Koszul Duality Patterns in Physics

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↑
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with D. Ben-Zvi, A. Neitzke, K. Costello

closely related to work of C. Beem, M. Bullimore, D. Gaiotto, J. Hilburn
Koszul Duality

Math: a duality $A \leftrightarrow A^!$ of certain kinds of algebras are not equivalent; rather, $A^!$ can be obtained from $A$ and vice versa, and their categories of modules are equivalent.
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rather, $A^!$ can be obtained from $A$ & vice versa,

and their categories of modules are equivalent.

Concept developed in 60's & 70's

Quillen, Sullivan (rat\textsuperscript{2} homotopy thy)

Priddy

Bernstein-Gelfand-Gelfand (algebra)

and (later, e.g.)

Ginzburg-Kapranov (operads)

Beilinson-Ginzburg-Serng (geometric rep'' thy, topology)

Goresky-Kottwitz-MacPherson

Beilinson-Drinfeld

Francis-Cahitgory

Lurie

Costello-Gwilliam

Ayala-Francis

Posielski

(Chiral algebras)

(Lie algebras, factorization alg, fact'' homology)

(Dg...)
Today I'd like to argue that Koszul duality is as fundamental and as prevalent in physics as it is in mathematics.
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Focus: 2d TQFT's and their 1d b.c.,

(twists of SUSY QFT's)

governed by K.D. for associative algebrae.
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(twists of SUSY QFT's) (dg)

governed by K.D. for associative algebras.

Major applications:

• gauging/ungauging
• gluing/ungluing
Today I'd like to argue that Koszul duality is as fundamental and as prevalent in physics as it is in mathematics.

Focus: 2d TQFT's and their 1d b.c.,
(twists of SUSY QFT's)
(governed by K.D. for associative algebras).

Major applications:
• gauging/ungauging
• gluing/ungluing

Hope that generalization (e.g. to $d>2$) will be clear, in principle
(cf. D. Ben-Zvi's talk
C. Teleman (3d $N=4$, G actions on Fukaya cat's)
K. Costello (AdS/CFT !))
Definition

Let \( A \) be a dg algebra
\[ \rho: A \to C \]  
"augmentation"

trivial dga

Note \( C \) (as a vec space) is a module for \( C \) (as a dga),
hence a module for \( A \).
Let $A$ be a dg algebra

$\sim$ a homomorphism $\rho : A \to C$   "augmentation."

trivial dga

Note $C$ (as a vec space) is a module for $C$ (as a dga), hence a module for $A$.

The Koszul dual dg algebra is

$A^! := \text{End}_A(C) = \text{Hom}_A(C, C)$

Complex  module  Complex, i.e. $\text{Ext}^*$
Basic setup for physics

Let $C$ be a dg category.

For any object $B$ there's a functor

$C \rightarrow \text{End}(B)\text{-mod}$

$X \mapsto \text{Hom}(B, X)$

\[ \text{End}(B) \]

\[ \text{Hilbert space } \text{Hom}(B, X) \]
Spoke there is a pair of objects $B$, $B'$ such that

1) \[ \text{End}(B) \text{-mod} \xrightarrow{C} \text{End}(B') \text{-mod} \]

both iso\$. i.e. both $B$ and $B'$ generate the entire category (they are large enough).
Suppose there is a pair of objects $B, B'$ such that

1) $\text{End}(B) \rightarrow \text{End}(B!)$ both iso.

$\text{End}(B)$-mod $\rightarrow \text{End}(B!)$-mod $\rightarrow C$

2) $\text{Hom}(B, B!)$ $\cong C$

i.e. $B, B'$ are transverse

i.e. both $B$ and $B'$ generate the entire category (they are large enough)

Trivial Id thy on a strip/sandwich
Suppose there is a pair of objects $B, B'$ such that

1) $\mathbf{End}(B) \text{-mod} \xrightarrow{\sim} \mathbf{End}(B') \text{-mod}$

   $A \xlongleftarrow{\sim} A'$. Both are isos.

   $\mathbf{End}(B) \text{-mod} \xrightarrow{\text{both iso}^3} \mathbf{End}(B') \text{-mod}$

   $A \xlongleftarrow{\sim} A'$. Both $B$ and $B'$ generate the entire category (they are large enough).

2) $\text{Hom}(B, B') = C$.

   i.e. $B, B'$ are transverse.

Then $A = \text{End}(B), A' = \text{End}(B')$ are Koszul dual algebras.

