GW Theory, FJRW Theory, and MSP Fields

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(based on joint work with
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Quintic Calabi-Yau 3-fold Q and its Mirror Family $\check{Q}_\psi$

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$$\{y_1^5 + \cdots + y_5^5 - 5\psi y_1 \cdots y_5 = 0\} \subset \mathbb{P}^4/G$$

$$G = \{(a_1, \ldots, a_5) \in (\mu_5)^5 : a_1 \cdots a_5 = 1\}/\{(a, \cdots, a) : a \in \mu_5\} \cong (\mu_5)^3$$

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$h^{1,1}(\check{Q}_\psi) = 101$, $h^{2,1}(\check{Q}_\psi) = 1$

B-model: Complex Moduli

The (compactified) complex moduli of $\check{Q}_\psi$ is $M = \mathbb{P}[5,1]$, obtained by gluing $\mathbb{C}_z$ and $[\mathbb{C}_\psi/\mu_5]$ along $\mathbb{C}^*$ by the transition function $z = (5\psi)^{-5}$.

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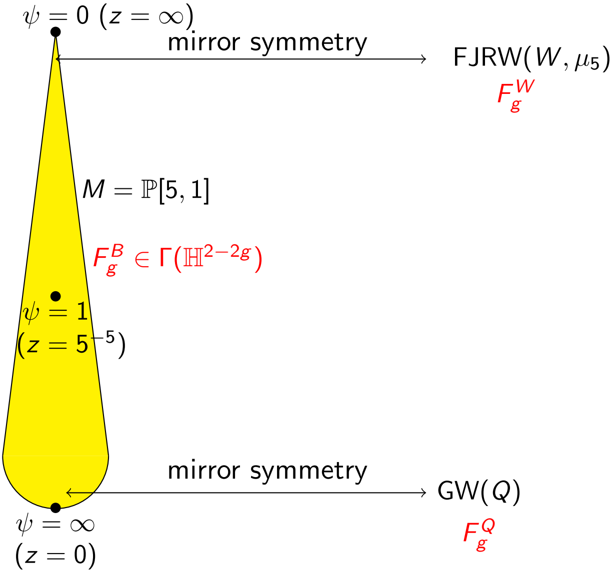
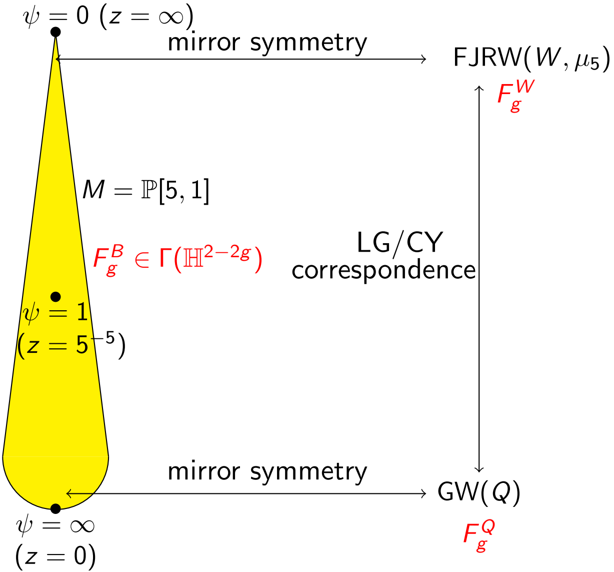
Hodge line bundle

$$\begin{array}{ccc} H^0(\check{Q}_\psi, \Omega^3_{\check{Q}_\psi}) & \subset & \mathbb{H} \\ \downarrow & & \downarrow \\ \psi & \in & M \end{array}$$

