BPS Particles, Superconformal Indices, and Chiral Algebras

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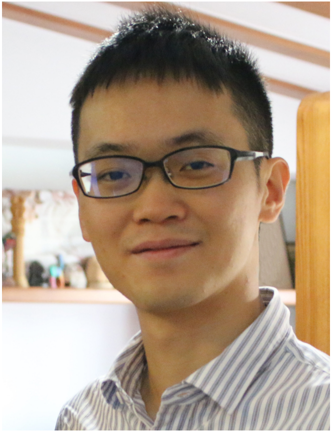
In Collaboration With:



(a) Shu-Heng Shao (IAS)

(b) Davide Gaiotto (Perimiter)

RG Flow and Protected Observables

$4d$ $\mathcal{N} = 2$ quantum field theory is at the center of a rich and enduring interplay between physics and mathematics

Many results flow from the combination of two principles

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If we compute a protected observable in the UV and IR we often obtain very different expressions for the same quantity

- Example: Donaldson Theory $\longrightarrow$ Seiberg-Witten Theory

Ingredients

In this talk we present a new conjectural $UV \longrightarrow IR$ relationship for an index $\mathcal{I}(q)$ (a q series) that can be associated to any $\mathcal{N} = 2$ quantum field theory. ($\mathcal{I}(q) \equiv$ Schur index)

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- $\mathcal{I}(q)$ can be expressed as a sum over local operators (UV data) in the QFT
- If the UV theory is conformal, $\mathcal{I}(q)$ is the a character of a chiral algebra (modular properties!)

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- If the UV theory is conformal, $\mathcal{I}(q)$ is the a character of a chiral algebra (modular properties!)

The IR formulation involves BPS particles (generalized and refined DT invariants). It has the following ingredients:

- rank r of the Coulomb branch (number of $U(1)$'s in IR)
- generating function of BPS particles $\mathrm{Tr}\left[\mathcal{O}(q)\right]$ built from the Kontsevich and Soibelman wall-crossing technology

Conjecture

Our main conjecture is a formula [Córdova-Shao]:

$$\underbrace{\mathcal{I}(q)}_{UV} = \underbrace{(q)_\infty^{2r}\ \mathrm{Tr}\left[\mathcal{O}(q)\right]}_{IR}$$

$((q)_\infty = \prod_{n=1}^{\infty}(1-q^n))$

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Applied to CFT we find that generating functions of BPS states (DT invariants) are equal to characters of chiral algebras

Related ideas: [Cecotti-Neitzke-Vafa] and [Iqbal-Vafa]

Physics Motivations

Why is this result interesting? A generic renormalization group flow starts in the UV at an asymptotically free or conformal theory and ends in a gapped or IR free theory.

- UV: characterized by the spectrum of local operators, and their operator product algebra
- IR: characterized by the spectrum of one-particle states

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- UV: characterized by the spectrum of local operators, and their operator product algebra
- IR: characterized by the spectrum of one-particle states

What is the relation between these concepts?

Our conjecture is an explicit formula relating supersymmetric local operators and BPS particles

Rough Intuition: Form Factors

One can get an intuition about why there should be a relation between particles and operators as follows

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$$|\Psi\rangle = \Phi(0)|0\rangle_{IR}$$

If the operator Φ is supersymmetric, so is the above state $|\Psi\rangle$

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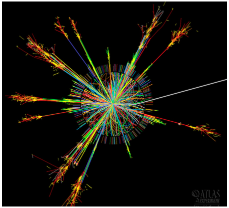
If the operator Φ is supersymmetric, so is the above state $|\Psi\rangle$

$|\Psi\rangle$ is a multiparticle state which we can think of as a jet of BPS particles

(This idea can be made precise for $2d$ QFTs. Interesting direction to explore in higher dimensions)



Mathematics Motivations

BPS particles have been widely studied

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  - the central charge can be varied in a finite dimensional complex moduli space
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What wall-crossing invariants exist and what are their properties?

This problem was essentially solved by [Kontsevich-Soibelman], (understood in physics by [Gaiotto-Moore-Neitzke]) who constructed an invariant $\mathcal{O}(q)$ valued in a quantum torus algebra

Our result links these ideas to vertex operator algebras

Definition of the Index

Every $\mathcal{N} = 2$ theory has a local operator spectrum graded by $SU(2)_R \times SU(2)_{J_1} \times SU(2)_{J_2}$. Define the index as a weighted sum over operators

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Properties:

- $\mathcal{I}(q)$ receives contributions only from 1/4 BPS local operators
- For superconformal theories, it coincides with a limit of the superconformal index, the so-called Schur index
- Explicitly computable for Lagrangian field theories
- It uses only symmetries that are present on the Coulomb branch, making it possible to compute in the IR