And $A \text{-mod} \simeq A' \text{-mod}$. (obvious)
As a picture:

simultaneously a module for $A \otimes A'$, commuting actions.

"Large enough" assumption guarantees that $A$ is the full commutant of $A'$ and vice versa.

$$A' = \text{End}_A(C)$$
$$A = \text{End}_{A'}(C)$$
Examples

1. \( A = \text{Sym}^* V \quad A^! = \Lambda^* V^* \)  
   \[
   \rightarrow \text{B model of target } V
   \]
   \[
   \text{Bernstein-Gelfand-Gelfand}
   \]

2. \( A = H^*_c (\mathcal{P}) \quad A^! = H^*_c (G) \)  
   \[
   \text{symmetric} \quad \text{exterior}
   \]
   \[
   \rightarrow \text{A-twist of pure } N=(2,2) \text{ gauge thy,}
   \text{gauging/ungauging in de Rham SQM,}
   \text{homological } G \text{-actions in SQM}
   \]
   \[
   \text{w/ Ben-Zvi & Neitzke}
   \]

3. \( A = U(g) \quad A^! = C^*(g) \)  
   \[
   \rightarrow \text{B-twist of } N=(2,2) \text{ gauge thy}
   \]
   \[
   (2d \text{ top YM}), \text{ holomorphic } G \text{-actions in SQM}
   \]
   \[
   \text{Costello, Bulloch-Yoo}
   \]

4. \( C = \text{BGG category } \mathcal{O}_g \equiv \text{highest weight } g \text{-modules} \)
   \[
   A = \text{End}(\oplus \text{simples}) \quad A^! = \text{End}(\oplus \text{projectives})
   \]
   \[
   \rightarrow \text{symplectic duality } \rightarrow 3d \text{ } \mathcal{N}=4 \text{ theories}
   \]
Consider boundary conditions that preserve $B$-type SUSY

and the $U(1)$ flavor/$R$ sym' that rotate $\phi$, $\psi$. 

B-twist of 2d $N=(2,2)$ thy of a free chiral multiplet

bulk local ops $\phi$, $\psi$

$C[\phi, \psi] = \text{polyvec fields on } C$
B-twist of 2d $N=(2,2)$ theory of a free chiral multiplet

\[ \text{bulk local ops } \phi, \bar{\psi} \]

\[ (C[\phi, \bar{\psi}] = \text{polyvex fields on } C) \]

Consider boundary conditions that preserve B-type SUSY

and the U(1) flavor/R sym that rotate $\phi, \bar{\psi}$.

Two basic ones:

1. Neumann: $\phi|_{\partial} = 0$

   \[ \text{End}(N) = C[\phi] \text{ sym} \]

2. Dirichlet: $\phi|_{\partial} = 0$

   \[ \text{End}(D) = C[\bar{\psi}] \text{ ext} \]
Two basic ones: Neumann \( \delta |_{\partial} = 0 \)

\[ \text{End}(N) = C[\phi] \text{ sym} \]

Dirichlet \( \phi |_{\partial} = 0 \)

\[ \text{End}(D) = C[\phi] \text{ exterior} \]

Note:

- these are *transverse*

(\( C \approx \text{a complex } C[\phi, q] \), with \( A = \phi \frac{\partial}{\partial t} \).)

- less trivially, with careful finiteness conditions

\[ C \approx C[\phi_1, d_1]-\text{mod} \approx C[\phi_2, d_2]-\text{mod} \]
What does $C[\phi] - (dg)_{\text{mod}} = C[\gamma] - dg_{\text{-mod}}$ mean?

Typical example of a bdy condition:

$X^{(n)} = \text{modification of } N \text{ that sets } \phi^n = 0$
What does \( \mathbb{C}[\phi]-dg\text{-mod} = \mathbb{C}[\mathcal{F}]-dg\text{-mod} \) mean?

Typical example of a bdy condition:
\[
X^{(n)} = \text{modification of } \mathcal{N} \text{ that sets } \phi^n = 0
\]

- As a \( \mathbb{C}[\phi]-\text{module} \):
  \[
  \begin{array}{c}
  \text{Hom}(\mathcal{N}, X^{(n)}) = \mathbb{C}[\phi]/(\phi^n)
  \\
  \end{array}
  \]
What does $\mathbb{C}[\phi]/(d)_{\mod} = \mathbb{C}[\phi]/(d)_{\mod}$ mean?

