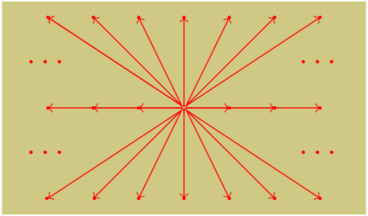
Riemann-Hilbert problems from Donaldson-Thomas theory

Tom Bridgeland

University of Sheffield



Preprints: 1611.03697 and 1703.02776.

Motivation

Motivation

Two types of parameters in string theory:

(I) Deformation parameters.

(II) Stability parameters.

Exchanged by mirror symmetry.

Motivation

Two types of parameters in string theory:

(i) Deformation parameters.

(ii) Stability parameters.

Exchanged by mirror symmetry.

The deformation space carries a variation of Hodge structures.

Can one construct similar geometric structures on stability space?

Motivation

Two types of parameters in string theory:

(I) Deformation parameters.

(II) Stability parameters.

Exchanged by mirror symmetry.

The deformation space carries a variation of Hodge structures.

Can one construct similar geometric structures on stability space?

The obvious data to use is Donaldson-Thomas invariants.

Motivation

Two types of parameters in string theory:

(i) Deformation parameters.

(ii) Stability parameters.

Exchanged by mirror symmetry.

The deformation space carries a variation of Hodge structures.

Can one construct similar geometric structures on stability space?

The obvious data to use is Donaldson-Thomas invariants.

Fundamental property: Kontsevich-Soibelman wall-crossing formula.

Strong analogy with Stokes factors from differential equations.

1. BPS structures.

The output of (unrefined) DT theory

The output of (unrefined) DT theory

A BPS structure (Γ, Z, Ω) consists of

(A) An abelian group $\Gamma \cong \mathbb{Z}^{\oplus n}$ with a skew-symmetric form

$$\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}$$

(B) A homomorphism of abelian groups $Z \colon \Gamma \to \mathbb{C}$,

(C) A map of sets $\Omega \colon \Gamma \to \mathbb{Q}$.

The output of (unrefined) DT theory

A BPS structure (Γ, Z, Ω) consists of

(A) An abelian group $\Gamma \cong \mathbb{Z}^{\oplus n}$ with a skew-symmetric form

$$\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}$$

(B) A homomorphism of abelian groups $Z \colon \Gamma \to \mathbb{C}$,

(C) A map of sets $\Omega \colon \Gamma \to \mathbb{Q}$.

satisfying the conditions:

(I) Symmetry: $\Omega(-\gamma) = \Omega(\gamma)$ for all $\gamma \in \Gamma$,

(II) Support property: fixing a norm $\| \cdot \|$ on the finite-dimensional vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, there is a $C > 0$ such that

$$\Omega(\gamma) \neq 0 \implies |Z(\gamma)| > C \cdot \|\gamma\|.$$

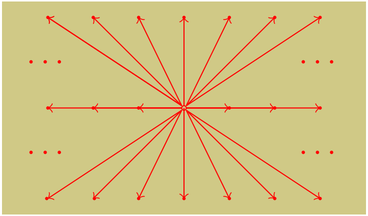
Example: conifold BPS structure

Example: conifold BPS structure

Take $\Gamma = \mathbb{Z}^{\oplus 2}$ with $\langle -, - \rangle = 0$ and $Z(r, d) = ir - d$. Set

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

This arises from DT theory applied to the resolved conifold.



Poisson algebraic torus

Poisson algebraic torus

Consider the algebraic torus with character lattice Γ:

$$\mathbb{T}_+ = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

$$\mathbb{C}[\mathbb{T}_+] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma \cong \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm n}].$$

The form $\langle -, - \rangle$ induces an invariant Poisson structure on $\mathbb{T}_+$:

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_\alpha \cdot x_\beta.$$

Poisson algebraic torus

Consider the algebraic torus with character lattice Γ:

$$\mathbb{T}_+ = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

$$\mathbb{C}[\mathbb{T}_+] = \bigoplus_{\gamma \in \Gamma} \mathbb{C} \cdot x_\gamma \cong \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm n}].$$

The form $\langle -, - \rangle$ induces an invariant Poisson structure on $\mathbb{T}_+$:

$$\{x_\alpha, x_\beta\} = \langle \alpha, \beta \rangle \cdot x_\alpha \cdot x_\beta.$$

More precisely we should work with an associated torsor

$$\mathbb{T}_- = \{g : \Gamma \to \mathbb{C}^* : g(\gamma_1 + \gamma_2) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} g(\gamma_1) \cdot g(\gamma_2)\},$$

which we call the twisted torus.