Indices and Chiral Algebras

A crucial result by [Beem-Lemos-Liendo-Peelaers-Rastelli-van Rees] implies that $\mathcal{I}(q)$ is vacuum character of a $2d$ chiral algebra

$$c_{2d} = -12 c_{4d} < 0 \ , \qquad\qquad (< T_{4d}(x) T_{4d}(0) > \sim \frac{c_{4d}}{|x|^8})$$

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- Free vector muliplet $\leftrightarrow$ b, c ghosts

$$\mathcal{I}(q) = (q)^2_\infty$$

- Free hypermuliplet $\leftrightarrow$ symplectic bosons ($z =$ flavor fugacity)

$$\mathcal{I}(q,z) = \frac{1}{\prod_{n=1}^{\infty}(1 + z q^{n+1/2})} \equiv E_q(z)$$

- $SU(2)$ $N_f = 4 \leftrightarrow \widehat{SO(8)}_{-2}$

$$\mathcal{I}(q) = \frac{1}{240q}\left(\frac{E_4'(q)}{(q)^{10}_\infty}\right)$$

BPS Particles and Wall-Crossing Invariants

Coulomb branch: $U(1)^r$ gauge theory + charged particles

Charges $\gamma \in \Gamma$, Dirac pairing $<,>: \Gamma x \Gamma \to \mathbb{Z}$

The BPS particles are encoded in a collection of positive integers

$$\Omega_n(\gamma) = \# \text{ of BPS particles of spin } n \text{ and charge } \gamma$$

The invariants $\Omega_n(\gamma)$ can jump as moduli are varied.

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Invariants were constructed by [Kontsevich-Soibelman]. They make use of non-commutative variables X_γ

$$X_\gamma X_{\gamma'} = q^{\frac{1}{2}<\gamma,\gamma'>} X_{\gamma+\gamma'} = q^{<\gamma,\gamma'>} X_{\gamma'} X_\gamma$$

BPS Particles and Wall-Crossing Invariants

For each charge γ introduce a factor

$$K_\gamma = \prod_n E_q((-1)^n q^{n/2} X_\gamma)^{(-1)^n \Omega_n(\gamma)}$$

Then consider the product, phase ordered by the central charge

$$\mathcal{O}(q) = \prod_\gamma^{\curvearrowright} K_\gamma$$

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The quantum torus valued $\mathcal{O}(q)$ is wall crossing invariant. We can obtain a simpler invariant q-series as

$$\mathrm{Tr}[\mathcal{O}(q)] \ , \qquad \mathrm{Tr}[X_\gamma] = \begin{cases} 1 & \gamma = 0 \\ 0 & \text{else} \end{cases}$$

First constructed and explored by [Cecotti-Neitzke-Vafa]

A_2 Argyres-Douglas Theory and a Virasoro Minimal Model

Non-trivial example [Argyres-Douglas] CFT. Geometrically:

$$\text{IIB on hypersurface} \qquad uv = y^2 + x^3$$

The central charge is

$$c_{2d} = -22/5 = c((2,5) \text{ Virasoro minimal model})$$

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On the Coulomb branch, there is a single $U(1)$ ($\Gamma \cong \mathbb{Z}^2$). BPS spectrum can be computed in a variety of ways

- special lagrangians on resolved hypersurface
- spectral network/WKB curves on pentagon
- representation theory of quiver $\bigcirc \longrightarrow \bigcirc$

There are two or three BPS particles depending on the chamber

A_2 Argyres-Douglas Theory and a Virasoro Minimal Model

Test the idea! (2,5) Virasoro minimal model vacuum character is a Rogers-Ramanujan function

$$\mathcal{I}(q) = q^{-\frac{27}{120}} \frac{1}{(q)_\infty} \sum_{\ell \in \mathbb{Z}} \left(q^{\frac{(20\ell - 3)^2}{40}} - q^{\frac{(20\ell + 7)^2}{40}} \right)$$

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$$(q)_\infty^2 \mathrm{Tr}[\mathcal{O}(q)] = (q)_\infty^2 \sum_{\ell_1, \ell_2 = 0}^{\infty} \frac{q^{\ell_1 + \ell_2 + \ell_1 \ell_2}}{[(q)_{\ell_1} (q)_{\ell_2}]^2}$$

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Smorgasbord

A range of other examples have been explicitly investigated

- (A_{N-1}, A_{M-1}) theory $(N, M$ coprime$) \leftrightarrow (N, N+M) \quad W_N$ minimal model

$$\text{IIB on hypersurface} \qquad uv = y^N + x^M$$

- Other simple Argyres-Douglas theories give rise to Bershadsky-Polyakov algebras, and $\widehat{SU(2)}$
- $SU(2)$ gauge theory with fundamental matter. (Realized in IIA by canonical bundle over (blowups of) $\mathbb{P}^1 \times \mathbb{P}^1$)

