Comments on Vertex Algebras for $\mathcal{N} = 2$ SCFTs

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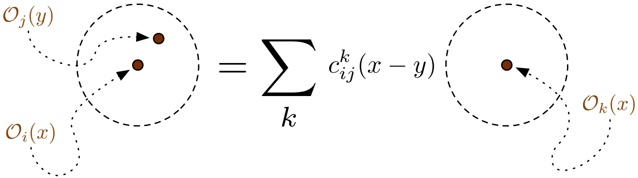
Introduction

I will be discussing several algebraic structures that can be associated to $\mathcal{N} = 2$ superconformal theories in four dimensions.

These are all shadows of the full CFT operator algebra:



The *CFT data* (*i.e.*, list of local operators and their OPE coefficients) is sufficient to determine any correlation function of local operators and is strongly constrained by OPE associativity (crossing symmetry).

Subsets of this CFT data are recycled in various simpler algebras we will discuss.

$\mathcal{N} = 2$ SCFTs: Local operators

Local operators in $\mathcal{N} = 2$ SCFTs are organized into representations of the superconformal algebra $\mathfrak{su}(2,2|2)$.

These representations are further subdivided into finite-dimensional representations of the subalgebra

$$\mathbb{R}_E \times \mathfrak{su}(2)_1 \times \mathfrak{su}(2)_2 \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$$

We label operators by their charges (E, j_1, j_2, R, r) under the Cartan subalgebra.

In addition, there are sixteen nilpotent, fermionic symmetries:

- Poincaré supercharges: $\mathcal{Q}^I_\alpha$ and $\widetilde{\mathcal{Q}}^I_{\dot\alpha}$ with $I = 1, 2$, $\alpha = \pm$, $\dot\alpha = \pm$.
- Special conformal supercharges: $\mathcal{S}_I^\alpha$ and $\widetilde{\mathcal{S}}_I^{\dot\alpha}$.

$\mathcal{N} = 2$ SCFTs: Subalgebras

There are many interesting and simpler algebras encoded in the full OPE algebra.

They can be defined by passing to cohomologies (or similar constructions) with respect to supercharges.

Coulomb chiral ring: $\mathcal{R}_C$

- Built from operators with $E = r$.
- Commutative, associative $\mathbb{C}$-algebra.
- Freely generated in examples: $\mathcal{M}_C = \mathrm{Spec}(\mathcal{R}_C)$ algebraically boring.
- Defined as cohomology of Donaldson-Witten supercharge

$$\mathcal{Q}_{DW} = \mathcal{Q}^1_+ + \mathcal{Q}^2_- \ .$$

- *cf.*, Ben-Zvi's talk.

$\mathcal{N} = 2$ SCFTs: Subalgebras

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They can be defined by passing to cohomologies (or similar constructions) with respect to supercharges.

Higgs chiral ring: $\mathcal{R}_H$

- Built from operators with $E = 2R$.
- Commutative, associative, Poisson algebra.
- $\mathcal{M}_H = \mathrm{Spec}(\mathcal{R}_H)$ a symplectic singularity (really hyperkähler).
- Defined as *simultaneous cohomology* of four supercharges: $\left\{\mathcal{Q}^1_{\alpha}\ , \widetilde{\mathcal{Q}}^1_{\dot{\alpha}}\right\}$.

$\mathcal{N} = 2$ SCFTs: Subalgebras

There are many interesting and simpler algebras encoded in the full OPE algebra.

They can be defined by passing to cohomologies (or similar constructions) with respect to supercharges.

Hall-Littlewood chiral ring: $\mathcal{R}_{HL}$

- Built from operators with $E = 2R + r$.
- Commutative, associative $\mathbb{C}$-algebra (actually Poisson).
- $\mathrm{Spec}(\mathcal{R}_{HL})$ is something like an (odd) coherent sheaf on $\mathcal{M}_H$.
- Simultaneous cohomology of three supercharges: $\left\{ \mathcal{Q}^1_\alpha \ , \widetilde{\mathcal{Q}}^1_{\dot{-}} \right\}$.

$\mathcal{N} = 2$ SCFTs: Subalgebras

There are many interesting and simpler algebras encoded in the full OPE algebra.

They can be defined by passing to cohomologies (or similar constructions) with respect to supercharges.

"Schur algebra": $\mathcal{V}_{\text{comm.}}$

- Built from *Schur operators* with $E = 2R + j_1 + j_2$.
- Commutative vertex algebra (really a vertex Poisson algebra).
- Geometry of this algebra has not been seriously studied as far as I know.
- Something like derived symplectic quotient of jet scheme of representation space for gauge theories.
- Defined by *holomorphic/topological twist*,
  cf. [Johansen], [Kapustin], [Costello]

$$\mathcal{Q}_{\text{Sch}} = Q^1_- + \widetilde{Q}^1_{\dot{-}}$$

$\mathcal{N}=2$ SCFTs: Subalgebras

The previous constructions all apply in the non-conformal case.