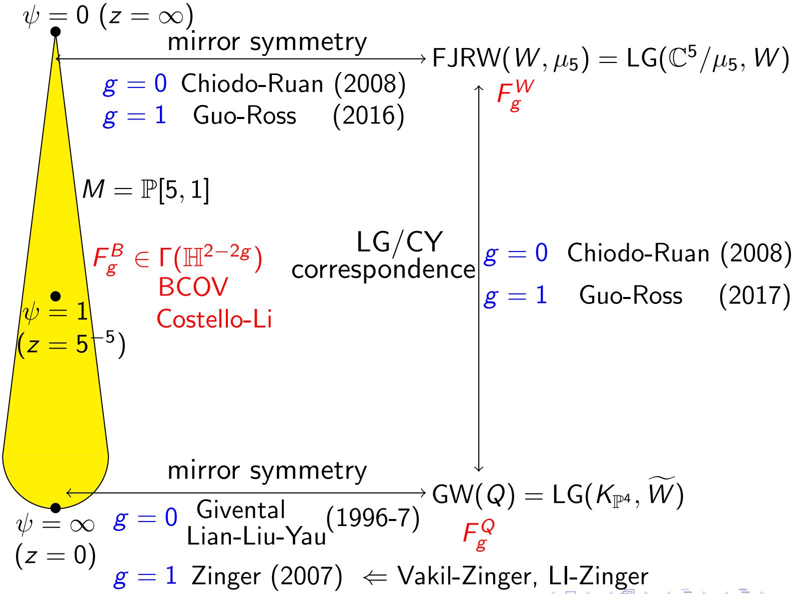
Motivation: Mirror Symmetry



Motivation: Mirror Symmetry and LG/CY Correspondence



Motivation and Overview



Gromov-Witten theory: GW(Q)

The genus g, degree d GW invariants of the quintic 3-fold Q is

$$N_{g,d} := \int_{[\overline{\mathcal{M}}_{g,0}(Q,d)]^{\mathrm{vir}}} 1 \in \mathbb{Q} \quad \text{where } (g,d) \neq (0,0), (1,0)$$

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$$F_g^Q(q) = \begin{cases} \frac{5}{6}(\log q)^3 + \sum_{d=1}^{\infty} N_{0,d} q^d, & g = 0; \\ -\frac{25}{12} \log q + \sum_{d=1}^{\infty} N_{1,d} q^d, & g = 1; \\ \sum_{d=0}^{\infty} N_{g,d} q^d, & g \geq 2. \end{cases}.$$

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Maulik-Pandharipande (2006): algorithm of evaluating $N_{g,d}$ based on degeneration (LI)

Stable maps with fields (Chang-LI)

Mathematical theory of LG$(K_{\mathbb{P}^4}, \widetilde{W} = p(x_1^5 + \cdots + x_5^5))$
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$(\varphi_1, \ldots, \varphi_5, \rho)$ quantum version of $(x_1, \ldots, x_5, p)$

Superpotential and Cosection

The **superpotential**

$$\widetilde{W} : K_{\mathbb{P}^4} \longrightarrow \mathbb{C}, \quad [x_1, \ldots, x_5, p] \mapsto p(x_1^5 + \cdots + x_5^5).$$

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σ	is the quantum version of	$\widetilde{W}$
$\mathcal{P}(\sigma) = \overline{\mathcal{M}}_{g,n}(Q, d)$	is the quantum version of	$\mathrm{Crit}(\widetilde{W}) = Q$

Cosection Localized Virtual Cycle

Applying Kiem-LI construction of cosection localized virtual cycle, Chang-LI obtain

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, d)]^{\mathrm{vir}}_{\sigma} \in \quad A_n(\overline{\mathcal{M}}_{g,n}(Q, d); \mathbb{Q})$$
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Moduli of 5-Spin Curves

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L representable line bundle on C,

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compact complex orbifold of dimension $3g - 3 + \ell$

Moduli of 5-spin Curves with Fields (Chang-LI-li)

$$\overline{\mathcal{M}}_{g,\gamma}^{1/5,5\varphi} = \Big\{[C,\vec{z},L,\varphi,\rho] : [C,\vec{z},L,\rho] \in \overline{\mathcal{M}}_{g,\gamma}^{1/5}, \\ \varphi = (\varphi_1,\ldots,\varphi_5) \in H^0(C, L^{\oplus 5})\Big\}$$

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Fan-Jarvis-Ruan-Witten theory: FJRW(W, μ_5)

Applying Kiem-LI's construction of cosection localized virtual cycles, Chang-LI-Li obtain the Witten's top Chern class

$$[\overline{\mathcal{M}}_{g,\gamma}^{1/5,5\varphi}]^{\mathrm{vir}}_{\sigma} \in \begin{array}{l} A_{d_\gamma}(\overline{\mathcal{M}}_{g,\gamma}^{1/5};\mathbb{Q}) \\ H_{2d_\gamma} \end{array}$$

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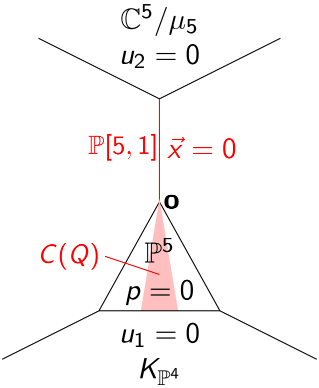
$$F_g^W(t) := \sum_{\ell\geq 0} \theta_{g,\ell}t^\ell \qquad (\theta_{1,0} := 0)$$

