Versality in mirror symmetry

Nick Sheridan

July 27, 2017

Introduction

- Aim: prove Kontsevich's homological mirror symmetry (HMS) conjecture.
- The following strategy for proving HMS is due to Seidel:
  1. Prove it in the large volume/complex structure limit;
  2. Extend to nearby points using a 'versality theorem'.
- We prove a general versality theorem of this type.
- (with Ivan Smith) Applications include proofs of HMS for:
  - All Greene–Plesser mirrors;
  - Related examples, including Kuznetsov's 'K3 category of the cubic fourfold' and the (rigid) 'Z-manifold'.

Homological mirror symmetry

- Let X and Y be mirror compact Calabi–Yaus.
- The *mirror map* is an isomorphism[1]

$$\Psi : \mathcal{M}_{\text{Käh}}(X) \xrightarrow{\sim} \mathcal{M}_{\text{cpx}}(Y).$$

- HMS predicts

$$D^b Fuk(X_q) \simeq D^b Coh(Y_{\Psi(q)})$$

for all $q \in \mathcal{M}_{\text{Käh}}(X)$.

[1] $\mathcal{M}_{\text{Käh}}(X) :=$ Kähler moduli space of X, with $T\mathcal{M}_{\text{Käh}}(X) \cong H^{1,1}(X)$; $\mathcal{M}_{\text{cpx}}(Y) :=$ complex moduli space of Y, with $T\mathcal{M}_{\text{cpx}}(Y) \cong H^1(TY)$.

Versality

Suppose we know HMS at *one* point in the moduli space:

$$D^b Fuk(X_q) \simeq D^b Coh(Y_p).$$

Then we have:

$$\left\{ \begin{array}{c} \text{Formal deformations} \\ \text{of } D^b Fuk(X_q) \end{array} \right\} \xleftrightarrow{\cong} \left\{ \begin{array}{c} \text{Formal deformations} \\ \text{of } D^b Coh(Y_p) \end{array} \right\}$$

Versality

Suppose we know HMS at *one* point in the moduli space:

$$D^b Fuk(X_q) \simeq D^b Coh(Y_p).$$

Then we have:

$$\begin{array}{ccc}
\text{Formal nbhd. of } q \in \mathcal{M}_{\text{Käh}}(X) & & \text{Formal nbhd. of } p \in \mathcal{M}_{\text{cpx}}(Y) \\
\downarrow & & \downarrow \\
\left\{ \begin{array}{c}\text{Formal deformations} \\ \text{of } D^b Fuk(X_q)\end{array} \right\} & \overset{\cong}{\longleftrightarrow} & \left\{ \begin{array}{c}\text{Formal deformations} \\ \text{of } D^b Coh(Y_p)\end{array} \right\}
\end{array}$$

Versality

Suppose we know HMS at *one* point in the moduli space:

$$D^b Fuk(X_q) \simeq D^b Coh(Y_p).$$

Then we have:

$$\begin{array}{ccc}
\text{Formal nbhd. of } q \in \mathcal{M}_{\text{Käh}}(X) & & \text{Formal nbhd. of } p \in \mathcal{M}_{\text{cpx}}(Y) \\
\downarrow \cong & & \downarrow \cong \\
\left\{ \begin{array}{c} \text{Formal deformations} \\ \text{of } D^b Fuk(X_q) \end{array} \right\} & \overset{\cong}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{Formal deformations} \\ \text{of } D^b Coh(Y_p) \end{array} \right\}
\end{array}$$

- *Versality*: these are isos.

Versality

Suppose we know HMS at *one* point in the moduli space:

$$D^b Fuk(X_q) \simeq D^b Coh(Y_p).$$

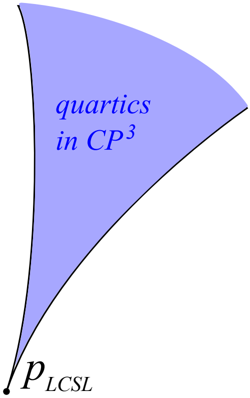
Then we have:

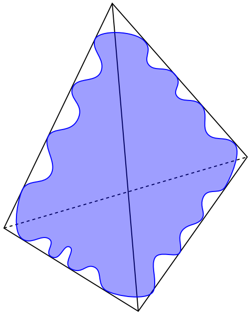
$$\begin{array}{ccc}
\begin{array}{c}\text{Formal nbhd.}\\ \text{of } q \in \mathcal{M}_{\text{Käh}}(X)\end{array} & \xrightarrow[\sim]{\Psi} & \begin{array}{c}\text{Formal nbhd.}\\ \text{of } p \in \mathcal{M}_{\text{cpx}}(Y)\end{array} \\
\downarrow \cong & & \downarrow \cong \\
\left\{\begin{array}{c}\text{Formal deformations}\\ \text{of } D^b Fuk(X_q)\end{array}\right\} & \overset{\cong}{\longleftrightarrow} & \left\{\begin{array}{c}\text{Formal deformations}\\ \text{of } D^b Coh(Y_p)\end{array}\right\}
\end{array}$$

- *Versality*: these are isos.
- Implies that HMS holds at all nearby points.

Large complex structure limit (LCSL)

- Compactify $\mathcal{M}_{\text{cpx}}(Y) \subset \bar{\mathcal{M}}_{\text{cpx}}(Y)$ by allowing some degenerate varieties.
- $p_{LCSL} \in \partial\bar{\mathcal{M}}_{\text{cpx}}(Y)$ is a 0-stratum.
- $Y_{p_{LCSL}}$ is 'maximally degenerate'.

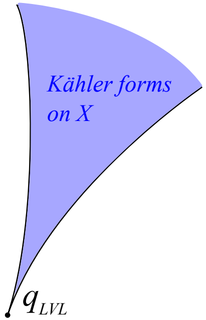


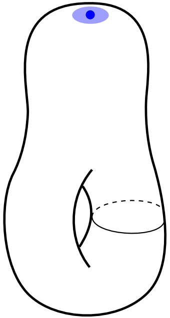


$$Y_{p_{LCSL}} = \{z_1 z_2 z_3 z_4 = 0\} \subset \mathbb{CP}^3.$$

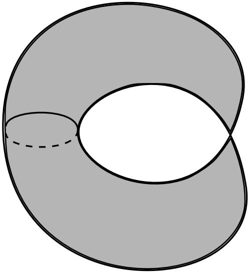
Large volume limit (LVL)

- Choose $D \subset X$ a normal-crossings divisor.
- Compactify $\mathcal{M}_{\text{Käh}}(X) \subset \bar{\mathcal{M}}_{\text{Käh}}(X)_D$ by allowing $[\omega] \to +\infty$ along components of D.
- $q_{LVL} \in \partial\bar{\mathcal{M}}_{\text{Käh}}(X)_D$ is the 0-stratum: $[\omega] = +\infty$ along D.
- $X_{q_{LVL}}$ corresponds to $X \setminus D$.

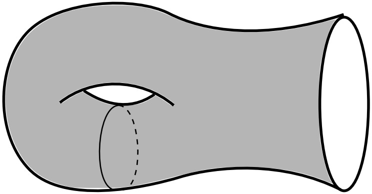




Example: elliptic curves



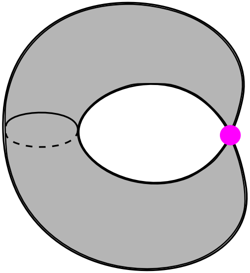
$Y_{p_{LCSL}}$ is a nodal elliptic curve.



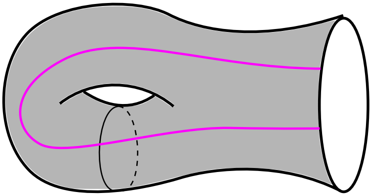
$X \setminus D$ is a punctured elliptic curve.

$$\begin{array}{ccc} Perf(Y_{p_{LCSL}}) & \xleftrightarrow{\sim} & D^b Fuk(X \setminus D) \\ \big\downarrow & & \big\downarrow \\ D^b Coh(Y_{p_{LCSL}}) & \xleftrightarrow{\sim} & D^b WFuk(X \setminus D) \end{array}$$

Example: elliptic curves



$Y_{p_{LCSL}}$ is a nodal elliptic curve.