With conformal invariance, get additional fermionic symmetries and can define new cohomologies.

Associated vertex operator algebra: $\mathcal{V}$
[Beem-van Rees-Lemos-Liendo-Peelaers, 2013]

- Non-commutative vertex operator algebra.
- This is a quantization of the Schur algebra.
- Arises upon taking cohomology of "funny" supercharge,

$$\mathbb{Q} = \mathcal{Q}_{-}^{1} + \widetilde{\mathcal{S}}_{1}^{\dot{-}} \ .$$

- (Should also be realizable as a novel Ω-deformation of holomorphic/topological twist in conformal theories.)
  cf. [Butson-Costello-Gaiotto]

VOA review: basics

For our purposes, a VOA is a meromorphic OPE algebra in two dimensions.

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_k \frac{{c_{12}}^k \mathcal{O}_k(w)}{(z-w)^{h_1+h_2-h_k}} ~.$$

Each operator can be expanded in a Laurent series,

$$\mathcal{O}(z) = \sum_{n=-\infty}^{\infty} z^n \mathcal{O}_{-h-n} ~, \qquad \mathcal{O}_n \in \mathrm{End}(\mathcal{V}) ~,$$

where we also denote by $\mathcal{V}$ the space of local operators at the origin.

There is a simple operator-state map in terms of these Laurent modes,

$$\mathcal{O}(0) \Leftrightarrow \mathcal{O}_{-h}|\Omega\rangle ~, \qquad \partial^n\mathcal{O}(0) \Leftrightarrow \mathcal{O}_{-h-n}|\Omega\rangle ~, \quad \ldots$$

VOA review: operations

The *normally ordered product* of two operators is the operator that is defined at the origin according to

$$\mathrm{NO}(a,b)(0) := a_{-h_a} b_{-h_b} |\Omega\rangle \ .$$

$\mathrm{NO}(\cdot,\cdot)$ is generally *non-commutative* and *non-associative*.

Can also introduce a secondary bracket (first of infinitely many n'th brackets)

$$\{a,b\} := a_{-h_a+1} b_{-h_b} |\Omega\rangle \ .$$

This acts by picking out the simple pole in the OPE,

$$\{a,b\}(w) = \oint \frac{dz}{2\pi i} a(z) b(w) \ .$$

VOA review: vocabulary

A *strong generator* of a VOA is an $\mathfrak{sl}(2)$ primary that cannot be written as a normally ordered product.

All operators in a VOA can be written as normally ordered products of the strong generators and their derivatives.

Conjecture

VOAs associated to $\mathcal{N} = 2$ SCFTs are strongly finitely generated

The associated VOA: construction

Our vector space $\mathcal{V}$ is the space of *Schur operators* in the SCFT.

Operators at general points in the plane are defined via *twisted translation*,

$$Y(\mathcal{O}, z) = \mathcal{O}(z) = \left[e^{zL_{-1}+\bar{z}(\bar{L}_{-1}+\mathcal{R}^-)}\mathcal{O}_{\mathrm{Sch}}(0)e^{-zL_{-1}-\bar{z}(\bar{L}_{-1}+\mathcal{R}^-)}\right]_{\mathbb{Q}}$$

In terms of a basis $\mathcal{O}^{(\alpha_1\cdots\alpha_{2R})}$ for the $\mathfrak{su}(2)_R$ representation,

$$\mathcal{O}(z) = \left[\mathcal{O}^{(+\cdots+)}(z,\bar{z}) + \bar{z}\mathcal{O}^{+\cdots+-}(z,\bar{z}) + \ldots + \bar{z}^{2R}\mathcal{O}^{(-\cdots-)}(z,\bar{z})\right]_{\mathbb{Q}}$$

The conformal weight of the vertex operator is given by

$$h = E - R = R + j_1 + j_2 \ .$$

Meromorphicity at the level of cohomology is guaranteed by the superconformal algebra.

The associated VOA: properties

I want to discuss a variety of structural properties of the associated VOAs of four-dimensional SCFTs.

