the point</td>
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<td>Underlying Parameter Space</td>
</tr>
<tr>
<td>Rational</td>
<td>Super Rational</td>
<td>Super Cochain Complexes</td>
<td>Super Dg-modules</td>
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</table>

Quillen 69, Sullivan 77: infinitesimal methods in homotopy theory
Homotopy Theory

plain spectra \rightarrow \text{parameterized spectra} \rightarrow \text{spaces}

rational super cochain complexes dg-modules dgc-algebras

super dgc-algebras ("FDA"s)

Nieuwenhuizen 82, D’Auria-Fré 82: FDAs efficiently construct SuGra-s
**Homotopy Theory**

<table>
<thead>
<tr>
<th>Stable</th>
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<td>underlying parameter space spaces</td>
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<tr>
<th>Rational</th>
<th>Cochain Complexes</th>
<th>Dgc-Algebras</th>
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<tbody>
<tr>
<td>Super rational</td>
<td>super cochain complexes</td>
<td>super dgc-algebras (“FDA”s)</td>
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</table>

_Schwede-Shipley 03_: stable homotopy theory subsumes homological algebra
Homotopy Theory

plain spectra \rightarrow \text{parameterized spectra} \rightarrow \text{spaces}

parameterized spectra

rational cochain complexes \rightarrow \text{dg-modules} \rightarrow \text{dgc-algebras}

super rational cochain complexes \rightarrow \text{super dg-modules}

Schlegel
<table>
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<td>super cochain complexes</td>
<td>super dg-modules</td>
<td>super dgc-algebras (&quot;FDA&quot;s)</td>
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</tbody>
</table>
Cohomology

extension

down

homotopy fiber

up

space cocycle

twisted
→
twist
→
parameterized

left

space

twist

right

space

cocycle

Nikolaus-Schreiber-Stevenson 12, Ando-Blumberg-Gepner-Hopkins-Rezk 14
We now work out
in *rational* super-homotopy theory
a tower of extensions,
each invariant wrt
automorphisms modulo R-symmetries.
We now work out
in \textit{rational} super-homotopy theory
a tower of extensions,
each invariant wrt
automorphisms modulo R-symmetries.

Beware:
Everything in the following holds
in (super-) \textit{rational} homotopy theory.
In the beginning
the atom of space:
the superpoint

$T^{0|1}$
Its maximal torus extension is the super-line.
its type II version: 
the $\mathcal{N} = 2$ superpoint
maximal torus extension:
\[ d = 3, \ N = 1 \]

super-Minkowski spacetime
type II version: 
\(d = 3, \ N = 2\)

super-Minkowski spacetime
maximal invariant torus extension:
\( d = 4, \ N = 1 \)
super-Minkowski spacetime
type II version:
\( d = 4, \ N = 2 \)
super-Minkowski spacetime
maximal invariant torus extension:
\( d = 6, \ N = 1 \)
super-Minkowski spacetime
type IIB version:
\( d = 6, \ N = (2, 0) \)
super-Minkowski spacetime.
$\mathbb{T}^5,1 | 8 + 8 \leftrightarrow \mathbb{T}^5,1 | 8 \leftrightarrow \mathbb{T}^5,1 | 8 + \bar{8}$

$\mathbb{T}^3,1 | 4 + 4 \leftrightarrow \mathbb{T}^3,1 | 4$

$\mathbb{T}^2,1 | 2 + 2 \leftrightarrow \mathbb{T}^2,1 | 2 \leftrightarrow \mathbb{T}^1 | 1$

$\mathbb{T}^0 | 1 + 1 \leftrightarrow \mathbb{T}^0 | 1$

**type IIA version:**

$d = 6, \ N = (1, 1)$

super-Minkowski spacetime.
maximal invariant torus extension: $d = 10, N = 1$
super-Minkowski spacetime
$T^{9,1|16+16} \leftrightarrow T^{9,1|16}$