$X \setminus D$ is a punctured elliptic curve.

$$\begin{array}{ccc} Perf(Y_{p_{LCSL}}) & \overset{\sim}{\longleftrightarrow} & D^b Fuk(X \setminus D) \\ \downarrow & & \downarrow \\ D^b Coh(Y_{p_{LCSL}}) & \overset{\sim}{\longleftrightarrow} & D^b WFuk(X \setminus D) \end{array}$$

Strategy (Seidel): use versality at LCSL/LVL to prove HMS

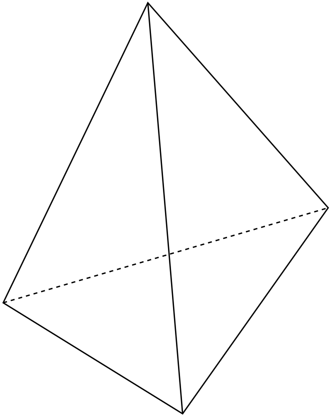
- It is easier to compute $D^b Fuk(X \setminus D)$ than $D^b Fuk(X)$:
  - *Finite* number of computations;
  - Seidel's Lefschetz fibration techniques;
  - Lagrangian skeleta techniques (Ganatra–Pardon–Shende);
  - Behaves nicely under taking covers, products.
- Hence easier to prove HMS 'at the limit':

$$Perf(Y_{p_{LCSL}}) \simeq D^b Fuk(X \setminus D).$$

- Hope a general versality result will allow us to conclude HMS over a formal neighbourhood of the limit.
- Problem: extra component of deformation space!

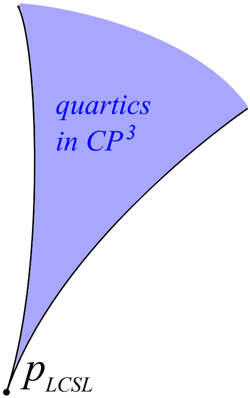
LCSL deformation space

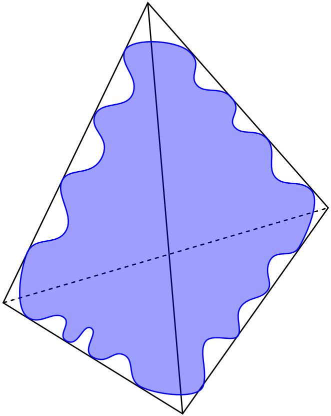
$\cdot\, p_{LCSL}$



$$Y_{p_{LCSL}} = \{z_1 z_2 z_3 z_4 = 0\} \subset \mathbb{CP}^3.$$

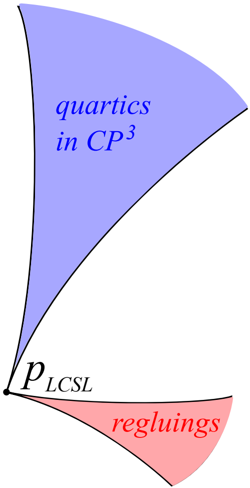
LCSL deformation space

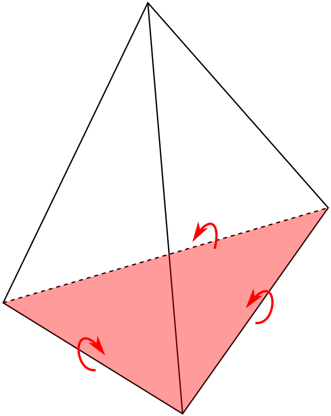




$$Y_{p_{LCSL}} = \{z_1 z_2 z_3 z_4 = 0\} \subset \mathbb{CP}^3.$$

LCSL deformation space



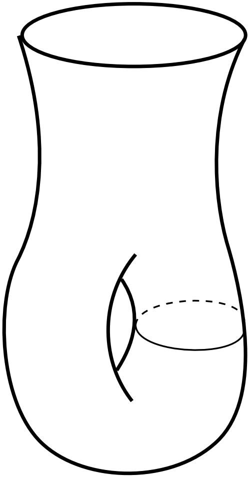


$$Y_{p_{LCSL}} = \{z_1 z_2 z_3 z_4 = 0\} \subset \mathbb{CP}^3.$$

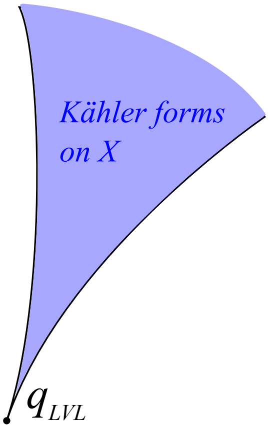
Deform by cutting out $\{z_4 = 0\}$ and gluing it back in with a twist.