DT Hamiltonians

DT Hamiltonians

The DT invariants $\mathrm{DT}(\gamma) \in \mathbb{Q}$ of a BPS structure are defined by

$$\mathrm{DT}(\gamma) = \sum_{\gamma = n\alpha} \frac{\Omega(\alpha)}{n^2}.$$

For any ray $\ell = \mathbb{R}_{>0} \cdot z \subset \mathbb{C}^*$ we consider the generating function

$$\mathrm{DT}(\ell) = \sum_{Z(\gamma) \in \ell} \mathrm{DT}(\gamma) \cdot x_\gamma.$$

A ray $\ell \subset \mathbb{C}^*$ is called active if this expression is nonzero.

DT Hamiltonians

The DT invariants $\mathrm{DT}(\gamma) \in \mathbb{Q}$ of a BPS structure are defined by

$$\mathrm{DT}(\gamma) = \sum_{\gamma = n\alpha} \frac{\Omega(\alpha)}{n^2}.$$

For any ray $\ell = \mathbb{R}_{>0} \cdot z \subset \mathbb{C}^*$ we consider the generating function

$$\mathrm{DT}(\ell) = \sum_{Z(\gamma) \in \ell} \mathrm{DT}(\gamma) \cdot x_\gamma.$$

A ray $\ell \subset \mathbb{C}^*$ is called active if this expression is nonzero.

We would like to think of the time 1 Hamiltonian flow of the function $\mathrm{DT}(\ell)$ as defining a Poisson automorphism $\mathrm{S}(\ell)$ of the torus $\mathbb{T}$.

Making sense of S(ℓ)

Making sense of $\mathrm{S}(\ell)$

Formal approach

Restrict to classes γ lying in a positive cone $\Gamma^+ \subset \Gamma$, consider

$$\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \supset \mathbb{C}[x_1, \cdots, x_n] \subset \mathbb{C}[[x_1, \cdots, x_n]],$$

and the automorphism $\mathrm{S}(\ell)^* = \exp\{\mathsf{DT}(\ell), -\}$ of this completion.

Making sense of $\mathrm{S}(\ell)$

Formal approach

Restrict to classes γ lying in a positive cone $\Gamma^+ \subset \Gamma$, consider

$$\mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \supset \mathbb{C}[x_1, \cdots, x_n] \subset \mathbb{C}[[x_1, \cdots, x_n]],$$

and the automorphism $\mathrm{S}(\ell)^* = \exp\{\mathsf{DT}(\ell), -\}$ of this completion.

Analytic approach

Restrict attention to BPS structures which are convergent:

$$\exists R > 0 \text{ such that } \sum_{\gamma \in \Gamma} |\Omega(\gamma)| \cdot e^{-R|Z(\gamma)|} < \infty.$$

Then on suitable analytic open subsets of $\mathbb{T}$ the sum $\mathsf{DT}(\ell)$ is absolutely convergent and its time 1 Hamiltonian flow $\mathrm{S}(\ell)$ exists.

Birational transformations

Often the maps $\mathrm{S}(\ell)$ are birational automorphisms of $\mathbb{T}$.

Birational transformations

Often the maps $\mathrm{S}(\ell)$ are birational automorphisms of $\mathbb{T}$. Note

$$\exp\left\{\sum_{n\geq 1}\frac{x_{n\gamma}}{n^2},-\right\}(x_\beta)=x_\beta\cdot(1-x_\gamma)^{\langle\beta,\gamma\rangle}.$$

Birational transformations

Often the maps $\mathrm{S}(\ell)$ are birational automorphisms of $\mathbb{T}$. Note

$$\exp\left\{\sum_{n\geq 1}\frac{x_{n\gamma}}{n^2},-\right\}(x_\beta)=x_\beta\cdot(1-x_\gamma)^{\langle\beta,\gamma\rangle}.$$

Whenever a ray $\ell\subset\mathbb{C}^*$ satisfies

(I) only finitely many active classes have $Z(\gamma_i)\in\ell$,

(II) these classes are mutually orthogonal $\langle\gamma_i,\gamma_j\rangle=0$,

(III) the corresponding BPS invariants $\Omega(\gamma_i)\in\mathbb{Z}$.

there is a formula

$$\mathrm{S}(\ell)^*(x_\beta)=\prod_{Z(\gamma)\in\ell}(1-x_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}.$$

Variation of BPS structures

Variation of BPS structures

A framed variation of BPS structures over a complex manifold S is a collection of BPS structures (Γ, Z_s, Ω_s) indexed by $s \in S$ such that

Variation of BPS structures

A framed variation of BPS structures over a complex manifold S is a collection of BPS structures (Γ, Z_s, Ω_s) indexed by $s \in S$ such that

(I) The numbers $Z_s(\gamma) \in \mathbb{C}$ vary holomorphically.