Typical example of a bdy condition:

\[ X^{(n)} = \text{modification of } N \text{ that sets } \phi^n = 0 \]

- As a $\mathbb{C}[\phi]$-module:
  \[
  \begin{array}{c}
  N \\
  \text{Hom}(N, X^{(n)}) = \mathbb{C}[\phi]/(\phi^n)
  \end{array}
  \]

- As a $\mathbb{C}[\phi]$-module?

Direct computation $\Rightarrow$ UV definition of $X^{(n)}$:

\[
\begin{array}{c}
D \\
\text{Hom}(D, X^{(n)}) = ?
\end{array}
\]
What does $C[\phi]-dg\text{-mod} \cong C[\chi]-dg\text{-mod}$ mean?

Typical example of a bdy condition:

\[ X^{(n)} = \text{modification of } N \text{ that sets } \phi^n = 0 \]

- As a $C[\phi]$-module:

\[
\begin{array}{c}
\text{Hom}(N, X^{(n)}) = C[\phi]/(\phi^n)
\end{array}
\]

- As a $C[\chi]$-module?

Direct computation of UV definition of $X^{(n)}$:

Better: use concrete isomorphism!
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Picture means:

- Start w/ $C[\phi]/(\phi^n)$
- Tensor w/ $(C[\phi, \Delta], \Delta = \phi \Delta)$, identifying the $\phi$'s w/ cohomology
Better: use concrete isomorphism!

\[ \text{Hom}(D, X^{(n)}) \]

\[ \text{Hom}(\alpha, X^{(m)}) \cong C[\phi]/(\phi^n) \]

Picture means:
- Start w/ \( C[\phi]/(\phi^n) \)
- Tensor w/ \( (C[\phi, \phi^4], Q = \phi \phi^4) \), identifying the \( \phi \)'s

\[ \Rightarrow \text{Hom}(D, X^{(n)}) = (C[\phi, \phi^4]/(\phi^n), Q = \phi \phi^4) \]

\[ = \left( \begin{array}{c}
C[\phi] \xrightarrow{\phi^4} C[\phi^2] \xrightarrow{\phi^4} C[\phi^3] \xrightarrow{\phi^4} \cdots \xrightarrow{\phi^4} C[\phi^n] \xrightarrow{\phi^4} C[\phi^{n-1}] \\
C^2
\end{array} \right) \]
Better: use concrete isomorphism!

Picture means:

- Start w/ \( C[\Phi]/(\Phi^n) \)
- Tensor w/ \( (C[\Phi,2], Q = \Phi^2) \), identifying the \( \phi \)'s

\[
\Rightarrow \text{Hom}(D, X^{(m)}) = (C[\Phi,2]/(\Phi^n), Q = \Phi^2)
\]

\[
\]

\( C[2] \) action:

- \( C[4] \)
- \( C[8] \)
- \( C[16] \)
- \( C[32] \)
- \( C[64] \)
- \( C[128] \)
Better: use \( \text{concrete isomorphism!} \)

Picture means:

- Start w/ \( C[\phi 3/\phi^n] \)
- Tensor w/ \( (C[\phi, \psi], Q = \phi \psi) \), identifying the \( \phi \)'s

\[
\text{Hom}(D, X^{(n)}) = \left( C[\phi, \psi]/(\phi^n), Q = \phi \psi \right)
\]

\[
= \left( C[\psi] \rightarrow C[\psi] \phi \rightarrow C[\psi] \phi^2 \rightarrow \ldots \rightarrow C[\psi] \phi^{n-1} \right)
\]

**\( C[\psi] \) action:**

**Cohomology:**

\[
C\langle \psi \rangle \quad (\psi = 0)
\]

But as a dg-module, can't pass to cohomology.
B-model example may have seemed abstract?
contrived?
It is not!

Perhaps a more natural setting for Koszul-dual b.c. is in gauge thy.

Consider: pure (maybe SUSY) $G$ gauge thy in d+1 dims
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Perhaps a more natural setting for Koszul-dual b.c.
is in gauge thy.