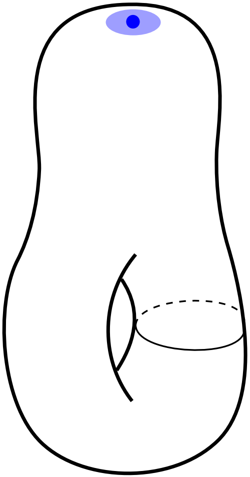
LVL deformation space

$\cdot\; q_{LVL}$

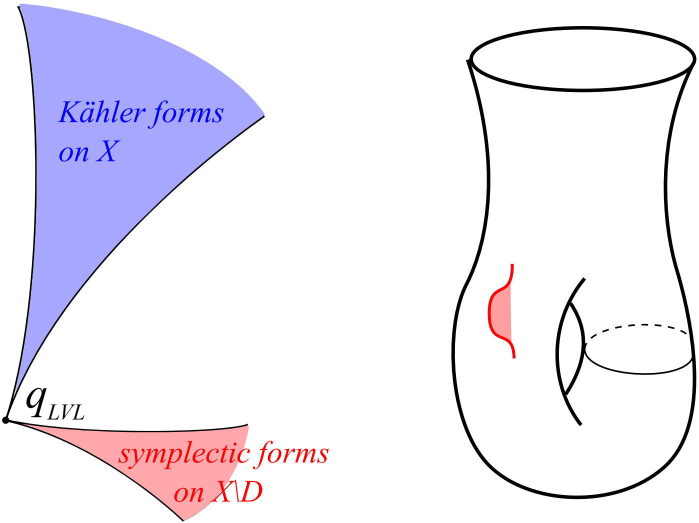


LVL deformation space





LVL deformation space



Deformation theory via L_∞ algebras

- Recall that an L_∞ algebra (or 'strong homotopy Lie algebra') is a pair $\mathfrak{g} := (V, \ell^j)$, where

$$\ell^j : \wedge^j V \to V[2-j] \quad \text{satisfy}$$

$$\sum_\sigma \pm \ell^*(\ell^*(v_{\sigma(1)}, \ldots), v_{\sigma(i)}, \ldots, v_{\sigma(j)}) = 0.$$

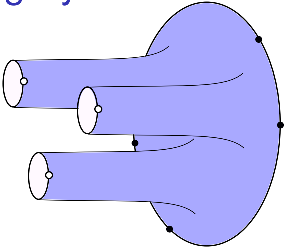
- One defines the *Maurer–Cartan space*:

$$MC(\mathfrak{g}) := \left\{ v \in V_1 : \sum_j \frac{1}{j!} \ell^j(v, \ldots, v) = 0 \right\} / \sim,$$

where $\sim$ denotes 'gauge equivalence'.

- In many deformation problems, the space of deformations up to equivalence is equal to the Maurer–Cartan space of some L_∞ algebra.

Deformation space of a category



- "Let $CC^*(C)$ = Hochschild cochains on category C. Then

$$\{\text{Def. of } C\} / \sim \quad \cong \quad MC(CC^*(C))."$$

- There is an L_∞ morphism

$$CO : SC^*(X) \dashrightarrow CC^*(D^b Fuk(X)),$$

in good situations a *quasi-isomorphism* $\Rightarrow$ same MC.

- Here $SC^*(X)$ is the *Floer homology*: Morse homology of the free loop space of X w.r.t. the action functional.

[1] See, e.g., Blanc–Katzarkov–Pandit: arXiv:1705.00655.