Variation of BPS structures

A framed variation of BPS structures over a complex manifold S is a collection of BPS structures (Γ, Z_s, Ω_s) indexed by $s \in S$ such that

(i) The numbers $Z_s(\gamma) \in \mathbb{C}$ vary holomorphically.

(ii) For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise ordered product

$$\mathrm{S}_s(\Delta) = \prod_{\ell \in \Delta} \mathrm{S}_s(\ell) \in \mathsf{Aut}(\mathbb{T})$$

is constant whenever the boundary of Δ remains non-active.

Part (ii) is the Kontsevich-Soibelman wall-crossing formula.

Variation of BPS structures

A framed variation of BPS structures over a complex manifold S is a collection of BPS structures (Γ, Z_s, Ω_s) indexed by $s \in S$ such that

(I) The numbers $Z_s(\gamma) \in \mathbb{C}$ vary holomorphically.

(II) For any convex sector $\Delta \subset \mathbb{C}^*$ the clockwise ordered product

$$\mathrm{S}_s(\Delta) = \prod_{\ell \in \Delta} \mathrm{S}_s(\ell) \in \mathrm{Aut}(\mathbb{T})$$

is constant whenever the boundary of Δ remains non-active.

Part (ii) is the Kontsevich-Soibelman wall-crossing formula.

The complete set of numbers $\Omega_s(\gamma)$ at some point $s \in S$ determines them for all other points $s \in S$.

Example: the A_2 case

Example: the A_2 case

Let $\Gamma = \mathbb{Z}^{\oplus 2} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\langle e_1, e_2 \rangle = 1$. Then

$$\mathbb{C}[\mathbb{T}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}], \quad \{x_1, x_2\} = x_1 \cdot x_2.$$

Example: the A_2 case

Let $\Gamma = \mathbb{Z}^{\oplus 2} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\langle e_1, e_2 \rangle = 1$. Then

$$\mathbb{C}[\mathbb{T}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}], \quad \{x_1, x_2\} = x_1 \cdot x_2.$$

A central charge $Z\colon \Gamma \to \mathbb{C}$ is determined by $z_i = Z(e_i)$. Take

$$S = \mathfrak{h}^2 = \{(z_1, z_2) : z_i \in \mathfrak{h}\}.$$

Example: the A_2 case

Let $\Gamma = \mathbb{Z}^{\oplus 2} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with $\langle e_1, e_2 \rangle = 1$. Then

$$\mathbb{C}[\mathbb{T}] = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}], \quad \{x_1, x_2\} = x_1 \cdot x_2.$$

A central charge $Z \colon \Gamma \to \mathbb{C}$ is determined by $z_i = Z(e_i)$. Take

$$S = \mathfrak{h}^2 = \{(z_1, z_2) : z_i \in \mathfrak{h}\}.$$

Define BPS invariants as follows:

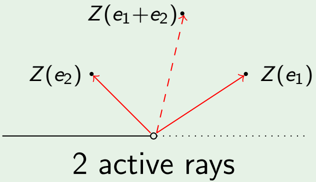
(A) $\operatorname{Im}(z_2/z_1) > 0$. Set $\Omega(\pm e_1) = \Omega(\pm e_2) = 1$, all others zero.

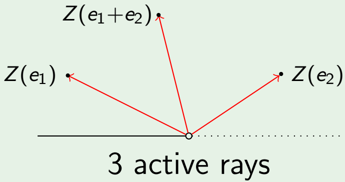
(B) $\operatorname{Im}(z_2/z_1) < 0$. Set $\Omega(\pm e_1) = \Omega(\pm(e_1 + e_2)) = \Omega(\pm e_2) = 1$.

Wall-crossing formula: A_2 case

Wall-crossing formula: A_2 case

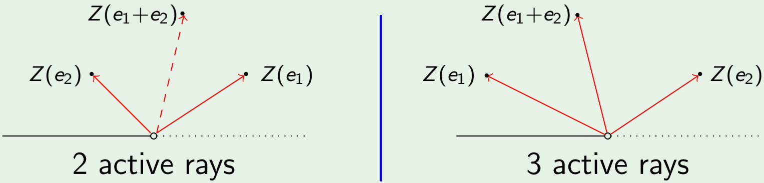
Two types of BPS structures appear, as illustrated below





Wall-crossing formula: A_2 case

Two types of BPS structures appear, as illustrated below



The wall-crossing formula is the cluster pentagon identity

$$C_{(0,1)} \circ C_{(1,0)} = C_{(1,0)} \circ C_{(1,1)} \circ C_{(0,1)}.$$

$$C_\alpha \colon x_\beta \mapsto x_\beta \cdot (1 - x_\alpha)^{\langle \alpha, \beta \rangle}.$$

2. The Riemann-Hilbert problem.

The Riemann-Hilbert problem

Fix a BPS structure (Γ, Z, Ω) and a point $\xi \in \mathbb{T}$.