Consider: pure (maybe SUSY) \( G \) gauge thy in \( d+1 \) dims

Two natural transverse b.c.

\[ N \]
\[ F_{\mu \perp} = 0 \]

\[ D \]
\[ A_{\mu} = 0 \]
Two natural transverse b.c.:

\[ N \quad F_{\mu \perp} = 0 \]
\[ D \quad A_{\mu} = 0 \]

Note: \( D \) has a global \( G \) symmetry (breaks gauge sym) \( \leftrightarrow \) inherited by local operators \( F_{\mu \perp} \) \( \leftrightarrow \) currents \( J_{\mu} \) \( X_{\text{global}}^{(d)} \)
Two natural transverse b.c.: \[ N \]
\[ F_{\mu 1} = 0 \]
\[ D \]
\[ A_\mu = 0 \]

Sandwich \[ X^{(d)} \]
\[ \downarrow \]
\[ X_{\text{gauged}} \]
\[ \downarrow \]
\[ X^{(d)}_{\text{global}} \]

Note: \( D \) has a global \( G \) symmetry (breaks gauge sym) \( \leftrightarrow \) inherited by local operators \( F_{\mu 1} \)

\[ \downarrow \]

\[ \downarrow \]

\( N \) has a gauged \( G \)-sym (may have ghosts as local ops)

\[ \sim X^{(d)}_{\text{gauged}}, \] a gauged version of \( X^{(d)}_{\text{global}} \).
Lesson for QFT's in d-dims:

if we can promote a thy \( X^{(d)} \) w/ \( G \)-symmetry to a bdy cond for \((d+1)\)-dim pure gauge thy,

can recover both gauged & ungauged versions of \( X^{(d)} \)

from collision w/ \( N, D \).
Lesson for QFT’s in d-dims:

if we can promote a thy $X^{(d)}$ w/ $G$-symmetry to a bdy cond
for $(d+1)$-dim $\psi$ pure gauge thy,

  can recover both gauged & ungauged versions of $X^{(d)}$

  from collision of $N,D$.

For TQFT, can do better:

If $N,D$ both generate the $(d)$-category of b.c.

  so that they’re Koszul dual,

  can freely gauge ungauged $G$-sym $\psi$ in d-dims.

Caveat: very subtle

  in $d > 2$!

Cf: Teleman
A precise realization in $d = 2$:

A twist of pure $\mathcal{N}=(2,2)$ $G$-gauge theory

w/ Ben-Zvi, Neitzke; physics of Goresky-Kottwitz-MacPherson
A precise realization in $d = 2$:

A twist of pure $N=(2,2)$ $G$ gauge thy

- if $G = U(1)$, almost a free twisted-chiral multiplet, will behave a lot like B-model to $C$.
- general $G$ more interesting

[Ben-Zvi, Neitzke; physics of Goresky-Kottwitz-MacPherson]
A precise realization in $d = 2$:

- if $G = U(1)$, almost a free twisted-chiral multiplet,
  will behave a lot like B-model to $C$.
- general $G$ more interesting

Any 1d $N = 2$ SQM (de Rham type) w/ global $G$ symmetry defines a bdy condition.

Consider a category $C$ generated by smooth, compact Riem"{a}mannifolds w/ $C$ with $G$.
A precise realization in \( d = 2 \):

- if \( G = U(1) \), almost a free twisted-chiral multiplet, will behave a lot like B-model to \( C \).
- general \( G \) more interesting.

Any \( 1d \ N = 2 \) SQM (de Rham type) w/ global \( G \) symmetry defines a bdy condition.

Consider a category \( C \) generated by smooth, compact Riem "manifolds \( X \) with \( C \).

Any such \( X \) can be coupled to 2d gauge fields (w/ SUSY).
Consider a category \( C \) generated by smooth, compact Riemannian manifolds \( X \) with

Any such \( X \) can be coupled to 2d gauge fields (w/ SUSY) \( \sim \)

\( \text{Homs:} \)

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\Rightarrow & \Rightarrow \\
G & \text{gauged} & \text{SQM on} \ X \times Y
\end{array}
\]
Consider a category \( \mathcal{C} \) generated by smooth, compact Riemannian manifolds \( X \) with compact SU(3), and any such \( X \) can be coupled to 2d gauge fields (w/ susy).

\( \text{Hom}_{\mathcal{C}}(X, Y) \approx X \times_{Y} G \) is the G-gauged SQM on \( X \times Y \).