Deformations of $D^b Fuk(X_q)$ ($\cong MC(SC^*(X_q))$)

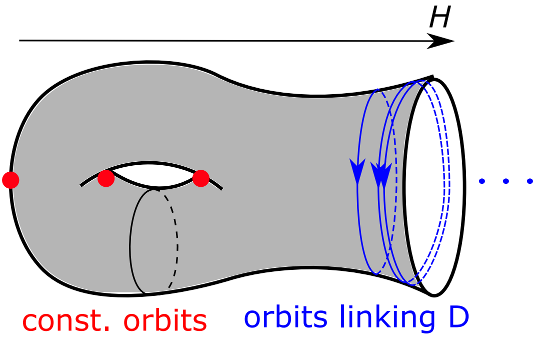
- If X is compact, $SC^*(X_q)$ 'localizes at constant loops':

$$SC^*(X_q) \simeq H^*(X),$$

  and the L_∞ structure on the RHS is trivial.
- Therefore the space of deformations of $D^b Fuk(X_q)$ is locally isomorphic to $H^2(X)$.
- If $H^2(X) \cong H^{1,1}(X)$, we have versality!

$SC^*(X \setminus D)$

- $X_{q_{LVL}} \sim X \setminus D$ is *not* compact: $SC^*(X \setminus D)$ is the Floer homology of the free loop space with respect to the action functional which is *deformed by a Hamiltonian H which goes to* $+\infty$ *near D* (Floer–Hofer).
- $0 \to C^*(X \setminus D) \to SC^*(X \setminus D) \to SC^*_+(X \setminus D) \to 0.$
- $C^*(X \setminus D)$ comes from constant loops; $SC^*_+(X \setminus D)$ comes from loops 'at infinity' (i.e., linking D).



Deformations of $D^b Fuk(X \setminus D)$ ($\cong MC(SC^*(X \setminus D))$)

$$\ldots \to H^2(X \setminus D) \to SH^2(X \setminus D) \to SH^2_+(X \setminus D) \to \ldots$$

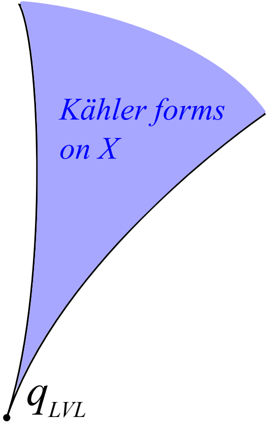
- Deforming $X \setminus D$ by a symplectic form ω corresponds to $[\omega] \in H^2(X \setminus D)$.
- If X Calabi–Yau, D 'large enough', then

$$SH^2_+(X \setminus D) \cong H^2(X, X \setminus D)$$

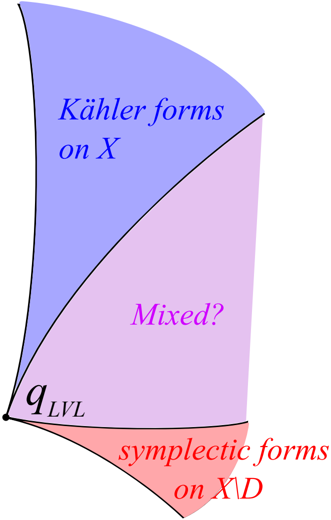
  (loop around D_i corresponds to $PD(D_i)$).
- Deforming $X \setminus D$ by compactifying with Kähler form ω corresponds to $[\omega] \in H^2(X, X \setminus D) \cong SH^2_+(X \setminus D)$.

Picture of $MC(SC^*(X \setminus D))$

What we want:



What we have:



What we need: $H^2(X \setminus D) = 0$.

Getting rid of the extra deformations

- Suppose $G \circlearrowright (X, D)$ and $H^2(X \setminus D)^G = 0$.
- The space of deformations that respect the action of G has no extra deformations.
- Useful to allow G to act by *anti*-holomorphic involutions, which act by *dualities*: $D^bFuk \xrightarrow{\sim} D^bFuk^{op}$.
- To apply to HMS, we need analogous group action on mirror: a natural duality is

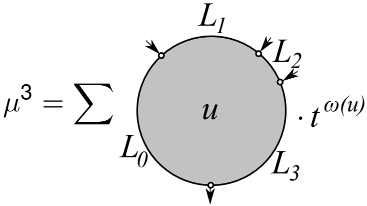
$$D^bCoh(Y) \xrightarrow{\sim} D^bCoh(Y)^{op}$$
$$\mathcal{E} \mapsto \mathcal{E}^\vee.$$

