Supersymmetric indices, partition functions and the A-twist

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String Math 2017, Universität Hamburg

Based on: 1605.06531 with H. Kim
1701.03171 and 1707.05774 with H. Kim and B. Willett
This talk will be about QFTs with four Poincaré supercharges.

The relevant supersymmetry algebras in dimensions $d = 2, 3$ and 4 are called:

$$2d \ n = (2, 2) \xleftarrow{\mathbb{S}^1} 3d \ n = 2 \xleftarrow{\mathbb{S}^1} 4d \ n = 1$$

We restrict ourselves to theories with a $U(1)_R$ symmetry:

$$[R, Q] = -Q, \quad [R, \bar{Q}] = \bar{Q},$$

Including but not limited to: SCFTs.
The twisted chiral ring

Consider the 2d case:

$$2\text{d } \mathcal{N} = (2, 2): \quad \{Q_-, \bar{Q}_-\} = P_z, \quad \{Q_+, \bar{Q}_+\} = -P_{\bar{z}},$$

$$\{Q_-, \bar{Q}_+\} = iZ, \quad \{Q_+, \bar{Q}_-\} = i\bar{Z},$$

with $Z$ a complex central charge that commutes with $R$.

$\mathcal{N} = (2, 2)$ theories contain interesting subsectors of protected local operators.

We are interested in the twisted chiral ring:

$$\omega : \quad [Q_-, \omega] = 0, \quad [\bar{Q}_+, \omega] \quad (\text{mod } Q \text{ or } \bar{Q}-\text{exact})$$

It is conveniently singled out by the topological $A$-twist.

[Witten, 1988]
Twisted chiral ring for $d = 3$ and $4$

In dimension $d > 2$, the twisted-chiral condition breaks $so(d)$ covariance down to $so(2)$.

Twisted chiral operators are extended operators of codimension 2:

$d = 3$: half-BPS line operators in 3d $\mathcal{N} = 2$ theories, $\mathcal{L}$. (For instance, supersymmetric Wilson loop operators.)

$d = 4$: half-BPS surface operators in 4d $\mathcal{N} = 1$ theories, $\mathcal{S}$. 
Parallel twisted chiral operators have non-singular OPE. They satisfy a fusion algebra. We must have:

$$\mathcal{L}_i \cdot \mathcal{L}_j = N_{ij}^k \mathcal{L}_k$$

for half-BPS line operators $\mathcal{L}$ in 3d, and similarly for $S$ in 4d.

These algebras have been discussed e.g. by:

[Kapustin, Willett, 2013; Cecotti, Gaiotto, Vafa, 2013]
A-models

Our setup will be:

- 3d $\mathcal{N} = 2$ theory on $\mathbb{R}^2 \times S^1$
- 4d $\mathcal{N} = 1$ theory on $\mathbb{R}^2 \times T^2$

with line operators on $S^1$, or surface operators on $T^2$.

$\mathbb{R}^2$ can be compactified to $\Sigma_g$ with the $A$-twist. We define the “A-model” of the 3d or 4d theory as the 2d TFT on $\Sigma_g$ obtained by going to the cohomology of $Q_-$ and $\bar{Q}_+$.

$A$-model observables:

$$\langle L_i L_j \cdots \rangle_{\Sigma_g \times S^1}, \quad \langle S_i S_j \cdots \rangle_{\Sigma_g \times T^2}$$

They capture the quantum ring structure constants $N_{ij}^k$. 
Another disclaimer

This talk will be exclusively concerned with ultraviolet (UV)-complete gauge theories with a UV Lagrangian: \footnote{My apologies.}

\[
\langle \mathcal{O} \rangle = \int [D\mathcal{V} D\Phi] e^{-\int d^3 x \sqrt{g} \mathcal{L}(\mathcal{V}, \Phi)} \mathcal{O}(\mathcal{V}, \Phi)
\]

For instance:

- 3d $\mathcal{N} = 2$ SQED
- 3d $\mathcal{N} = 2^*$ quivers
- 4d $\mathcal{N} = 1$ $SU(N_c)$ SQCD
- \ldots
Supersymmetric indices and partition functions

In the last 10 years, there has been a lot of interest in computing supersymmetric partition functions and indices:

\[ Z_{\mathcal{M}_3}[\mathcal{T}_{3d \mathcal{N}=2}] \], \quad Z_{\mathcal{M}_3 \times S^1}[\mathcal{T}_{4d \mathcal{N}=1}] = I_{\mathcal{M}_3} \]

This can be done using supersymmetric localization in many examples. [Pestun, 2007; Kapustin, Willett, Yaakov, 2010; Jafferis, 2010; Hama, Hosomichi, Lee, 2010; Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014; Assel, Cassani, Martelli, 2014; ...]