\( \Rightarrow \text{Hom}(X, Y) = H^*_C(X \times Y) \) (Hilbert space of gauged SQM)
Homs: \[ X \xrightarrow{\text{g}} Y \cong X \text{sym} \quad \text{G-gauged SQM on } X \times Y \]

\[ \Rightarrow \quad \text{Hom} (X, Y) = H^c_c (X \times Y) \quad \text{(Hilbert space of gauged SQM)} \]

\[ \text{N.b. : } \quad X = p \quad \text{preserves } G \text{ sym (acts trivially)} \]
Homs: \[ X \overset{\pi}{\to} Y \cong X \overset{\xi}{\to} Y \]  \hspace{1cm} \text{G-gauged SQM on } X \times Y

\[ \Rightarrow \quad \text{Hom} (X,Y) = H^*_G (X \times Y) \]  \hspace{1cm} \text{(Hilbert space of gauged SQM)}

\text{N\text{\textdegree} b.c.: } X = \pi \quad \text{preserves } G \text{ sym (act trivially)}

\text{check: } \quad \Sigma \overset{\pi}{\underset{p}{\simeq}} X \quad \Rightarrow \quad \text{Hom}(\pi_X, X) = H^*_G (\pi_X \times X) = H^*_G (X)

\text{Hilb. Space of G-gauged SQM to } X \checkmark
\[ \text{Homs: } \quad \begin{array}{c|c|c} \end{array} \quad \Rightarrow \quad \text{Hom} (X, Y) = H^*_c (X \times Y) \quad \text{(Hilbert space of gauged SQM)} \]

\[ \text{N b.c.: } \quad X = p \quad \text{preserves } \text{G sym (acting trivially)} \]

\[ \text{check: } \quad \begin{array}{c|c|c} \end{array} \quad \text{Hom} (p, X) = H^*_c (p \times X) = H^*_c (X) \quad \text{Hilb. space of G-gauged SQM to } X \checkmark \]

\[ \text{End} (\mathbb{N}) = \text{Hom} (p, p) = H^*_c (p \times p) = H^*_c (p) \approx \mathbb{C} [\mathfrak{g}]^G, \quad \sigma \in \mathfrak{g} \in \text{symmetric algebra, vector multiplet scalar} \]

\[ \text{Recall: } \quad H^*_c (p) \subset H^*_c (X) \]

\[ \begin{array}{c|c|c} \end{array} \]
Examples:

1) $X = S^1 \quad G = U(1) \quad \sigma \in \mathbb{C}$

$$H^*_\text{low}(S^1) \cong H^*(\mathbb{C}/U(1)) = H^*(\mathbb{C}) \cong \mathbb{C}$$

as a module (free action)
Examples:

1) $X = S^1$, $G = U(1)$, $\sigma \in C$

\[ H^{\omega_0}(S^1) \cong H^\cdot(S^1/U(1)) = H^\cdot(pt) = C \] (free action)

\[ \cong \mathbb{C}[\sigma]/\langle \sigma \rangle \] as a module

2) $X = S^2$, $G = U(1)$

\[ H^{\omega_0}(S^2) \cong \mathbb{C}^2 \otimes \mathbb{C}[\sigma] \]

\[ \cong \mathbb{C}^{\omega_0}(S^1) \] (fixed pt localization)
Examples:

1) $X = S^1 \quad G = U(1) \quad \sigma \in \mathbb{C}$

\[ H^*_\text{ucl}(S^1) \cong H^*(S^1/\mathbb{C}) = H^*(\mathbb{R}) = \mathbb{C} \quad \text{(free action)} \]

\[ \cong \mathbb{C}[\sigma]/(\sigma^2) \quad \text{as a module} \]

2) $X = S^2 \quad G = U(1)$

\[ H^*_\text{ucl}(S^2) \cong \mathbb{C}^2 \otimes \mathbb{C}[\sigma] \quad \text{(fixed pt localization)} \]

\[ \cong \mathbb{C}[\sigma]/(\sigma^2) \quad \text{as a module} \]

3) $X = S^3 \quad \mathbb{S}^1 \to S^3 \quad \mathbb{S}^1 \to S^3$ \quad \text{(free again)}

\[ H^*_\text{ucl}(S^3) \cong H^*(S^2) = \mathbb{C}^2 \]

\[ \cong \mathbb{C}[\sigma]/(\sigma^2) \quad \text{as a module} \]
D b.c. : \( X = G \) (claim)

"Spontaneously breaks" \( G \) symmetry (intuitively, not literally... )
D b.c. : $X = G$

check: $\frac{D}{G} \longrightarrow X$

“Spontaneously breaks” $G$ symmetry
(intuitively, not literally...)