Fukaya category, $Fuk(X, \omega)$

- (X, ω) a compact symplectic manifold.
- Coefficient field: $\Lambda := \mathbb{C}((t^{\mathbb{R}}))$.
- Objects: Lagrangian submanifolds $L \subset X$.
- Morphisms: $hom^*(L_0, L_1) := \Lambda\langle L_0 \cap L_1 \rangle$.
- Composition maps:

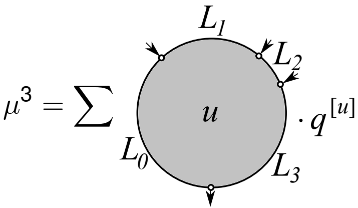
$$\mu^s : hom^*(L_0, L_1) \otimes \ldots \otimes hom^*(L_{s-1}, L_s) \to hom^*(L_0, L_s)$$

count holomorphic discs $u : \mathbb{D} \to X$ weighted by $t^{\omega(u)} \in \Lambda$:



Relative Fukaya category, $Fuk(X, D)$

- Let $D = \cup_i D_i \subset X$ be simple normal-crossings.
- Coefficient ring: let $NE \subset H_2(X, X \setminus D)$ be the cone of all u s.t. $u \cdot E \geq 0$ for any effective ample divisor E supported on D. Define $R := \mathbb{C}[\![NE]\!]$.
- Objects: compact exact Lagrangians $L \subset X \setminus D$;
- Composition maps: count holomorphic discs $u : \mathbb{D} \to X$ weighted by $q^{[u]} \in R$.



Relationship between $Fuk(X, \omega)$ and $Fuk(X, D)$

- Think of $Fuk(X, D)$ as a family of categories over $\bar{\mathcal{M}}_{\text{Käh}}(X)_D := Spec(R)$.
- $q_{LVL} \in \bar{\mathcal{M}}_{\text{Käh}}(X)_D$ is maximal ideal corresponding to vertex of NE; we have

$$Fuk(X, D)_{q_{LVL}} \simeq Fuk(X \setminus D).$$

- Kähler form ω with $\omega|_{X \setminus D} = d\theta$ defines map

$$\begin{aligned} R &\to \Lambda \\ q^u &\mapsto t^{\omega(u) - \theta(\partial u)}, \end{aligned}$$

  which we regard as a Λ-point $q_\omega \in \bar{\mathcal{M}}_{\text{Käh}}(X)_D$.
- There is an embedding

$$Fuk(X, D)_{q_\omega} \hookrightarrow Fuk(X, \omega).$$

Versality theorem

Theorem (S., 2017)

Suppose that:

- *G acts on* (X, D) *(anti-)holomorphically, and* $H^2(X \setminus D)^G = 0$*;*
- *D 'supports enough effective ample divisors';*
- *X is Calabi–Yau, and the map*

$$CO : SH^2(X \setminus D) \to HH^2(Fuk(X \setminus D))$$

  is surjective.

Then for any other deformation B of $Fuk(X \setminus D)$ *over R with same first-order behaviour, respecting G-action, there exists an isomorphism* $\Psi^* : R \to R$ *and an* A_∞ *isomorphism* $Fuk(X, D) \simeq \Psi^* B$*.*

Application: the cubic fourfold (joint with I. Smith)

- Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface.
- There is a semi-orthogonal decomposition

$$D^b Coh(Y) \simeq \langle \mathcal{A}_Y, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle.$$

- The *Kuznetsov category* $\mathcal{A}_Y$ can be regarded as a noncommutative K3 surface.