Certain general properties follow more or less directly:

- $\frac{1}{2}\mathbb{Z}$-valued conformal grading (uncorrelated with parity).
- Four dimensional theory local $\Longrightarrow$ $\mathcal{V} \supseteq \mathrm{Vir}_{-12c_{4d}}$ $(c_{4d} > 0)$.
- Continuous G global symmetry $\Longrightarrow$ $\mathcal{V} \supseteq V_{-k_{4d}/2}(\mathfrak{g})$ $(k_{4d} > 0)$.
- $\mathcal{R}_{HL}$ generators $\Longrightarrow$ *strong* $\mathcal{V}$-generators .
- $\mathcal{R}_{HL}$ elements $\Longrightarrow$ Virasoro primaries .
- Schur index $\Longrightarrow$ VOA character,

$$\mathcal{I}_{\mathrm{Sch}}(q) := q^{c_4 d/2}\mathrm{STr}_{\mathcal{H}(\mathbb{S}^3)} q^{E-R} = \mathrm{STr}_{\mathcal{V}} q^{L_0 - c/24} =: \chi_{\mathcal{V}}(q)$$

- Null states are removed (*i.e.*, always in simple quotient) .

The associated VOA: filtration

- $\mathcal{V}$ is triply graded as a vector space by R, r, and h:

$$\mathcal{V} = \bigoplus_{h,R,r} \mathcal{V}_{h,R,r} \ .$$

- OPE violates R conservation but is compatible with *filtration*

$$\mathcal{F}_{h,R,r} = \bigoplus_{k \geqslant 0} \mathcal{V}_{h,R-k,r} \ .$$

- Normally ordered product is filtered.

$$\mathrm{NO}(\mathcal{F}_{h_1,R_1,r_1}, \mathcal{F}_{h_2,R_2,r_2}) \in \mathcal{F}_{h_1+h_2,R_1+R_2,r_1+r_2} \ .$$

- Secondary bracket is filtered of degree -1,

$$\{\mathcal{F}_{h_1,R_1,r_1}, \mathcal{F}_{h_2,R_2,r_2}\} \in \mathcal{F}_{h_1+h_2-1,R_1+R_2-1,r_1+r_2} \ .$$

The associated VOA: filtration

This is a *good filtration*, so the associated graded (with the operations inherited from normally ordered products and the secondary bracket) is a vertex Poisson algebra. [H. Li]

$$\mathrm{gr}_F \mathcal{V} \cong \mathcal{V}_{\mathrm{comm}} \ .$$

By further restricting to subspaces with $h = R$ or $h = R + r$, we recover $\mathcal{R}_H$ and $\mathcal{R}_{HL}$ (as Poisson algebras).

This is great! Associated VOA plus filtration gives a lot of physics!

Problem:

No general construction for filtration... maybe not intrinsic to VOA?

Easier problem:

Only construct $\mathcal{R}_H$; then we don't need the whole filtration.

Higgs branch recovery

To discuss the Higgs branch, let me introduce some of vector subspaces of $\mathcal{V}$:

$$\begin{aligned} \overline{\mathcal{V}_H} &:= \bigoplus_{h>R} \mathcal{V}_{h,R,r} = \bigoplus_{h} F_{h,h-1,r} \ , \\ \mathcal{V}_H &:= \bigoplus \mathcal{V}_{R,R,0} \cong \mathcal{V}/\overline{\mathcal{V}_H} \ . \end{aligned}$$

In addition, the following subspace can be defined intrinsically in the VOA,

$$C_2(\mathcal{V}) := \mathrm{span}\,\{a_{-h_a-1}b \ , \quad a \ , b \in \mathcal{V}\}$$

$C_2(\mathcal{V})$ (and obviously $\overline{\mathcal{V}_H}$) are two-sided ideals for the normally ordered product.

We thus have a chain of ideals with respect to the normally ordered product:

$$\mathcal{V} \supset \overline{\mathcal{V}_H} \supset C_2(\mathcal{V}) \ .$$

Higgs branch recovery

Direct calculation shows that the commutator and associator of the normally ordered product lie within $C_2(\mathcal{V})$, so in $\overline{\mathcal{V}_H}$

$$\begin{aligned} [\mathcal{V},\mathcal{V}] &\subset C_2(\mathcal{V}) \subseteq \overline{\mathcal{V}_H}\ , \\ [\mathcal{V},\mathcal{V},\mathcal{V}] &\subset C_2(\mathcal{V}) \subseteq \overline{\mathcal{V}_H}\ . \end{aligned}$$

Also the symmetrizer and Jacobiator of the secondary bracket lie in $C_2(\mathcal{V})$

$$\begin{aligned} \{\mathcal{V},\mathcal{V}\}_+ &\subset C_2(\mathcal{V}) \subseteq \overline{\mathcal{V}_H}\ , \\ \{\mathcal{V},\mathcal{V},\mathcal{V}\} &\subset C_2(\mathcal{V}) \subseteq \overline{\mathcal{V}_H}\ . \end{aligned}$$

So $\mathrm{NO}(\ ,\ )$ and the secondary bracket induce the structure of a commutative Poisson algebra on $\mathcal{V}/C_2(\mathcal{V})$ as well as on $\mathcal{R}_H$.