$\sim \text{Hom}(D, X) = \text{Hom}(G, X) = H^*_G(G \times X)$

$\sim H^*(X)$

Hilbert space of SQM to $X$
($G$ global sym)
D b.c.: \( X = G \)

\[
\begin{align*}
\text{claim) } & \quad \text{ spontaneous breaks } G \text{ symmetry} \\
& \quad \text{(intuitively, not literally...)} \\
\text{check:} & \quad \frac{D}{G} \bigg| \frac{X}{=} \bigg| \text{symmetry} \\
\sim & \quad \text{Hom}(D,X) = \text{Hom}(G,X) = H^*_G(G \times X) \\
& \quad \cong H^*(X) \\
\text{Hilbert space of SQM to } X & \quad (G \text{ global sym})
\end{align*}
\]

ops? \quad \frac{X}{=} \bigg| \text{symmetry}

\[
\begin{align*}
\text{End}(D) & = \text{Hom}(G,G) = H^*_G(G \times G) \\
& \cong [H^*(G)]
\end{align*}
\]
\(D\text{ b/c: } X = G\)

\(\text{check: } D \Bigg/ \frac{G}{G} \xrightarrow{\sim} \text{Hom}(D, X) = \text{Hom}(G, X) = H^\ast_c(G \times X) = H^\ast(X)\)

Hilbert space of SQM to \(X\) (\(G\) global sym)

\(\text{End}(D) = \text{Hom}(G, G) = H^\ast_c(G \times G) \supseteq \square \overset{?}{\rightarrow} H^\ast(G)\)

Implies that Hilb. space of SUSY ground states in SQM w/ target \(X\) should be a \(\mathsf{Hilb}^{\ast}\text{-dg-module}\).\(\text{End}(G) \supseteq H^\ast(G)\) (better)
D b.c. : \( X = G \)

check: \( \frac{D}{G} \rightarrow X \)

\[
\sim \text{Hom}(D, X) = \text{Hom}(G, X) = H^*_c(G \times X) \\
\sim H^*(X)
\]

Hilbert space of SQM to X
(G global sym)

ops ?

\[
\text{End}(D) = \text{Hom}(G, G) = H^*_c(G \times G) \\
\sim \boxed{H^*(G)}
\]

Implies that Hilb. space of SUSY ground states
in SQM w/ target X should be a \( H^*_c \)-dg-module.

\[
\Rightarrow \boxed{\text{Homological } G\text{-action in SQM}}
\]

It's there, & it's important.
Homological $G$-action

Recall: $SQM \rhd$ target $X$

full $\mathcal{H} = \Omega^*(X)$

$\mathcal{Q} \leftrightarrow d$

$\mathcal{Q}^+ \leftrightarrow d^+$

$H \leftrightarrow \Delta = \{d, d^+\}$

$R$-charge $\leftrightarrow$ form degree

(form #) $\leftrightarrow$ form degree

$H_{susy} \approx$ harmonic forms $\approx H^*(X)$
Riemann geometry $G \subset X$ induces a flavor symmetry that commutes with SUSY.

Infinitesimal action of $g$ generated by

$\text{Noether change } \leftrightarrow L_V \quad (V \in g^*)$
It is given that $H = \mathcal{H}^*(X)$. For the target $X$, we have $\alpha \leftrightarrow d$, $\alpha^+ \leftrightarrow d^+$, $H \leftrightarrow \Delta = \{d, d^+\}$.

$R$-change $\leftrightarrow$ form degree

$\text{fermion } \# \leftrightarrow \text{form degree}$

Riemannian geometry $G \subset X$ induces a flavor symmetry $\text{commutes w/ SUSY}$.

Infinitesimal action of $\xi$ generated by $\text{Noether change} \leftrightarrow \mathcal{L}_V$

$\text{Commute w/ SUSY:} \quad [\mathcal{L}_V, d] = [\mathcal{L}_V, d^+] = 0$

$\Rightarrow \mathcal{L}_V \in H_{\text{susy}}$
Riemannian geometry $G \subset X$ induces a flavor symmetry $\text{commutes w/ SUSY}$

Infinitesimal action of $G$ generated by $\text{Noether change} \leftrightarrow L_V \quad (V \in g^*)$

Commute w/ SUSY: $[L_V, d] = [L_V, d^+] = 0$

$\Rightarrow L_V \circ H_{\text{susy}}$

$\text{BUT: } L_V = \{d, L_V\} \Rightarrow \boxed{L_V \equiv 0}$ on $H_{\text{susy}}$

check: $\omega$ closed $\Rightarrow L_V \omega = d(L_V \omega)$ is exact.
BUT: \[ L \nu = \{ d, \psi \} \Rightarrow [L \nu \equiv 0] \text{ on } H_{\text{susy}} \]

check: \( \omega \) closed \( \Rightarrow L \nu \omega = d(\nu \omega) \) is exact.