The Riemann-Hilbert problem

Fix a BPS structure (Γ, Z, Ω) and a point $\xi \in \mathbb{T}$.

Find a piecewise holomorphic function $\Phi\colon \mathbb{C}^* \to \mathbb{T}$ satisfying:

The Riemann-Hilbert problem

Fix a BPS structure (Γ, Z, Ω) and a point $\xi \in \mathbb{T}$.

Find a piecewise holomorphic function $\Phi \colon \mathbb{C}^* \to \mathbb{T}$ satisfying:

(i) (Jumping): When t crosses an active ray ℓ clockwise,

$$\Phi(t) \mapsto \mathrm{S}(\ell)(\Phi(t)).$$

The Riemann-Hilbert problem

Fix a BPS structure (Γ, Z, Ω) and a point $\xi \in \mathbb{T}$.

Find a piecewise holomorphic function $\Phi \colon \mathbb{C}^* \to \mathbb{T}$ satisfying:

(I) (Jumping): When t crosses an active ray ℓ clockwise,

$$\Phi(t) \mapsto \mathrm{S}(\ell)(\Phi(t)).$$

(II) (Limit at 0): Write $\Phi_\gamma(t)) = x_\gamma(\Phi(t))$. As $t \to 0$,

$$\Phi_\gamma(t) \cdot e^{Z(\gamma)/t} \to x_\gamma(\xi).$$

The Riemann-Hilbert problem

Fix a BPS structure (Γ, Z, Ω) and a point $\xi \in \mathbb{T}$.

Find a piecewise holomorphic function $\Phi\colon \mathbb{C}^* \to \mathbb{T}$ satisfying:

(I) (Jumping): When t crosses an active ray ℓ clockwise,

$$\Phi(t) \mapsto \mathrm{S}(\ell)(\Phi(t)).$$

(II) (Limit at 0): Write $\Phi_\gamma(t)) = x_\gamma(\Phi(t))$. As $t \to 0$,

$$\Phi_\gamma(t) \cdot e^{Z(\gamma)/t} \to x_\gamma(\xi).$$

(III) (Growth at ∞): For any $\gamma \in \Gamma$ there exists $k > 0$ with

$$|t|^{-k} < |\Phi_\gamma(t)| < |t|^k \text{ as } t \to \infty.$$

THE A_1 EXAMPLE

The A_1 example

Consider the following BPS structure

(i) The lattice $\Gamma = \mathbb{Z} \cdot \gamma$ is one-dimensional. Thus $\langle -, - \rangle = 0$.

(ii) The central charge $Z \colon \Gamma \to \mathbb{C}$ is determined by $z = Z(\gamma) \in \mathbb{C}^*$,

(iii) The only non-vanishing BPS invariants are $\Omega(\pm\gamma) = 1$.

The A_1 example

Consider the following BPS structure

(i) The lattice $\Gamma = \mathbb{Z} \cdot \gamma$ is one-dimensional. Thus $\langle -, - \rangle = 0$.

(ii) The central charge $Z \colon \Gamma \to \mathbb{C}$ is determined by $z = Z(\gamma) \in \mathbb{C}^*$,

(iii) The only non-vanishing BPS invariants are $\Omega(\pm\gamma) = 1$.

Then $\mathbb{T} = \mathbb{C}^*$ and all automorphisms $\mathrm{S}(\ell)$ are the identity.

$$\Phi_\gamma(t) = \xi \cdot \exp(-z/t) \in \mathbb{T} = \mathbb{C}^*.$$

The A_1 example

Consider the following BPS structure

(I) The lattice $\Gamma = \mathbb{Z} \cdot \gamma$ is one-dimensional. Thus $\langle -, - \rangle = 0$.

(II) The central charge $Z \colon \Gamma \to \mathbb{C}$ is determined by $z = Z(\gamma) \in \mathbb{C}^*$,

(III) The only non-vanishing BPS invariants are $\Omega(\pm\gamma) = 1$.