For \( \mathcal{M}_3 \approx \Sigma_g \times S^1 \), it is a A-model observable:

\[ Z_{\Sigma_g \times S^1} = \langle 1 \rangle_{\Sigma_g \times S^1} \]


Can we understand more general partition functions as A-model observables?
Supersymmetric indices and partition functions

One of the best-known example is the three-sphere index:

\[ I_{S^3} = \text{Tr}_{S^3} \left[ (−1)^F p^{J_3 + J'_3 + \frac{1}{2} R} q^{J_3 - J'_3 + \frac{1}{2} R} \prod_F y_{QF} \right] \]

In 3d, this reduce to the ("squashed") \( S^3_b \) partition function.

In this talk: \( p = q \equiv q \), \( \leftrightarrow \) \( b = 1 \)

The index can be computed in terms of an elliptic hypergeometric integral:

\[ I_{S^3} = q^{E_{S^3}} \frac{(q; q)_{2\text{rk}(G)}}{|W_G|} \int \prod_{a=1}^{\text{rk}(G)} \frac{dx_a}{2\pi i x_a} \prod_{\rho,\omega} \Gamma_0(x^\rho y^\omega q^{r_{\rho,\omega} - 1}; q) \prod_\alpha \Gamma_0(x^\alpha q^{-1}; q) \]

\[ \text{[Romelsberger, 2005; Dolan, Osborn, 2008]} \]
Supersymmetric indices and partition functions

Localization computations can be very subtle.

In this talk, I’ll explain a different method to compute:

\[ Z_{\mathcal{M}_3} , \quad Z_{\mathcal{M}_3 \times S^1} , \]

for 3d and 4d gauge theories, for a relatively simple family of \( \mathcal{M}_3 \) backgrounds allowed by supersymmetry.

This will teach us some interesting lessons about these objects, and will allow us to compute a few new interesting observables in theories with 4 supercharges.

Previous works: [Ohta, Yoshida, 2012; Nishioka, Yaakov, 2014]
Outline

3d $\mathcal{N} = 2$ theories
Outline

3d $\mathcal{N} = 2$ theories

4d $\mathcal{N} = 1$ theories
Outline

3d $\mathcal{N} = 2$ theories

4d $\mathcal{N} = 1$ theories

Applications
<table>
<thead>
<tr>
<th>Introduction</th>
<th>3d $\mathcal{N} = 2$ theories</th>
<th>4d $\mathcal{N} = 1$ theories</th>
<th>Applications</th>
<th>Outlook</th>
</tr>
</thead>
</table>

3d $\mathcal{N} = 2$ theories
Consider 3d $\mathcal{N} = 2$ supersymmetric Yang-Mills-Chern-Simons-matter theories, with:

- **Vector multiplet** $\mathcal{V}$ for a gauge group $G$, with $\text{Lie}(G) = \mathfrak{g}$.
- **Chiral multiplets** $\Phi_i$ in representations $\mathcal{R}_i$ of $\mathfrak{g}$.
- $R$-symmetry-preserving superpotential $W(\Phi)$.
- A choice of CS interactions for $G \times G_F$:

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}_3} d^3x \sqrt{g} (i \epsilon^{\mu\nu\rho} (a_\mu \partial_\nu a_\rho - \frac{2i}{3} a_\mu a_\nu a_\rho) - 2\sigma D + 2i \tilde{\lambda} \lambda)$$

We have the CS level $k \in \mathbb{Z}$, and $\mathcal{M}_3$ must be a spin manifold.
Circle compactification

Consider the theory on $\mathbb{R}^2 \times S^1$, with $S^1$ a circle of radius $\beta$. Using the Kaluza-Klein (KK) expansion:

$$\phi = \sum_{n \in \mathbb{Z}} \phi_n(z, \bar{z}) e^{in\psi},$$

we can consider the 3d theory as a 2d theory with an infinite number of fields, in 2d $\mathcal{N} = (2, 2)$ supermultiplets.