So the action of \( \frak{g}_\nu \) on \( H_{\text{susy}} \) is trivial.
But the action of \( \frak{g} \) is not!
BUT: \[ L_v = \{ d, \omega \} \Rightarrow L_v \equiv 0 \] on \( H_{susy} \)

Check: \( \omega \) closed \( \Rightarrow L_v \omega = d(\omega v) \) is exact.

So the action of \( \mathfrak{g} \) on \( H_{susy} \) is trivial.
But the action of \( G \) is not!

To proceed, use topological descent on \( G \)

\[ L_v \omega = d(\omega v) \quad \text{view } \omega v \in g^* \otimes \Omega^1(X) \]

as a 1-form on \( G \)
But: \[ L^v = \{ \text{d}, \psi \emptyset \} \Rightarrow L^v \equiv 0 \text{ on } H_{\text{susy}} \]

Check: \( \omega \) closed \( \Rightarrow L^v \omega = d(\psi \omega) \) is exact.

So the action of \( \mathcal{O}_G \) on \( H_{\text{susy}} \) is trivial.

But the action of \( G \) is not!

To proceed, use topological descent on \( G \):

\[
L^v \omega = d(\psi \omega) \quad \text{view } \psi \omega \in g^* \otimes \Omega_c^1(X) \]

as a 1-form on \( G \).

Given any 1-cycle \( \gamma \in G \), define \( \gamma \cdot \omega := \oint_{\gamma} \psi \omega \).

(hidden push-forward / averaging over \( \gamma \))
But: \[ L^v = \{ d, L^3 \} \Rightarrow [L^v = 0] \text{ on } H_{\text{susy}} \]

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Given any 1-cycle \( \gamma \subset G \), define

\[ \gamma \cdot \omega := \oint_\gamma \omega \]

\[ \text{hidden push-forward / averaging over } \gamma \]

- \( \gamma : \Omega^0(X) \to \Omega^{p-1}(X) \)
- \( \{ \gamma, d3 \} = 0 \)

\( \Rightarrow \gamma \circ H_{\text{susy}} = H^0(X) \)
Given any 1-cycle $\gamma \in G$, define

$$Y \cdot \omega := \int_{\gamma} \omega$$

(hidden push-forward / averaging over $\gamma$)

- $Y : \mathcal{R}^p(X) \to \mathcal{R}^{p-1}(X)$
- $\{ Y, d \} = 0$

$\Rightarrow \quad Y \in \mathcal{R}_\text{susy} = H^0(X)$.

Similarly, every $\gamma \in H_k(G)$ defines $Y : H^0(X) \to H^{0-k}(X)$. 
Given any 1-cycle $\gamma \in G$, define

$$\gamma \cdot \omega := \oint_{\gamma} \omega$$

(hidden push-forward / averaging over $\gamma$

- $\gamma : \Omega^p(X) \to \Omega^{p+1}(X)$
- $\{\gamma, \partial \} = 0$

$\Rightarrow \gamma \in \mathfrak{h}_{\text{susy}} = H^0(X)$.

Similarly, every $\gamma \in H_k(G)$ defines $\gamma : H^0(X) \to H^{0-k}(X)$.

$\Rightarrow \gamma \in \mathfrak{h}_{\text{CSH.I.T.}}$ endowed with structure of exterior algebra (built from fermionic ops $v$).
Example: \( G = U(1) \) \( H(G) = \mathbb{C}^{1,1} = \mathbb{C}[x] \quad (x^2 = 0) \)
Example:

\[ G = U(1) \quad H.(G) = \mathbb{C} < 1, \chi > = \mathbb{C}[\chi] \quad (\chi^2 = 0) \]

1) \( X = S^1 \subseteq U(1) \)

\[ \omega = a(\theta) d\theta + b(\theta) \quad \Rightarrow \quad \oint_{S^1} a(\theta) d\theta \]

\[ H^*:\quad \mathbb{C} \xrightarrow{\partial = 0} \mathbb{C} \quad (\chi^2 = 0) \]
Example: \( G = U(1) \quad H(G) = \mathbb{C} < 1, \chi > = \mathbb{C}[\chi] \quad (\chi^2 = 0) \)

1) \( X = S^1 \subseteq U(1) \quad \omega = a(\theta) d\theta + b(\theta) \quad \overset{\tau}{\rightarrow} \quad \oint_{S^1} a(\theta) d\theta \)

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\( H^0 : \quad \mathbb{C} \xrightarrow{d=0} \mathbb{C} \quad (\chi^2 = 0) \)