Then $\mathbb{T} = \mathbb{C}^*$ and all automorphisms $\mathrm{S}(\ell)$ are the identity.

$$\Phi_\gamma(t) = \xi \cdot \exp(-z/t) \in \mathbb{T} = \mathbb{C}^*.$$

Now double the BPS structure: take the lattice $\Gamma \oplus \Gamma^\vee$ with canonical skew form, and extend Z and Ω by zero. Consider

$$y(t) = \Phi_{\gamma^\vee}(t) \colon \mathbb{C}^* \to \mathbb{C}^*.$$

Doubled A_1 case

Doubled A_1 case

Consider the case $\xi = 1$. The map $y\colon \mathbb{C}^* \to \mathbb{C}^*$ should satisfy

Doubled A_1 case

Consider the case $\xi = 1$. The map $y\colon \mathbb{C}^* \to \mathbb{C}^*$ should satisfy

(I) y is holomorphic away from the rays $\mathbb{R}_{>0} \cdot (\pm z)$ and has jumps

$$y(t) \mapsto y(t) \cdot (1 - x(t)^{\pm 1})^{\pm 1}, \quad x(t) = \exp(-z/t),$$

as t moves clockwise across them.

Doubled A_1 case

Consider the case $\xi = 1$. The map $y\colon \mathbb{C}^* \to \mathbb{C}^*$ should satisfy

(I) y is holomorphic away from the rays $\mathbb{R}_{>0} \cdot (\pm z)$ and has jumps

$$y(t) \mapsto y(t) \cdot (1 - x(t)^{\pm 1})^{\pm 1}, \quad x(t) = \exp(-z/t),$$

as t moves clockwise across them.

(II) $y(t) \to 1$ as $t \to 0$.

Doubled A_1 case

Consider the case $\xi = 1$. The map $y\colon \mathbb{C}^* \to \mathbb{C}^*$ should satisfy

(I) y is holomorphic away from the rays $\mathbb{R}_{>0} \cdot (\pm z)$ and has jumps

$$y(t) \mapsto y(t) \cdot (1 - x(t)^{\pm 1})^{\pm 1}, \quad x(t) = \exp(-z/t),$$

as t moves clockwise across them.

(II) $y(t) \to 1$ as $t \to 0$.

(III) there exists $k > 0$ such that

$$|t|^{-k} < |y(t)| < |t|^{k} \text{ as } t \to \infty.$$

Solution: the Gamma function

Solution: the Gamma function

The doubled A_1 problem has the unique solution

$$y(t) = \Delta\left(\frac{\pm z}{2\pi i t}\right)^{\mp 1} \quad \text{where} \quad \Delta(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w-\frac{1}{2}}},$$

in the half-planes $\pm \operatorname{Im}(t/z) > 0$.

Solution: the Gamma function

The doubled A_1 problem has the unique solution

$$y(t) = \Delta\left(\frac{\pm z}{2\pi i t}\right)^{\mp 1} \quad \text{where} \quad \Delta(w) = \frac{e^w \cdot \Gamma(w)}{\sqrt{2\pi} \cdot w^{w-\frac{1}{2}}},$$

in the half-planes $\pm \operatorname{Im}(t/z) > 0$.

This is elementary: all you need is

$$\Gamma(w) \cdot \Gamma(1-w) = \frac{\pi}{\sin(\pi w)}, \quad \Gamma(w+1) = w \cdot \Gamma(w),$$

$$\log \Delta(w) \sim \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g-1)} w^{1-2g}.$$

The tau function

The tau function

Suppose given a framed variation of BPS structures (Γ, Z_p, Ω_p) over a complex manifold S such that

$$\pi \colon S \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) = \mathbb{C}^n, \quad s \mapsto Z_s,$$

is a local isomorphism. Taking a basis $(\gamma_1, \cdots, \gamma_n) \subset \Gamma$ we get local co-ordinates $z_i = Z_s(\gamma_i)$ on S.

The tau function

Suppose given a framed variation of BPS structures (Γ, Z_p, Ω_p) over a complex manifold S such that

$$\pi\colon S \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) = \mathbb{C}^n, \quad s \mapsto Z_s,$$

is a local isomorphism. Taking a basis $(\gamma_1, \cdots, \gamma_n) \subset \Gamma$ we get local co-ordinates $z_i = Z_s(\gamma_i)$ on S.

Suppose we are given analytically varying solutions $\Phi_\gamma(z_i, t)$ to the Riemann-Hilbert problems associated to (Γ, Z_s, Ω_s).

The tau function

Suppose given a framed variation of BPS structures (Γ, Z_p, Ω_p) over a complex manifold S such that

$$\pi\colon S \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) = \mathbb{C}^n, \quad s \mapsto Z_s,$$

is a local isomorphism. Taking a basis $(\gamma_1, \cdots, \gamma_n) \subset \Gamma$ we get local co-ordinates $z_i = Z_s(\gamma_i)$ on S.