In particular, we have a 2d vector multiplet that includes a complex scalar:

$$u = i\beta \sigma - a^{(0)}, \quad a^{(0)} = \frac{1}{2\pi} \int_{S^1} a_\mu dx^\mu$$

with $\sigma$ the real scalar in $\mathcal{V}$ in 3d. We take $u$ dimensionless.
Consider giving an expectation values to the scalar $u$ in the 2d vector multiplet. This corresponds to the classical Coulomb branch of the 3d theory:

$$u = \text{diag}(u_a), \quad a = 1, \cdots, \text{rk}(G)$$

Due to large gauge transformations along $S^1$, we have:

$$u_a \sim u_a + 1, \quad \mathcal{M} \cong (\mathbb{C}^*)^{\text{rk}(G)}/W_G$$

We will also use the variables:

$$x_a \equiv e^{2\pi i u_a}$$
Circle compactification

At a generic point on $\mathcal{M}$, the 3d gauge group is Higgsed to:

$$\mathbf{G} \rightarrow \mathbf{H} \cong \prod_{a=1}^{\text{rk}(\mathbf{G})} U(1)_a$$

We can also think in terms of diagonalization of the 2d vector multiplet. In the path integral language, we should still sum over topological sectors, using a functional Weyl integral formula. [Blau, Thompson, 1992, 1993].

We integrate out all massive fields and write down an effective field theory for the low-energy modes $u_a$ and its superpartners, in twisted chiral multiplets $U_a$. That is our “$A$-model.”
Mass parameters: background $G_F$ vector multiplets

We are considering the theory in the presence of arbitrary supersymmetry-preserving background fields for the flavor symmetry $G_F$ with maximal torus:

$$H_F = \prod_\alpha U(1)_\alpha \subset G_F$$

We have the flavor parameters:

$$\nu_\alpha = i \beta m^F_\alpha - a^{(0)}_\alpha, \quad y_\alpha \equiv e^{2\pi i \nu_\alpha}$$

One may call $\nu$ and $y$ the “chemical potentials” and “fugacities”, respectively.
Consider the $A$-twisted theory. This is equivalent to “curved-space supersymmetry” [Festuccia, Seiberg, 2012] on $\Sigma_g \times S^1$.

Up to $Q$-exact terms, the Lagrangian of the effective field theory on $\mathcal{M}$ reads:

$$
S_{\text{TQFT}} = \int_{\Sigma_g} \left( -i f_a \frac{\partial \mathcal{W}(u, \nu)}{\partial u_a} + \tilde{\Lambda}^a \Lambda^b \frac{\partial^2 \mathcal{W}(u, \nu)}{\partial u_a \partial u_b} \right)
+ \frac{i}{2} \int_{\Sigma_g} d^2x \sqrt{g} \Omega(u, \nu) R
$$

with $f_a$ the abelian field strength of $a^a_{\mu}$ and $R$ the Ricci scalar.

[Witten, 1993; Nekrasov, Shatashvili, 2014]
Effective twisted superpotential and effective dilaton

The A-model is fully determined by the two holomorphic potentials:
\[ \mathcal{W}(u, \nu), \quad \Omega(u, \nu) \]

The effective twisted superpotential takes the schematic form:
\[ \mathcal{W} = \frac{k}{2} u(u + 1) + \frac{k^F}{2} \nu(\nu + 1) + \frac{k_g}{24} + \frac{1}{(2\pi i)^2} \sum_{(\rho, \omega) \in (\mathbb{R}, \mathbb{R}_F)} \text{Li}_2(x^\rho y^\omega) \]

The classical contribution is from CS terms, including background CS terms.

The dilog is a one-loop correction from integrating out the matter fields. This result is for the so-called $U(1)_{-\frac{1}{2}}$ quantization of a 3d Dirac fermion, which preserves gauge invariance but breaks parity.

[CC, Kim, Willett, 2017]
Effective twisted superpotential and effective dilaton

Similarly, the effective dilaton takes the form:

\[
\Omega = k^a R u_\alpha + k^{\alpha R} \nu_\alpha + \frac{1}{2} k^{R R}
\]

\[
- \frac{1}{2\pi i} \sum_{(\rho,\omega) \in (\mathcal{R},\mathcal{R}_F)} (r_\omega - 1) \log(1 - x^\rho y^\omega)
\]

\[
- \frac{1}{2\pi i} \sum_{\alpha \in g} \log(1 - x^\alpha)
\]

The CS contributions are supersymmetric CS terms involving the \(U(1)_R\) background gauge field [CC, Dumitrescu, Festuccia, Komargodski, Seiberg, 2012].