The C_2 algebra

$C_2(\mathcal{V})$ defined intrinsically in the VOA, so we have an intrinsic construction of a commutative Poisson algebra, *Zhu's C_2-algebra*:

$$\mathcal{R}_{\mathcal{V}} := (\mathcal{V}/C_2(\mathcal{V})\ ,\ \mathrm{NO}(\ ,\ )\ ,\ \{\cdot,\cdot\}\ )\ .$$

On the other hand, our four-dimensional construction tells us we can reconstruct the Higgs chiral ring given $\overline{\mathcal{V}_H}$:

$$\mathcal{R}_H = \left(\mathcal{V}/\overline{\mathcal{V}_H},\ \mathrm{NO}(\ ,\ )\ ,\ \{\cdot,\cdot\}\ \right)\ .$$

What is the difference between $\mathcal{R}_{\mathcal{V}}$ and $\mathcal{R}_{\mathcal{H}}$?

$$\mathcal{R}_H := \mathcal{R}_{\mathcal{V}}/\mathcal{I}\ , \qquad \mathcal{I} = \overline{\mathcal{V}_H}/C_2(\mathcal{V})\ .$$

What is this ideal $\mathcal{I}$?

Recovering $\mathcal{R}_H$

$\mathcal{R}_H$ is a *reduced* commutative $\mathbb{C}$-algebra, *i.e.*, it has no nilpotents.

$\mathcal{R}_{\mathcal{V}}$ has no reason to be reduced; in many VOAs it is not. So $\mathcal{I} \supseteq \mathrm{Nil}(\mathcal{R}_{\mathcal{V}})$.

Conjecture

Nilpotent elements are the only obstruction to identifying $\mathcal{R}_H$ with $\mathcal{R}_{\mathcal{V}}$. That is to say,

$$\mathcal{I} = \mathrm{Nil}(\mathcal{R}_{\mathcal{V}})$$

In other words, we are suggesting that

$$\mathcal{M}_H = \mathrm{Spec}(\mathcal{R}_{\mathcal{V}})_{\mathrm{red}} = \text{“Associated Variety”} \quad X_{\mathcal{V}}$$

The associated variety has been studied by Tomoyuki Arakawa and collaborators.

VOAs whose associated varieties are symplectic are dubbed *quasi-Lisse*.

Examples: empty Higgs branch

When the vector space $\mathcal{R}_{\mathcal{V}}$ is finite, a VOA is called *C_2-cofinite*.

C_2-cofiniteness is a necessary condition for *rationality*, so any SCFT whose associated VOA is rational must have an empty Higgs branch.

Example: (A_1, A_{2n}) AD Theories

$\mathcal{V}_{(A_1,A_{2n})} = Vir(2, 2n+3)$:

$$\begin{aligned} \mathcal{R}_{\mathcal{V}} &= \mathbb{C}[t]/\langle t^{2n+2}\rangle \ , \quad \{t,t\} = 0 \ , \\ \mathcal{R}_{\mathcal{H}} &= \mathbb{C} \ . \end{aligned}$$

Remark

C_2-cofinite algebras, which for some time were the primary object of study in the literature, are an incredibly special case in the world of associated VOAs. Most SCFTs have Higgs branches.

Non-empty Higgs branch

Example: (A_1, D_{2n+1}) Argyres-Douglas Theories

$\mathcal{V}_{(A_1,D_{2n+1})} = V_{-4n/(2n+1)}(\mathfrak{su}(2))$:

$$\mathcal{R}_{\mathcal{V}} = \mathbb{C}[j^1, j^2, j^3]/\langle j^A \Omega^n \rangle \ , \quad \{j^A, j^B\} = f^{AB}{}_C j^C \ .$$
$$\mathcal{R}_{\mathcal{H}} = \mathbb{C}[j^1, j^2, j^3]/\langle \Omega \rangle \cong \mathbb{C}^2/\mathbb{Z}_2 \ .$$

Example: (A_1, A_5) Argyres-Douglas Theory

$\mathcal{V}_{(A_1,A_5)} = \mathcal{BP}_{c=-23/2}$.

$$\mathcal{R}_{\mathcal{V}} = \mathbb{C}[x,y,z,t]/\langle xy + z^3 - \tfrac{3}{2}tz, t^3 \rangle \ , \quad \{z,x\} = 1 \ , \ \{z,y\} = -y \ , \ \{x,y\} = 3z^2 \ .$$
$$\mathcal{R}_{\mathcal{H}} = \mathbb{C}[x,y,z]/\langle xy + z^3 \rangle \ , \quad \{z,x\} = 1 \ , \ \{z,y\} = -y \ , \ \{x,y\} = 3z^2 \ .$$
$$\mathcal{R}_{\mathcal{H}} \cong \mathbb{C}^3/\mathbb{Z}_3 \ .$$

Consequences

Observe that any strong $\mathcal{V}$-generator that is not a Higgs chiral ring generator must be *nilpotent* in $\mathcal{R}_{\mathcal{V}}$. In particular this includes the stress tensor.