Suppose we are given analytically varying solutions $\Phi_\gamma(z_i, t)$ to the Riemann-Hilbert problems associated to (Γ, Z_s, Ω_s).

Define a function $\tau = \tau(z_i, t)$ by the relation

$$\frac{\partial}{\partial t} \log \Phi_{\gamma_k}(z_i, t) = \sum_{j=1}^{n} \epsilon_{jk} \frac{\partial}{\partial z_j} \log \tau(z_i, t), \quad \epsilon_{jk} = \langle \gamma_j, \gamma_k \rangle.$$

Solution in uncoupled case

Solution in uncoupled case

In the A_1 case the τ-function is essentially the Barnes G-function.

$$\log \tau(z, t) \sim \sum_{g \geq 1} \frac{B_{2g}}{2g(2g-2)} \left(\frac{2\pi i t}{z} \right)^{2g-2}.$$

Solution in uncoupled case

In the A_1 case the τ-function is essentially the Barnes G-function.

$$\log \tau(z, t) \sim \sum_{g \geq 1} \frac{B_{2g}}{2g(2g-2)} \left(\frac{2\pi i t}{z} \right)^{2g-2}.$$

Whenever our BPS structures are uncoupled

$$\Omega(\gamma_i) \neq 0 \implies \langle \gamma_1, \gamma_2 \rangle = 0,$$

we can try to solve the RH problem by superposition of A_1 solutions. This works precisely if only finitely many $\Omega(\gamma) \neq 0$.

Solution in uncoupled case

In the A_1 case the τ-function is essentially the Barnes G-function.

$$\log \tau(z,t) \sim \sum_{g \geq 1} \frac{B_{2g}}{2g(2g-2)} \left(\frac{2\pi i t}{z} \right)^{2g-2}.$$

Whenever our BPS structures are uncoupled

$$\Omega(\gamma_i) \neq 0 \implies \langle \gamma_1, \gamma_2 \rangle = 0,$$

we can try to solve the RH problem by superposition of A_1 solutions. This works precisely if only finitely many $\Omega(\gamma) \neq 0$.

$$\log \tau(z,t) \sim \sum_{g \geq 1} \sum_{\gamma \in \Gamma} \frac{\Omega(\gamma) \cdot B_{2g}}{2g(2g-2)} \left(\frac{2\pi i t}{Z(\gamma)} \right)^{2g-2}$$

GEOMETRIC CASE: CURVES ON A CY_3

Can apply this to coherent sheaves on a compact Calabi-Yau threefold supported in dimension ≤ 1.

Geometric case: curves on a CY_3

Can apply this to coherent sheaves on a compact Calabi-Yau threefold supported in dimension ≤ 1. We have

$$\Gamma = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}, \quad Z(\beta, n) = 2\pi(\beta \cdot \omega_{\mathbb{C}} - n).$$

$$\Omega(\beta, n) = \mathrm{GV}_0(\beta), \quad \Omega(0, n) = -\chi(X).$$

Geometric case: curves on a CY_3

Can apply this to coherent sheaves on a compact Calabi-Yau threefold supported in dimension ≤ 1. We have

$$\Gamma = H_2(X, \mathbb{Z}) \oplus \mathbb{Z}, \quad Z(\beta, n) = 2\pi(\beta \cdot \omega_{\mathbb{C}} - n).$$

$$\Omega(\beta, n) = \mathrm{GV}_0(\beta), \quad \Omega(0, n) = -\chi(X).$$

Since $\chi(-,-) = 0$ these BPS structures are uncoupled.

Geometric case: curves on a CY_3

Can apply this to coherent sheaves on a compact Calabi-Yau threefold supported in dimension ≤ 1. We have

$$\Gamma = H_2(X,\mathbb{Z}) \oplus \mathbb{Z}, \quad Z(\beta,n) = 2\pi(\beta\cdot\omega_{\mathbb{C}} - n).$$

$$\Omega(\beta,n) = \mathrm{GV}_0(\beta), \quad \Omega(0,n) = -\chi(X).$$

Since $\chi(-,-) = 0$ these BPS structures are uncoupled.

$$\tau(\omega_{\mathbb{C}},t) \overset{pos.\ deg}{\sim} \sum_{g\geq 2} \frac{\chi(X)\, B_{2g}\, B_{2g-2}}{4g\,(2g-2)\,(2g-2)!} \cdot (2\pi t)^{2g-2}$$

$$+ \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{k\geq 1} \mathrm{GV}_0(\beta) \frac{e^{2\pi i\omega\cdot k\beta}}{4k} \sin^{-2}(i\pi tk).$$

Matches degenerate contributions from genus 0 GV invariants.