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Important Point:

Conjecture is general. Applies to any $\mathcal{N} = 2$ quantum field theory, independent of how it is constructed

An Intuitive Physical Argument

How do we understand the formula?

$$\underbrace{\mathcal{I}(q)}_{UV} = \underbrace{(q)_\infty^{2r}\ \mathrm{Tr}\left[\mathcal{O}(q)\right]}_{IR}$$

Idea:

Try to compute the index $\mathcal{I}(q)$ in the IR treating the BPS particles as independent fields. In other words, compute in $U(1)^r$ QED

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Since BPS particles are both electrically and magnetically charged there is no Lagrangian, however we can still use this idea to produce $\mathcal{I}(q)$. Similar to [Gopakumar-Vafa]

An Intuitive Physical Argument

Begin with the index for free fields

$$\mathcal{I}_{U(1)}(q) = (q)_\infty^2 \ , \qquad \mathcal{I}_H(q,z) = E_q(z)$$

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Can write a simple expression for $U(1)$ QED

$$\mathcal{I}_{QED}(q) = (q)_\infty^2 \quad \underbrace{\oint \frac{dz}{2\pi i z}}_{\text{selects gauge invariants}} \quad \underbrace{E_q(z)E_q(z^{-1})}_{\text{words in matter fields}}$$

Goal:

Write a similar expression taking into account magnetic charges!

An Intuitive Physical Argument

Let H_γ be a hypermultiplet field of charge γ. Need to consistently assign quantum numbers to composite operators

$$H_\gamma H_{\gamma'} \qquad \begin{cases} \text{charge} & \gamma + \gamma' \\ \text{spin} & \frac{1}{2}\langle \gamma, \gamma' \rangle \end{cases}$$

The unexpected angular momentum takes into account Dirac angular momentum in the electromagnetic field

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These charge assignments are taken into account in index calculations using the non-commutative quantum torus variables

$$X_\gamma X_{\gamma'} = q^{\frac{1}{2}\langle \gamma, \gamma' \rangle} X_{\gamma+\gamma'}$$

q counts angular momentum in the index

An Intuitive Physical Argument

Assemble the pieces, to obtain an IR calculation of the index

$$\mathcal{I}(q) = (q)_\infty^{2r} \ \mathrm{Tr}\left[\prod_{\gamma}^{\curvearrowright} E_q(X_\gamma)\right]$$

- $(q)_\infty^{2r}$ is the contribution of the $U(1)^r$ vector multiplets
- $E_q(X_\gamma)$ factors account for hypermultiplets. (crucial phase ordering ansatz!)
- Trace projects onto gauge invariant combinations

This is the desired IR formula (simple generalization to spin)

Defectarama: Lines

$\mathcal{N} = 2$ theories admit a class of half-BPS line defects L

We build an index $\mathcal{I}_L(q)$ that counts quasi-local operators that can end the defect. (this is like counting charged fields)

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Extend the UV-IR conjecture to lines [Córdova-Gaiotto-Shao]

$$\mathcal{I}_L(q) = (q)_\infty^{2r}\ \mathrm{Tr}\left[\ \underbrace{F_L(X_\gamma)}_{\text{contribution from } L}\quad \mathcal{O}(q)\right]$$

$F_L(X_\gamma)$ is a generating function of framed BPS states [Gaiotto-Moore-Neitzke]

- Ordinary BPS particles bound to the heavy line
- Related to framed quiver representations and DT invariants

Defectarama: Lines

The logic of the conjecture is identical

- Under RG flow, the line defect L flows to a sum of IR lines. These are dyons and can be thought of as the X_γ
- The linear combination is controlled by the framed BPS states, and given by the generating function $F_L(X_\gamma)$
- To compute the index in the IR, we compute a combination of IR dyonic indices and sum them. This is accomplished by inserting $F_L(X_\gamma)$ into the trace

Defectarama: Lines

BPS line defects admit a topological OPE

$$L_i * L_j = \bigoplus_k c_{ij}^k(q) L_k$$

The OPE coefficients can be determined by multiplying the associated framed BPS generating functions $F_{L_i}(X_\gamma)$

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Experimental Fact

The OPE algebra of line defects reproduces the Verlinde algebra of the associated chiral algebra at the level of indices

$$\mathcal{I}_{L_i * L_j}(q) = \sum_k v_{ij}^k \mathcal{I}_{L_k}(q)$$

(v_{ij}^k Verlinde coefficients)