We conclude there must always exist a null vector $\mathcal{N}$ in the vacuum module of the VOA of the form

$$\mathcal{N}_T = (L_{-2})^k|\Omega\rangle + \varphi~, \qquad \varphi \in C_2(\mathcal{V})~.$$

As in RCFT, can derive differential constraints on correlation functions by inserting the null vector and demanding that the result vanish.

We apply this to the case of the torus partition function, $i.e.$, vacuum character, *i.e.*, Schur index of the four-dimensional theory.

Precisely this situation has been studied in various places in the literature.
[Mathur, Mukhi, Sen (1988)], [Zhu (1996)], [Gaberdiel, Keller (2008)], [Gaberdiel, Lang (2008)]

Consequences

Disclaimer

The following derivation sketch is morally true, but not technically accurate.

The full result, which is only guaranteed to hold for quasi-Lisse VOAs, was proven recently by [Arakawa-Kawasetsu (2016)].

In their proof, the primary trick is showing that in the quasi-Lisse case one can evade an obstruction that I will be ignoring.

Modular recursion

We require

$$\mathrm{STr}_{\mathcal{V}}(o(\mathcal{N}_T)q^{L_0-c/24}) = 0 \ ,$$

where $o(a) = a_0$ for $a \in \mathcal{V}$.

For our null vector, this gives

$$\mathrm{STr}_{\mathcal{V}}(o((L_{-2})^k)q^{L_0-c/24}) = \mathrm{STr}_{\mathcal{V}}(o(\varphi)q^{L_0-c/24}) \ .$$

The trick is to evaluate the left and right hand sides differently in terms of operations on the vacuum character.

Modular recursion

Left hand side:

$$\mathrm{STr}_{\mathcal{V}}\left(o(L_{[-2]}^{k}\Omega)q^{L_0-\frac{c}{24}}\right)=\mathcal{P}_k(D)\mathrm{STr}_{\mathcal{V}}\left(q^{L_0-\frac{c}{24}}\right) \ ,$$

where $\mathcal{P}_k(D)$ is modular covariant differential operator of order k and weight $2k$.

$$\begin{aligned}
\mathcal{P}_2(D) &= D_q^{(1)} \ , \\
\mathcal{P}_4(D) &= D_q^{(2)}+\tfrac{c}{2}\mathbb{E}_4(q) \ , \\
\mathcal{P}_6(D) &= D_q^{(3)}+\left(8+\tfrac{c}{2}\right)\mathbb{E}_4(q)D_q^{(1)}+10c\mathbb{E}_6(q) \ , \\
&\cdots \\
\text{where} & \\
D_q^{(k)}f &= \partial_{(2k-2)}\circ\cdots\circ\partial_{(2)}\circ\partial_{(0)} \ . \\
\partial_{(k)} &= \left(q\partial_q+k\mathbb{E}_2(q)\right) \ .
\end{aligned}$$

Modular recursion

Right hand side:

$$\mathrm{STr}_{\mathcal{V}}\left(o(a_{[-h_a-1]}b)q^{L_0-\frac{c}{24}}\right)=\sum_{k\geqslant 1}{}'(1-2k)\mathbb{E}_{2k}\begin{bmatrix}e^{2\pi i h_a}\\ 1\end{bmatrix}(q)\,\mathrm{Tr}_{\mathcal{V}}\left(o(a_{[2k-h_a]}b)q^{L_0-\frac{c}{24}}\right)\ .$$

Applying formula reduces dimension of the operator whose zero mode appears in the trace, so this will eventually terminate. When stress tensors show up, evaluate them in terms of the same differential operators as for the left hand side.

Twisted Eisenstein series appear when half-integer graded operators are involved.

Thus we get modular differential equations for $\Gamma^0(2)\subset\mathrm{PSL}(2,\mathbb{Z})$ for half-integer cases.

Consequences: Modularity

We are led to an extraordinary claim:

The Schur index of any $\mathcal{N} = 2$ SCFT is a solution of a finite order linear modular differential equation whose coefficients are polynomials in (twisted) Eistenstein series.

Schur indices are thus an enormous source of vector-valued (pseudo-)modular forms of weight zero.

Additionally, as there is a finite-dimensional space of such operators for any given weight, it is a simple matter to test this claim (for a given weight) in examples where the Schur index is known.

Consequences: Modularity

Example: Rank one F-theory SCFTs.

These are theories of single D3 branes probing singular fibers of elliptically fibered K3 in F-theory, labelled by $\mathfrak{a}_0$, $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{d}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$.