Resolved conifold again

Resolved conifold again

Take $\Gamma = \mathbb{Z}^{\oplus 2}$ with $\langle -, - \rangle = 0$ and

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Resolved conifold again

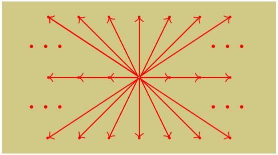
Take $\Gamma = \mathbb{Z}^{\oplus 2}$ with $\langle -, - \rangle = 0$ and

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm(1, d) \text{ for some } d \in \mathbb{Z}, \\ -2 & \text{if } \gamma = (0, d) \text{ for some } 0 \neq d \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

We get a variation of BPS structures over

$$\left\{ (v, w) \in \mathbb{C}^2 : w \neq 0 \text{ and } v + dw \neq 0 \text{ for all } d \in \mathbb{Z} \right\} \subset \mathbb{C}^2$$

by setting $Z(r, d) = rv + dw$.



Non-perturbative partition function

The corresponding RH problems have unique solutions, which can be written explicitly in terms of Barnes double and triple sine functions.

Non-perturbative partition function

The corresponding RH problems have unique solutions, which can be written explicitly in terms of Barnes double and triple sine functions.

$$\tau(v,w,t) = H(v,w,t)\cdot\exp(R(v,w,t)),$$

$$H(v,w,t) = \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{vs}-1}{e^{ws}-1}\cdot\frac{e^{ts}}{(e^{ts}-1)^2}\cdot\frac{ds}{s}\right),$$

$$R(v,w,t) = \Big(\frac{w}{2\pi it}\Big)^2\big(\operatorname{Li}_3(e^{2\pi iv/w}) - \zeta(3)\big) + \frac{i\pi}{12}\cdot\frac{v}{w}.$$

The function H is a non-perturbative closed-string partition function.

Finite-dimensional analogy

Finite-dimensional analogy

Matrix differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$

$$\frac{d}{dt}X(t) = \left(\frac{U}{t^2} + \frac{V}{t}\right)X(t), \quad U, V \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}).$$

Take $U = \mathrm{diag}(u_1, \cdots, u_n)$ with $u_i \neq u_j$ and V skew-symmetric.

Finite-dimensional analogy

Matrix differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$

$$\frac{d}{dt}X(t) = \left(\frac{U}{t^2} + \frac{V}{t}\right)X(t), \quad U, V \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}).$$

Take $U = \mathrm{diag}(u_1, \cdots, u_n)$ with $u_i \neq u_j$ and V skew-symmetric.

The Stokes rays are $\ell_{ij} = \mathbb{R}_{>0} \cdot (u_i - u_j)$.

Fact: in any half-plane centered on a non-Stokes ray there exists a unique solution such that $X(t) \cdot \exp(U/t) \to 1$ as $t \to 0$.

Finite-dimensional analogy

Matrix differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$

$$\frac{d}{dt}X(t) = \left(\frac{U}{t^2} + \frac{V}{t}\right)X(t), \quad U, V \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}).$$

Take $U = \mathrm{diag}(u_1, \cdots, u_n)$ with $u_i \neq u_j$ and V skew-symmetric.

The Stokes rays are $\ell_{ij} = \mathbb{R}_{>0} \cdot (u_i - u_j)$.

Fact: in any half-plane centered on a non-Stokes ray there exists a unique solution such that $X(t) \cdot \exp(U/t) \to 1$ as $t \to 0$.

The Stokes factors $S_{\ell_{ij}} \in G$ describe how these solutions jump.

Finite-dimensional analogy

Matrix differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{GL}_n(\mathbb{C})$

$$\frac{d}{dt}X(t) = \left(\frac{U}{t^2} + \frac{V}{t}\right)X(t), \quad U, V \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}).$$

Take $U = \mathrm{diag}(u_1, \cdots, u_n)$ with $u_i \neq u_j$ and V skew-symmetric.

The Stokes rays are $\ell_{ij} = \mathbb{R}_{>0} \cdot (u_i - u_j)$.

Fact: in any half-plane centered on a non-Stokes ray there exists a unique solution such that $X(t) \cdot \exp(U/t) \to 1$ as $t \to 0$.

The Stokes factors $S_{\ell_{ij}} \in G$ describe how these solutions jump.

Iso-Stokes deformation: as U varies we can vary V in a unique way so that the product of Stokes factors in any fixed sector is constant.