$T^{5,1|8+8} \leftrightarrow T^{5,1|8} \leftrightarrow T^{5,1|8+\bar{8}}$

$T^{3,1|4+4} \leftrightarrow T^{3,1|4}$

$T^{2,1|2+2} \leftrightarrow T^{2,1|2} \leftrightarrow T^{1|1}$

$T^{0|1+1} \leftrightarrow T^{0|1}$

\begin{align*}
\text{type IIB version:} \\
\bar{d} = 10, \quad N = (2,0) \\
\text{super-Minkowski spacetime}
\end{align*}
and its type IIA version:
\( d = 10, \mathcal{N} = (1, 1) \)
super-Minkowski spacetime
maximal invariant torus extension:

$d = 11, N = 1$

super-Minkowski spacetime
In summary:

**Theorem (Huerta-Schreiber 17):**
There exists a diagram as shown of maximal torus extensions at each stage invariant with respect to the semi-simple part of automorphisms modulo R-symmetry which happens to be the Lorentzian Spin-groups.
the higher invariant extensions: superstrings condense
these extensions are classified by WZW-term for the GS-Superstring
\[ \mu_{F1} = i\bar{\psi} \wedge \Gamma_a \psi \wedge e^a \]
(Green-Schwarz 81, Henneaux-Mezincescu 85)
the \textit{next higher} invariant extensions: D-branes condense
the next higher invariant extensions: D-branes condense
these extensions are classified by
WZW-terms for the super D-branes

$$\exp(F_2) \sum c_p \overline{\psi} \wedge \Gamma_{a_1 \cdots a_p} \psi \wedge e_{a_1} \wedge \cdots \wedge e_{a_p}$$

Azcárraga et al. 99, Sakaguchi 00
Fiorenza-Sati-Schreiber 13
D0-brane condensate is 11-th dimension via homotopy pullback
M2-branes condense its own higher invariant extension: M2-branes condense
its next higher extension: M5-branes condense
spacetime and M-branes have emerged from the superpoint as iterated higher invariant extensions
Perhaps we need to understand the nature of time itself better. [...] understand in what sense time itself is an emergent concept, [...] how pseudo-Riemannian geometry can emerge from more fundamental and abstract notions such as categories of branes.

Spacetime and M-branes have emerged from the superpoint as iterated higher invariant extensions.
consider the M-brane sector
the M2-extension is classified by a 4-cocycle:
the GS-WZW-term of the M2-brane

\[ \mu_{M2} = \frac{i}{2} \bar{\psi} \wedge \Gamma_{a_1 a_2} \psi \wedge e^{a_1} \wedge e^{a_2} \]

D’Auria-Fré 82, Bergshoeff-Sezgin-Townsend 87, Fiorenza-Sati-Schreiber 13
the M5-extension is classified by a 7-cocycle:
the GS-WZW-terms of the M5-brane

\[
\mu_{M5} = \frac{1}{5!} \bar{\psi} \wedge \Gamma_{a_1 \cdots a_5} \psi \wedge e^{a_1} \wedge \cdots \wedge e^{a_5} + \frac{1}{2} c_3 \wedge \frac{1}{2} \bar{\psi} \wedge \Gamma_{a_1 a_2} \psi \wedge e^{a_1} \wedge e^{a_2}
\]

D’Auria-Fré 82, Pasti-Sorokin-Tonin 97, Fiorenza-Sati-Schreiber 13
to descend this means to ask for analogous fiber sequence on the coefficients

Nikolaus-Schreiber-Stevenson 12
this comes out to be:
quaternionic Hopf fibration
(rationally)

Fiorenza-Sati-Schreiber 15
M5-cocycle descends:
unified M2/M5-cocycle

Fiorenza-Sati-Schreiber 15
\[
\begin{align*}
m_{5}\text{brane} \quad & \quad \downarrow \text{hofib}(\mu_{M5}) \quad \mu_{M5} \quad \rightarrow S^7 \\
m_{2}\text{brane} \quad & \quad \downarrow \text{hofib}(\mu_{M2}) \quad \mu_{M2} \quad \rightarrow S^4 \\
\mathbb{T}^{10,1|32} \quad & \quad \downarrow \mu_{M2/M5} \quad \rightarrow S^4 \\
& \quad \downarrow B^{4}\mathbb{Q} \\
\end{align*}
\]

dgc-model for \( S^4 \):
\[
\begin{align*}
d\omega_4 & = 0 \\
d\omega_7 & = -\frac{1}{2}\omega_4 \wedge \omega_4 \\
\end{align*}
\]