Find uniform second order equation,

$$\mathcal{D}^{\mathfrak{g}} = D^{(2)} - 5(h^\vee + 1)(h^\vee - 1)\mathbb{E}_4(q) \ .$$

where for $\mathfrak{a}_0$ we formally set $h^\vee = 6/5$.

Solutions have integer coefficients more generally for $\mathfrak{g}$ in "Deligne-Cvitanović exceptional series":

$$\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{g}_2 \subset \mathfrak{d}_4 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_{7+\frac{1}{2}} \subset \mathfrak{e}_8 \ .$$

No SCFTs for $\mathfrak{f}_4$, $\mathfrak{g}_2$ (or $\mathfrak{e}_{7+\frac{1}{2}}$) yet, though VOAs exist.
(Impossible? *cf.* [Shimizu, Tachikawa, Zafrir (2017)])

Consequences: Modularity

$\mathcal{N}=4$ SYM with low rank $\mathfrak{su}(n)$ gauge algebra

N	$\mathrm{ord}(\mathcal{D})$	Modular Group	Dimensions h_i	Conjugate dimensions $\tilde{h}_i$
2	2	$\Gamma^0(2)$	$-\frac{1}{2}, 0$	$(-\frac{3}{8})_2$
3	4	Γ	$(-1)_3, 0$	——
4	6	$\Gamma^0(2)$	$(-2)_2, (-\frac{3}{2})_3, 0$	$(-\frac{15}{8})_4, (-\frac{7}{8})_2$
5	9	Γ	$(-3)_5, (-2)_3, 0$	——
6	12	$\Gamma^0(2)$	$(-\frac{9}{2})_3, (-4)_5, (-\frac{5}{2})_3, 0$	$(-\frac{35}{8})_6, (-\frac{27}{8})_4, (-\frac{11}{8})_2$
7	16	Γ	$(-6)_7, (-5)_5, (-3)_3, 0$	——

Consequences: Modularity

A_1 theories of class $\mathcal{S}$

$\mathcal{C}_{g,s}$	$\mathrm{ord}(\mathcal{D})$	Modular Group	Indicial roots h_i	Conjugate roots $\tilde{h}_i$	$\dim V_{\mathcal{D}}$
$\mathcal{C}_{0,3}$	1	$\Gamma^0(2)$	0	$(-\frac{1}{2})$	0
$\mathcal{C}_{0,4}$	2	Γ	$-1, 0$	—	0
$\mathcal{C}_{0,5}$	4	$\Gamma^0(2)$	$(-1)_3, 0$	$(-\frac{3}{2}), (-\frac{1}{2})_3$	0
$\mathcal{C}_{0,6}$	6	Γ	$-2, (-1)_4, 0$	—	0
$\mathcal{C}_{0,7}$	13	$\Gamma^0(2)$	$(-2)_5, (-1)_3, 0, (\star)_4$	$-\frac{5}{2}, (-\frac{3}{2})_5, -(\frac{1}{2})_3, (\star)_4$	2
$\mathcal{C}_{0,8}$	16	Γ	$-3, (-2)_6, (-1)_4, 0, (\star)_4$	—	0
$\mathcal{C}_{1,1}$	2	$\Gamma^0(2)$	$-\frac{1}{2}, 0$	$(-\frac{1}{2})_2$	0
$\mathcal{C}_{1,2}$	4	Γ	$(-1)_2, -\frac{1}{3}, 0$	—	0
$\mathcal{C}_{1,3}$	6	$\Gamma^0(2)$	$-\frac{3}{2}, (-1)_3, -\frac{1}{2}, 0$	$(-\frac{3}{2})_2, (-\frac{1}{2})_4$	0
$\mathcal{C}_{1,4}$	9	Γ	$(-2)_2, (-1)_5, (0)_2$	—	0
$\mathcal{C}_{2,0}$	6	Γ	$(-1)_4, (0)_2$	—	0
$\mathcal{C}_{2,1}$	11	$\Gamma^0(2)$	$(-\frac{3}{2})_4, (-1)_3, 0, (\star)_3$	$(-\frac{3}{2})_4, (-\frac{1}{2})_4, (\star)_3$	1

Intepretation of additional solutions

The derivation of the LMDE goes through without trouble for characters of nontrivial modules as long as they have

- Finite-dimensional L_0 weight spaces (or generalized weight spaces).
- Bounded below conformal dimension.

$\mathcal{N} = (2, 2)$ superconformal surface operators necessarily furnish modules for the VOA (*cf.* Clay's talk).

Expect these to fill out the modular representation of the vacuum character.

Warning

The above two conditions are *not* generally necessary for a healthy surface defect.

Intepretation of additional solutions

There may be reasonably nice modules with infinite-dimensional L_0 eigenspaces, but with finite-dimensional weight spaces upon further refining by additional flavor fugacities. (*cf.*, admissible-level affine current algebras).