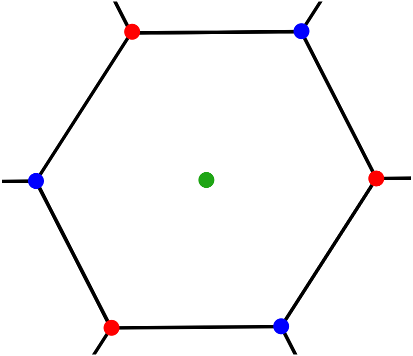
Conjecture (Kuznetsov)

Y is rational if and only if

$$\mathcal{A}_Y \simeq D^b Coh(S)$$

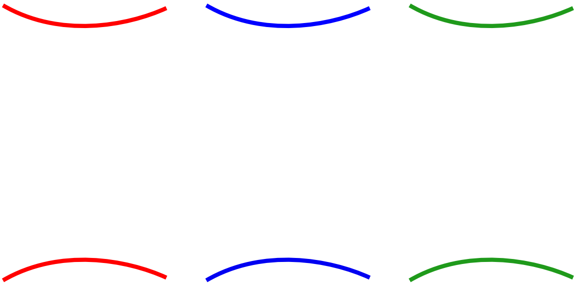
for some K3 surface S.

Mirror to the cubic fourfold (Batyrev–Borisov)



- $\mathbb{Z}/3 \circlearrowright E$ (elliptic curve).
- p_1, p_2, p_3 fixed points.
- $X' := E \times E/(x, y) \sim (\zeta \cdot x, \zeta^{-1} \cdot y)$ has 9 A_2 singularities.
- Resolve them to get X.

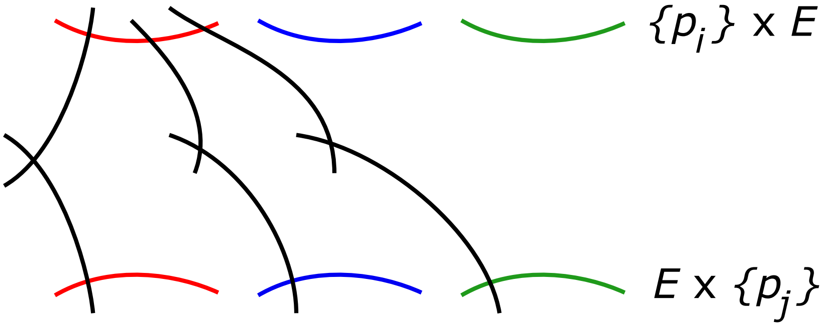
Divisors on X



$\{p_i\} \times E$

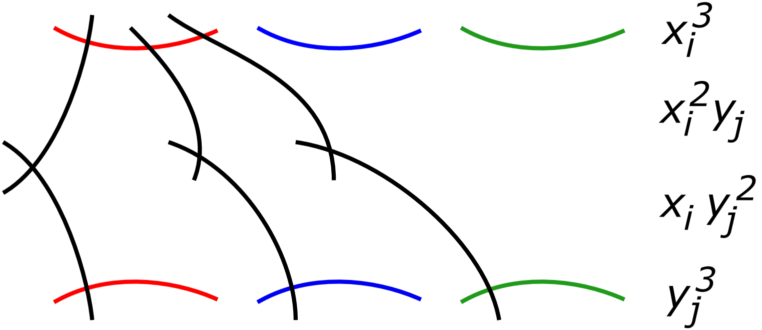
$E \times \{p_j\}$

Divisors on X



- Let $\{D_k\}_{k \in K}$ be the set of divisors: $|K| = 24$.

Monomial–divisor correspondence



- Define cubic monomials $(x, y)^k \in \Lambda[x_1, x_2, x_3, y_1, y_2, y_3]$ corresponding to each divisor D_k.

Homological mirror symmetry

Theorem (S.–Smith 2017)

If $[\omega] = \sum_{k \in K} \lambda_k \cdot [D_k]$ *is an 'ambient' Kähler form on* X*, then there exist*

$$p_k = t^{\lambda_k}(1 + \mathcal{O}(t)) \in \Lambda$$

such that the noncommutative K3 associated to

$$Y_p := \left\{ -x_1 x_2 x_3 - y_1 y_2 y_3 + \sum_{k \in K} p_k \cdot (x, y)^k = 0 \right\} \subset \mathbb{P}^5_\Lambda$$

is mirror to (X, ω)*: i.e.,*

$$D^\pi Fuk(X, \omega) \simeq \mathcal{A}_{Y_p}.$$

Proof: first show $D^\pi Fuk(X \setminus D) \simeq \mathcal{A}_{Y_0}$, then extend to a formal neighbourhood by 'versality'.