2) \( X = S^2 \quad \quad \quad \quad N^0 \xrightarrow{d} N^1 \xrightarrow{d} N^2 \)
Example: \( G = U(1) \) \( H(G) = \mathbb{C} \langle 1, \chi \rangle = \mathbb{C} \{ \chi \} \quad (\chi^2 = 0) \)

1) \( X = S^1 \subset U(1) \) \( \omega = a(\theta) d\theta + b(\theta) \)
\[ \Rightarrow \frac{\partial}{\partial \theta} a(\theta) \]
\[ \mathbb{H}^*: \quad \mathbb{C} \rightarrow \mathbb{C} \quad (\chi^2 = 0) \]

2) \( X = S^2 \)
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3) \( X = S^3 \) (Hopf)
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Example: \( G = U(1) \) \( H(G) = \mathbb{C} < 1, \gamma > = \mathbb{C}[\gamma] \) \( (\gamma^2 = 0) \)

1) \( X = S^1 \subset U(1) \) \( \omega = a(\theta) d\theta + b(\theta) \) \( \mapsto \oint_{S^1} a(\theta) d\theta \)

\( J^0 \overset{d}{\rightarrow} J^1 \)

\( H^* : \mathbb{C} \overset{d=0}{\rightarrow} \mathbb{C} \) \( (\gamma^2 = 0) \)

2) \( X = S^2 \)

\( J^0 \overset{d}{\rightarrow} J^1 \overset{d}{\rightarrow} J^2 \)

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3) \( X = S^3 \) (Hopf)

\( J^0 \overset{d}{\rightarrow} J^1 \overset{d}{\rightarrow} J^2 \overset{d}{\rightarrow} J^3 \)

\( H^* : \mathbb{C} \overset{d=0}{\rightarrow} \mathbb{C} \)

Here: \( J^*(S^3) \) is a dg-module for \( \mathbb{C}[\gamma] \).

Can't simplify it too far!

Or if pass to \( H^* \), there are [A° operations].
Summary:

- Gauged SQM to $X$ gives $H^*_c(X) \in H^*_c(p)\text{-mod}$
- Global SQM to $X$ gives $C^*(X) \in H_*(G)\text{-dg-mod}$

\[ \text{Diagram} \]
Summary:

- Gauged SQM to $X$ gives $H^0_c(X) \in H^0_c(p)\text{-mod}$
- Global SQM to $X$ gives $C^0(X) \in H^0_c(g)\text{-dg-mod}$

» Module structures are the key to gauging & ungauging.
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Module structures are the key to gauging & ungauging.

Duality functor:

$C = H^*_c(p \times G) \simeq H^*_c(G)$

$\Rightarrow$ replace $C$ complex $C^*_c(G)$, tensor with it to get duality.
Summary:

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Module structures are the key to gauging & ungauging.

Duality functor:

$$\mathcal{P} = N$$

$$\mathcal{C} = D$$

$$\mathcal{G} = H^*_c(p \times G) \simeq H^*_c(G)$$

\Rightarrow replace 1 complex $C^*_c(G)$, tensor with it to get duality.

Physical note: the $\gamma$ operations are (fermionic) disorder ops.
For $G = U(1)$, they come from the bulk;
they are fermionic modes of vortices.
Gluing & Ungluing

A common scenario in physics:

We'd like to manipulate a QFT on a half-space,

e.g. dualize on a half-space to get a duality interface.

To this end (and others), it's useful to be able to slice open the QFT.
Gluing & Un-gluing

A common scenario in physics:

We'd like to manipulate a QFT on a half-space,

e.g. dualize on a half-space to get a duality interface.

To this end (and others), it's useful to be able to slice open the QFT.

Namely: find a pair of b.i. $B, B'$ that can be coupled
to each other, s.t. the resulting coupled interface is
trivial/transparent.
Example: in (pure) gauge thy, expect $N$, $D$ to be such a pair.
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Why? $D$ has a global $G$ symmetry.

The coupling to $N$ is done by using the gauge fields on $N$ to gauge this global symmetry.

Result is a single, connected gauge thy.
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Why? $D$ has a global $G$ symmetry. The coupling to $N$ is done by using the gauge fields on $N$ to gauge this global symmetry. Result is a single, connected gauge thy.

Appearance of $N$, $D$ is no coincidence!

In TQFT, Koszul-dual pairs of brs. turn out to have precisely the right properties to allow slicing-and-regluing. (Can be made precise algebraically. Transversality & generation both important.)