In this case a couple of things can happen:

- Taking sums and differences of simple characters — treated as analytic functions rather than formal power series — yields a quantity that is finite when flavor fugacity is set to zero.
- A regularization of the singular behavior of the characters at zero flavor fugacity yields a "fake" character that may contain logarithms even if the original characters did not.

Consequences: Cardy behavior

Suppose that under an S-transformation $q \to \tilde{q}$, we have

$$\chi_{\mathcal{V}}(\tilde{q}) = \sum_i \mathcal{S}_{0i} \tilde{\chi}_i(q) \ , \qquad \tilde{\chi}_i(q) \sim q^{-c/24+h_i}(1+\ldots) \ .$$

The $\tilde{\chi}_i(q)$ are the full set of solutions of the modular equation (in the case of $\mathrm{PSL}(2,\mathbb{Z})$) or of the *conjugate modular equation* (in the case of $\Gamma^0(2)$).

This gives us control over the $q \to 1$ limit of the vacuum character,

$$\lim_{\tau\to 0} \log \chi_{\mathcal{V}}(q) \sim \frac{\pi i c_{\text{eff}}}{12\tau} + \ldots \ , \qquad c_{\text{eff}} := c_{2d} - 24 \min_i(\tilde{h}_i) \ .$$

This same limit is controlled by the Weyl anomaly coefficients of the four-dimensional theory by a generalization of arguments of Di Pietro and Komargodski.

$$\lim_{\tau\to 0} \log \mathcal{I}_{\text{Schur}}(q) \sim \frac{4\pi i(c_{4d} - a_{4d})}{\tau} \ .$$

Consequences: Cardy behavior

So the smallest (conjugate) character weight determines the a_{4d}-anomaly:

$$a_{4\mathrm{d}} = \frac{h_{\min}}{2} - \frac{5c_{2d}}{48} \ .$$

Combined with unitarity bounds of Hofman-Maldacena, this gives a constraint,

$$\frac{c_{2d}}{8} \leqslant h_{\min} \leqslant 0 \ .$$

Observation:

Indications that $h_i < 0$ for all non-vacuum characters appearing in the modular orbit as well.

This is already enough to eliminate certain VOAs from consideration as associated VOAs, *e.g.*, $(5,8)$, $(7,11)$, $(8,13)$, and $(9,14)$ Virasoro VOAs just to name a few.
[More extensive results to this effect for affine current algebras due to Cordova & Shao]

A complete example

For $\mathcal{N}=4$ super Yang-Mills with $\mathfrak{su}(2)$ gauge algebra, the associated VOA is the small $\mathcal{N}=4$ superconformal algebra at $c=-9$,

$$
\begin{aligned}
T(z)T(w) &\sim \frac{\frac{-9}{2}}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{T'(w)}{z-w}\ ,\\
J^A(z)J^B(w) &\sim \frac{\frac{-3}{4}\kappa^{AB}}{(z-w)^2}+\frac{f^{AB}{}_C J^C(w)}{z-w}\ ,\\
J^A(z)G^\alpha(z) &\sim \frac{(\sigma^A)_\beta{}^\alpha G^\beta(w)}{z-w}\ ,\\
J^A(z)\tilde{G}^\alpha(z) &\sim \frac{(\sigma^A)_\beta{}^\alpha \tilde{G}^\beta(w)}{z-w}\ ,\\
G^\alpha(z)\tilde{G}^\beta(w) &\sim \frac{-3\varepsilon^{\alpha\beta}}{(z-w)^3}+\frac{-4(\sigma_A)^{\alpha\beta}J^A(w)}{(z-w)^2}+\frac{\varepsilon^{\alpha\beta}T(w)-2(\sigma_A)^{\alpha\beta}J^A(w)}{z-w}\ .
\end{aligned}
$$

For this value of the central charge, $T(z)$ is actually given by the Sugawara construction with the $\mathfrak{su}(2)_R$ currents.

A complete example

There are a variety of null states at dimensions $h = 5/2$ and $h = 3$ encoding various chiral ring relations.