Generalizes observations of [Cecotti-Neitzke-Vafa]

Defectarama: Surfaces

We can also extend to BPS surface defects. Our results intertwine with the [Cecotti-Vafa] formula for the elliptic genus of a (2,2) theory in terms of the BPS solitons after relevant deformations

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The $2d$ defect has K massive vacua. There are solitons (going from vacuum i to j) and particles (going from vacuum i to i). They can carry bulk electromagnetic charge. These are $2d$-$4d$ BPS states [Gaiotto-Moore-Neitzke]

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We conjecture [Córdova-Gaiotto-Shao]

$$\mathcal{I}_S(q) = (q)_\infty^{2r}\ \mathrm{Tr}\left[\mathcal{O}_{2d-4d}(q)\right]$$

where now $\mathcal{O}_{2d-4d}(q)$ is a $K \times K$ matrix of wall-crossing operators

Defectarama: Surfaces

Conformal surface defects S interplay with the chiral algebra

- Local operators on S form a module for the chiral algebra, and the index $\mathcal{I}_S(q)$ is a character (not usually the vacuum)
- Natural operations like spectral flow and Drinfeld-Sokolov reduction have interpretations in surface defects related to monodromy defects, and vortex surface defects

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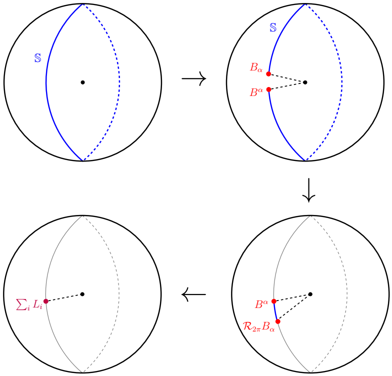
Simple example again consider A_2 Argyres-Douglas (related to the (2,5) minimal model). Canonical surface defect yields

$$(q)_\infty^2 \, \mathrm{Tr}\left[\mathcal{O}_{2d-4d}(q)\right] = (q)_\infty^2 \sum_{\ell_1,\ell_2=0}^{\infty} \frac{q^{\ell_1+\ell_2+\ell_1\ell_2}}{[(q)_{\ell_1}(q)_{\ell_2}]^2}(2-q^{\ell_1}) = \chi_\Phi^{(2,5)}(q)$$

Defectarama: Surfaces to Lines

We can use our understanding of indices to cut surfaces into lines (shown on S^3)



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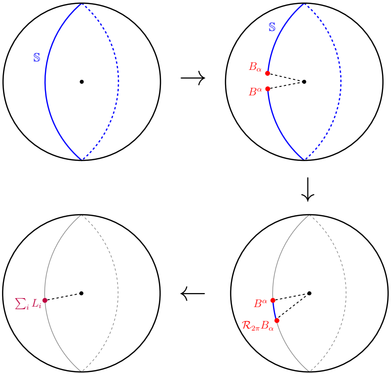

This implies a relationship

$$\text{character} = \mathcal{I}_S(q) = \sum_i \mathcal{I}_{L_i}(q)$$

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By exploring defects, we are interpreting more components of the quantum torus valued wall-crossing operator $\mathcal{O}(q)$ as indices

Natural to ask if we can see all matrix elements of $\mathcal{O}(q)$. This is achieved through boundary conditions

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Let B be a UV BPS boundary condition that flows to Dirichlet boundary conditions for the $U(1)$ gauge fields

- On the boundary we get flavor electric and magnetic charges
- Assemble the indices into a generating function. The result is the half-space wall-crossing operator ($\mathcal{O} = \mathcal{S}\mathcal{S}^{-T}$)

$$\mathcal{I}_B(q) = (q)_\infty^r \mathcal{S}(q)$$

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Let B be a UV BPS boundary condition that flows to Dirichlet boundary conditions for the $U(1)$ gauge fields

- On the boundary we get flavor electric and magnetic charges
- Assemble the indices into a generating function. The result is the half-space wall-crossing operator ($\mathcal{O} = \mathcal{S}\mathcal{S}^{-T}$)

$$\mathcal{I}_B(q) = (q)_\infty^r \mathcal{S}(q)$$

This gives a completely new physical interpretation of the wall-crossing operator as an intrinsic quantity in the UV QFT

Open Problems

Many questions remain:

- Prove the $UV - IR$ formula! (Promising work interpreting the index as a partition function [Dumitrescu-Festuccia-del Zotto])
- See the chiral algebras directly in geometry. (Related work by [Beem-Peelaers-Rastelli-van Rees], [Arakawa])
- Construct the full chiral algebra, not just characters, from BPS particles

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Thanks for listening!