11d SuGra \( C \)-field equation of motion:
\[
dG_7 + \frac{1}{2}G_4 \wedge G_4 = 0
\]
consider this
unified M-brane cocycle
Ext\((\mathbb{T}^9,1|16+\overline{16})\) $\rightarrow$ $\mathbb{T}^{10,1|32}$ $\rightarrow$ $S^4$

$\mu_{M2/\overline{M5}}$

$\mu_{M2}$

$B^4\mathbb{Q}$

remember that

11d spacetime
is (maximal invariant) extension of
type IIA spacetime
\[
\text{Ext}(\mathbb{T}^{9,1|16+1\overline{16}}) \cong \mathbb{T}^{10,1|32} \xrightarrow{\mu_{M2/M5}} S^4 \xrightarrow{\mu_{M2}} B^4 Q \xrightarrow{S^4} \text{Ext}(S^4/S^1)
\]

Similarly, $S^4$ is homotopy extension of its $S^1$ homotopy quotient via canonical $SU(2)$-action on $S^4 \cong S(\mathbb{R} \oplus \mathbb{H})$. 
This orbifold $S^4/C_n \rightarrow S^4/S^1$ happens to be the same as in the near-horizon geometry of the black M5-brane at an A-type singularity

Medeiros, Figueroa-O’Farrill 10
hence the unified M2/M5-cocycle is really of this form
Theorem (Fiorenza-Sati-Schreiber 17): Ext has a derived right adjoint

\[
\begin{array}{ccc}
\text{SuperHomotopyTypes} & \xrightarrow{\text{Extension}} & \text{SuperHomotopyTypes} / _{/ BS^1} \\
\downarrow & & \downarrow \text{Cyclification} \\
B^4 \mathbb{Q} & \xleftarrow{\mu_{M2}} & \mu_{M2/M5} & \xrightarrow{\mu_{M2}} & \text{Ext}(S^4 / S^1)
\end{array}
\]

given by passing to twisted loop spaces / cyclic cohomology
apply the right adjoint
\[
\begin{array}{c}
\mathbb{T}^{9,1|16+16} \\
\downarrow \eta_{\mathbb{T}^{9,1|16+16}} \\
\text{CycExt}(\mathbb{T}^{9,1|16+16}) \\
\downarrow \text{Cyc}(\mu_{M2/M5}) \\
\mathbb{S}^3 \\
\end{array}
\rightarrow
\begin{array}{c}
\mathbb{S}^4/\mathbb{S}^1 \\
\downarrow \eta_{\mathbb{S}^4/\mathbb{S}^1} \\
\text{CycExt}(\mathbb{S}^4/\mathbb{S}^1) \\
\end{array}
\]

and compose with the adjunction unit.
to obtain the
Ext ⊣ Cyc-adjunct
of the unified M-brane cocycle
Theorem (Fiorenza-Sati-Schreiber 17): This is the Green-Schwarz WZW term of the double dimensional reduction of M2/M5 to $F_{I/IIA}/D0/D2/D4/NS5$:

\[
\begin{align*}
\text{dgc-algebra for } & \text{CycExt}(S^4/S^1): \\
\begin{cases}
    dH_3 = 0, & dH_7 = F_2 \wedge F_6 - \frac{1}{2} F_4 \wedge F_4 \\
    dF_2 = 0, & dF_4 = H_3 \wedge F_2, & dF_6 = H_3 \wedge F_4
\end{cases}
\end{align*}
\]
This gives rise to two questions:
1) Where are the $D(p \geq 6)$-branes (gauge enhancement)?
2) Is there a dashed lift as above?
let us first make some room...
consider the Goodwillie-linearized lifting problem:
form the fiberwise suspension spectrum over $S^3$
to obtain an $S^3$ parameterized spectrum
$\mathbb{T}^{9,1|\mathbf{16}+\bar{\mathbf{16}}} \xrightarrow{\eta_{\mathbb{T}^{9,1|\mathbf{16}+\bar{\mathbf{16}}}}} \text{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\bar{\mathbf{16}}}) \xrightarrow{\text{Cyc}(\mu_{M_2/M_5})} \Omega_{S^3}^\infty \Sigma_{S^3}^\infty (S^4/S^1) \xrightarrow{\Omega_{S^3}^\infty \Sigma_{S^3}^\infty (\eta_{S^4/S^1})} \Omega_{S^3}^\infty \Sigma_{S^3}^\infty \text{CycExt}(S^4/S^1) \xrightarrow{} S^3
Theorem (Roig-Saralegi 00): rationally, a direct summand of $\Omega_{S^3}^\infty \Sigma_{S^3}^\infty (S^4/S^1)$ is twisted connective K-theory $\text{ku}/B^2\mathbb{Z}$.
and now there is a lift:
the unified cocyle of all the type IIA D-branes

dgc-algebra for $B^3 \mathbb{Z} \cong_\mathbb{Q} S^3$: $dH_3 = 0$
dg-module for $\text{ku}/B^2 \mathbb{Z}$: $dF_{2p+2} = H_3 \wedge F_{2p}$ \quad p \in \mathbb{N}
Conclusion:

Double dimensional reduction of unified M-brane cocycle via cyclification is unified IIA-brane cocycle
Conclusion:

Double dimensional reduction of unified M-brane cocycle via cyclification is unified IIA-brane cocycle
we repeat the process:

and consider the double dimensional reduction of the IIA-cocycle to 9d super-spacetime $\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}}$
\[
\begin{align*}
\text{Cyc}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) & \xrightarrow{\text{Cyc}(\mu_{F1/Dp}^{\text{IIA}})} \text{Cyc}(\mathbb{KU}/B^2\mathbb{Z}) \\
\text{CycExt}_{\text{IIA}}(\mathbb{T}^{8,1|\mathbf{16}+\overline{\mathbf{16}}}) & \\
\end{align*}
\]

hence apply cyclification
and compose
with the adjunction unit
to obtain
the double dimensional reduction
but there was also
the type IIB extension
whatever cocycle it carries
has itself a double dimensional reduction
by adjunction
this defines $\mu$
in terms of $\mu^{\text{IIA}}$
Theorem A: (Fiorenza-Sati-Schreiber 17):
This is the cocycle in twisted $K^1$ for the F1/Dp-branes in type IIB.
Theorem B: (Fiorenza-Sati-Schreiber 17):
The commutativity of this diagram is equivalently the Buscher rules for the RR-fields
(Hori 99)
\textbf{Theorem C:} \cite{Fiorenza-Sati-Schreiber17}:
The commutativity of this diagram is equivalently the rules of \textit{“topological T-duality”} \cite{Bouwknegt-Evslin-Mathai04, Bunke-Rumpf-Schick08} rationally.
Theorem D: (Fiorenza-Sati-Schreiber 17):
The homotopy fiber is the doubled generalized geometry 10d super-spacetime
Theorem E: (Fiorenza-Sati-Schreiber 17):
The homotopy pullback of type II doubled super-spacetime back to 11d super-spacetime is the local model for an F-theory fibration.
Conclusion:
A fair bit of
the expected structure of M-theory
emerges out of the superpoint
in rational super-homotopy theory.

Evident Conjecture:
The full theory emerges
once passing beyond the rational approximation
in full super-geometric homotopy theory.

Epilogue

In full super-geometric homotopy theory
the superpoint $\mathbb{R}^{0|1}$ itself
emerges from $\emptyset$

\[
\begin{array}{ccc}
id & \to & \text{id} \\
\vee & \vee & \\
\Rightarrow & \to & \rightsquigarrow \to \mathbb{R}^{0|1} \\
\vee & \vee & \\
\mathbb{R} & \to & D \to E \\
\vee & \vee & \\
\mathbb{R} & \to & b \to \# \\
\vee & \vee & \\
\emptyset & \to & * \\
\end{array}
\]

(Schreiber 16, FOMUS proceedings)