The Master Space



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$$M := (\mathbb{C}^6 \times \mathbb{P}^1)//\mathbb{C}^* = (\mathbb{C}^6 \times \mathbb{P}^1 - Z)/\mathbb{C}^*$$

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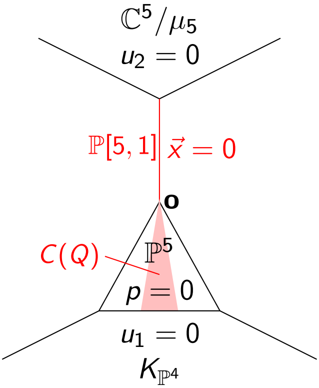

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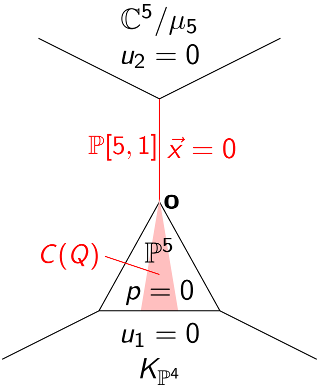

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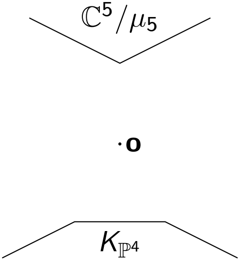
$$\mathrm{Crit}(\widetilde{W}) = \{p = \sum_{i=1}^5 x_i^5 = 0\} \cup \{\vec{x} = 0\} = C(Q) \cup \mathbb{P}[5,1]$$

Torus action

Consider the action of $T(\cong \mathbb{C}^*)$ on M by

$$t \cdot [x_1, \ldots, x_5, p, [u_1, u_2]] = [x_1, \ldots, x_5, p, [tu_1, u_2]].$$



The T fixed locus

$$M^T = K_{\mathbb{P}^4} \cup \mathbf{o} \cup (\mathbb{C}^5/\mu_5)$$

$$\mathrm{Crit}(\widetilde{W})^T = \quad Q \cup \mathbf{o} \cup (\{0\}/\mu_5)$$

Mixed Spin P (MSP) Fields (Chang-LI-Li-L)

Mathematical Theory of LG$(M, \widetilde{W} = p(x_1^5 + \cdots + x_5^5))$.
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The superpotential $\widetilde{W}$ determines a cosection $\sigma : \mathcal{O}b \to \mathcal{O}_{\mathcal{W}}$, where $\mathcal{O}b$ is the obstruction sheaf on $\mathcal{W} = \mathcal{W}_{g,\gamma,d_0,d_\infty}$, with degeneracy locus $\mathcal{W}^-$.

Cosection Localized Virtual Cycle

$T = \mathbb{C}^*$ acts on $\mathcal{W}$ by scaling ν_1. Everything is T-equivariant.

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Theorem (Chang-LI-Li-L)

$\mathcal{W}(\sigma)$ is closed, proper, and of finite type.

We obtain the cosection localized virtual cycle (Kiem-Li)

$$[\mathcal{W}_{g,\gamma,d_0,d_\infty}]^{\mathrm{vir}}_{\sigma} \in A^T_{d^{\mathrm{vir}}}(\mathcal{W}^-_{g,\gamma,d_0,d_\infty};\mathbb{Q}).$$

where

$$d^{\mathrm{vir}} = d_0 + d_\infty + 1 - g + \ell - \frac{4}{5}\sum_{i=1}^{\ell} m_i \text{ if } \gamma_i = \zeta^{m_i}.$$

Torus Localization

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$\Longrightarrow$ T-equivariant MSP theory can be reduced to

- $\mathrm{LG}(K_{\mathbb{P}^4}, \widetilde{W}) = \mathrm{GW}(Q)$
- GW theory of a point (known)
- $\mathrm{LG}(\mathbb{C}^5/\mu_5, W) = \mathrm{FJRW}(W, \mu_5)$

Vanishing and Polynomial Equations

When $d^{\mathrm{vir}} > 0$,

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Theorem (Chang-LI-Li-L)

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   - $\Theta_{g',\ell}$ for $g' < g$ and $\ell \leq 2g - 4$
   - $\Theta_{g,\ell}$ for $\ell \leq 2g - 2$.

Vanishing and Polynomial Equations

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Guo-Ross (2016): mirror formula of F_1^W