Wall-crossing formula = iso-Stokes

Wall-crossing formula = iso-Stokes

Differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{Aut}(\mathbb{T})$

$$\frac{d}{dt}X(t) = \left(\frac{Z}{t^2} + \frac{F}{t}\right)X(t),$$

with $Z \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and $F = \sum_{\gamma\in\Gamma} f_\gamma \cdot (x_\gamma + x_{-\gamma})$, where

$$\mathfrak{g} = \mathrm{Vect}(\mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \oplus \bigoplus_{\gamma\in\Gamma} \mathbb{C} \cdot x_\gamma.$$

Wall-crossing formula = iso-Stokes

Differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{Aut}(\mathbb{T})$

$$\frac{d}{dt}X(t) = \left(\frac{Z}{t^2} + \frac{F}{t}\right)X(t),$$

with $Z \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and $F = \sum_{\gamma\in\Gamma} f_\gamma \cdot (x_\gamma + x_{-\gamma})$, where

$$\mathfrak{g} = \mathrm{Vect}(\mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \oplus \bigoplus_{\gamma\in\Gamma} \mathbb{C}\cdot x_\gamma.$$

Have Stokes rays are $\mathbb{R}_{>0}\cdot Z(\gamma)$ and Stokes factors $\mathrm{S}(\ell) \in G$.

Wall-crossing formula = iso-Stokes

Differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{Aut}(\mathbb{T})$

$$\frac{d}{dt}X(t) = \left(\frac{Z}{t^2} + \frac{F}{t}\right)X(t),$$

with $Z \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and $F = \sum_{\gamma\in\Gamma} f_\gamma \cdot (x_\gamma + x_{-\gamma})$, where

$$\mathfrak{g} = \mathrm{Vect}(\mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \oplus \bigoplus_{\gamma\in\Gamma} \mathbb{C}\cdot x_\gamma.$$

Have Stokes rays are $\mathbb{R}_{>0}\cdot Z(\gamma)$ and Stokes factors $\mathrm{S}(\ell) \in G$.

Wall-crossing formula is iso-Stokes condition.

Wall-crossing formula = iso-Stokes

Differential equation for $X\colon \mathbb{C}^* \to G = \mathrm{Aut}(\mathbb{T})$

$$\frac{d}{dt}X(t) = \left(\frac{Z}{t^2} + \frac{F}{t}\right)X(t),$$

with $Z \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C})$ and $F = \sum_{\gamma\in\Gamma} f_\gamma \cdot (x_\gamma + x_{-\gamma})$, where

$$\mathfrak{g} = \mathrm{Vect}(\mathbb{T}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}) \oplus \bigoplus_{\gamma\in\Gamma} \mathbb{C}\cdot x_\gamma.$$

Have Stokes rays are $\mathbb{R}_{>0}\cdot Z(\gamma)$ and Stokes factors $\mathrm{S}(\ell) \in G$.

Wall-crossing formula is iso-Stokes condition.

Note that given $\xi \in \mathbb{T}$ there is a map $\mathrm{eval}_\xi\colon G \to \mathbb{T}$.

Further directions

(i) Theories of class S with $G = \mathrm{SL}_2(\mathbb{C})$. Variation of BPS structures over space of meromorphic quadratic differentials. Monodromy of projective structures gives map to space of framed local systems. Fock-Goncharov co-ordinates give solutions to RH problem (joint with D. Allegretti).

Further directions

(i) Theories of class S with $G = \mathrm{SL}_2(\mathbb{C})$. Variation of BPS structures over space of meromorphic quadratic differentials. Monodromy of projective structures gives map to space of framed local systems. Fock-Goncharov co-ordinates give solutions to RH problem (joint with D. Allegretti).

(ii) Analogy with Stokes data in finite-dimensional case. Allow ξ to vary to get RH problem with values in $G = \mathrm{Aut}(\mathbb{T})$. Uncoupled case can be solved following Gaiotto (joint with A. Barbieri).

Further directions

(i) Theories of class S with $G = \mathrm{SL}_2(\mathbb{C})$. Variation of BPS structures over space of meromorphic quadratic differentials. Monodromy of projective structures gives map to space of framed local systems. Fock-Goncharov co-ordinates give solutions to RH problem (joint with D. Allegretti).

(ii) Analogy with Stokes data in finite-dimensional case. Allow ξ to vary to get RH problem with values in $G = \mathrm{Aut}(\mathbb{T})$. Uncoupled case can be solved following Gaiotto (joint with A. Barbieri).

(iii) Our current formalism gives the partition function without the terms in t^{2g-2} for $g = 0, 1$. In examples, these additional terms make τ satisfy a difference equation. How to understand this? Can we quantize the RH problem? (J. Calabrese).