$$
\begin{aligned}
(\mathcal{N}_{JG})^{\alpha} &= \left((\sigma_A)_{\beta}{}^{\alpha} J^A_{-1} G^{\beta}_{-3/2} - \tfrac{1}{2} G^{\alpha}_{-5/2}\right)\Omega\ . \\
(\mathcal{N}_{J\tilde{G}})^{\alpha} &= \left((\sigma_A)_{\beta}{}^{\alpha} J^A_{-1} \tilde{G}^{\beta}_{-3/2} - \tfrac{1}{2} \tilde{G}^{\alpha}_{-5/2}\right)\Omega\ . \\
(\mathcal{N}_{G\tilde{G}})^{A} &= \left((\sigma^A)_{\alpha\beta} G^{\alpha}_{-3/2}\tilde{G}^{\beta}_{-3/2} + 2f^A{}_{BC} J^B_{-2} J^C_{-1} + 2J^A_{-3} - 2L_{-2}J^A_{-1}\right)\Omega\ , \\
\mathcal{N}_{G\tilde{G}} &= \left(\varepsilon_{\alpha\beta} G^{\alpha}_{-3/2}\tilde{G}^{\beta}_{-3/2} + L_{-3}\right)\Omega\ , \\
\mathcal{N}_{GG} &= \varepsilon_{\alpha\beta}\left(G^{\alpha}_{-3/2} G^{\beta}_{-3/2}\right)\Omega\ , \\
\mathcal{N}_{\tilde{G}\tilde{G}} &= \varepsilon_{\alpha\beta}\left(\tilde{G}^{\alpha}_{-3/2}\tilde{G}^{\beta}_{-3/2}\right)\Omega\ ,
\end{aligned}
$$

Along with a dimension $h = 4$ “modular null”:

$$
\mathcal{N}_T = \Big(\,(L_{-2})^2 + \varepsilon_{\alpha\beta}\left(\tilde{G}^{\alpha}_{-5/2} G^{\beta}_{-3/2} - G^{\alpha}_{-5/2}\tilde{G}^{\beta}_{-3/2}\right) - \kappa_{AB}\left(J^A_{-2} J^B_{-2}\right) - \tfrac{1}{2} L_{-4}\Big)\Omega\ .
$$

A complete example

The C_2 algebra is given by

$$\mathcal{R}_{\mathcal{V}} = \mathbb{C}[j^A, \omega^\alpha, \tilde{\omega}^\alpha, t]/\mathcal{I} \ ,$$

with

$$\mathcal{I} = \langle 2(j \otimes j)_{\mathbf{0}} - t, t^2, (j \otimes \omega)_{\mathbf{1/2}}, (j \otimes \tilde{\omega})_{\mathbf{1/2}}, (\omega \otimes \tilde{\omega}), (\omega \otimes \omega), (\tilde{\omega} \otimes \tilde{\omega})\rangle$$

We recover the Higgs chiral ring by removing the nilradical,

$$\mathcal{R}_{\mathcal{H}} = \mathbb{C}^2/\mathbb{Z}_2 \ .$$

The Poisson bracket comes along as well.

A complete example

The modular null (responsible for $t^2 = 0$) gives rise to a second order modular differential operator annihilating the vacuum character,

$$\mathcal{D}^{\mathcal{N}=4}_{\mathfrak{su}(2)} = D_q^{(2)} - 2\mathbb{E}_2\left[\begin{smallmatrix}-1\\+1\end{smallmatrix}\right](\tau)D_q^{(1)} - 18\mathbb{E}_4(\tau) + 18\mathbb{E}_4\left[\begin{smallmatrix}-1\\+1\end{smallmatrix}\right](\tau)\ .$$

The second character annihilated by this operator has $h = -1/2$ and is logarithmic. However this logarithm is resolved upon including flavor fugacities.
[C.B., W. Peelaers]

The conjugate differential operator whose solutions control the $q \to 1$ limit is

$$\widetilde{\mathcal{D}}^{\mathcal{N}=4}_{\mathfrak{su}(2)} = D_q^{(2)} - 2\mathbb{E}_2\left[\begin{smallmatrix}+1\\-1\end{smallmatrix}\right](\tau)D_q^{(1)} - 18\mathbb{E}_4(\tau) + 18\mathbb{E}_4\left[\begin{smallmatrix}+1\\-1\end{smallmatrix}\right](\tau)\ .$$

This has solutions with $\tilde{h}_{\min} = -3/8$, which correctly reproduces the $a_{4d} = 3/4$ Weyl anomaly.

Conclusions

Extensions

- Can include flavor fugacities and get differential equations for flavored indices.
  This can resolve issue of finite-dim'l weight spaces for modules.
- Modify recursion relations to account for global symmetry twists.
  Many surface operators, including canonical surface operators in class $\mathcal{S}$, give rise to twisted modules.

Open Questions

- How to predict the order of the LMDE/dimension of modular representation?
  Is it meaningful? The flavor-refined case is apparently related to the three-dimensional Coulomb branch [Fredrickson, Pei, W. Yan, Ye (2017)], [Neitzke, F. Yan]
- Can we extract the HL chiral ring from the VOA?
- Intrinsic way to find the R-filtration/grading on $\mathcal{V}$?
  Partial progress by J. Song (2016), work ongoing.
  This would allow the imposition of very strong constraints from unitarity.

Danke!