Note the contribution from the W-bosons.
Effective twisted superpotential and effective dilaton

The twisted superpotential is only defined modulo the ambiguity:

$$\mathcal{W} \sim \mathcal{W} + n^a u_a + n^\alpha \nu_\alpha + n^0, \quad n^a, n^\alpha, n^0 \in \mathbb{Z},$$

due to the sum over topological sectors. Similarly, we have:

$$\Omega \sim \Omega + n, \quad n \in \mathbb{Z}$$

This corresponds to branch cut ambiguities in the variables $u, \nu$. One the other hand, well-defined $A$-model operators will be holomorphic in $u, \nu$. 
The Bethe equations

Let us define the so-called Bethe equations:

\[ \Pi_a(u, \nu) \equiv \exp \left( 2\pi i \frac{\partial W}{\partial u_a} \right) = 1, \]

[Ne'kra'sov, Shatashvili, 2009]

The vacua of the $A$-model are two-dimensional vacua, the Bethe vacua:

\[ S_{BE} = \left\{ \hat{u}_a \mid \Pi_a(\hat{u}, \nu) = 1, \ \forall a, \ w \cdot \hat{u} \neq \hat{u}, \ \forall w \in W_G \right\} / W_G \]

Importantly, we must exclude would-be “non-abelian vacua”—solutions of $\Pi_a = 1$ not acted on freely by the Weyl group—by hand.
A-model defect operators

Given $\mathcal{W}$, $\Omega$, we can define a number of “canonical” defect operators, which are local on $\Sigma_g$.

These operators probably have an explicit construction in the three-dimensional UV theory, but we will only focus on their $A$-model “low energy” description.
The flavor flux operators

In the presence of flavor symmetries $U(1)_\alpha \subset G_F$, we can turn on generic background vector multiplets $V^F_\alpha$, as long as we preserve the $A$-twist supercharges.

In particular, we have chemical potentials and background fluxes

$$\nu_\alpha, \quad \frac{1}{2\pi} \int_{\Sigma_g} f_\alpha = n_\alpha \in \mathbb{Z}$$

This adds a piece to the $A$-model action:

$$S_{\text{flux}} = \int_{\Sigma_g} \left( -i f_\alpha \frac{\partial \mathcal{W}(u, \nu)}{\partial \nu_\alpha} \right)$$
The flavor flux operators

The background field $f_\alpha$ can have arbitrary profile over $\Sigma_g$:

If we concentrate its flux near a point, we obtain a singularity, describable by a local operator.
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In particular, if we take

\[ f_\alpha = 2\pi n_\alpha \delta^2(x - x_0), \]

turning on background flux is equivalent to the insertion of a local operator:

\[ \Pi_\alpha(u, \nu)^{n_\alpha} \]

in the path integral, with

\[ \Pi_\alpha(u, \nu) = \exp \left( 2\pi i \frac{\partial W(u, \nu)}{\partial \nu_\alpha} \right) \]

We call \( \Pi_\alpha \) the flux operator for the flavor symmetry \( U(1)_\alpha \).
The handle-gluing operator

Any 2d TQFT has a “handle-gluing operator” $\mathcal{H}$:

The explicit form of $\mathcal{H}$ for simple LG models is given by [Vafa, 1990]. The generalization to 2d gauge theories was investigated more recently. [Melnikov, Plesser, 2005; Nekrasov, Shatashvili, 2014]
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The handle-gluing operator

In the A-twisted gauge theory, the handle-gluing operator can be seen as a “flux operator for $U(1)_R$".

It is given explicitly by:

$$
\mathcal{H}(u, \nu) = e^{2\pi i \Omega(u, \nu)} \det_{ab} \left( \frac{\partial^2 \mathcal{W}(u, \nu)}{\partial u_a \partial u_b} \right)
$$

[Nekrasov, Shatashvili, 2014]

The appearance of the Hessian determinant of $\mathcal{W}$ is due to the presence of fermionic zero modes from the gauginos.
\[ \Sigma_g \times S^1 \text{ correlators and the 3d twisted index} \]

In any 2d TQFT, we have:

\[
\langle O \rangle_{\Sigma_g} = \langle O \mathcal{H}^g \rangle_{\mathbb{C}P^1} = \text{Tr}_V \left( \mathcal{H}^{g-1} O \right)
\]

where \( V \) is the TQFT Hilbert space. For us, \( V \cong S_{\text{BE}} \).

For the 3d \( A \)-model, we then find: [Nekrasov, Shatashvili, 2014]

\[
\left\langle \mathcal{L} \right\rangle_{\Sigma_g \times S^1} = \sum_{\hat{x} \in S_{\text{BE}}} \mathcal{L}(\hat{x}) \mathcal{H}(\hat{x},y)^{g-1} \Pi_\alpha(\hat{x},y)^{n_\alpha}
\]

for any line operator \( \mathcal{L} \) on \( \Sigma_g \times S^1 \), in the presence of background fluxes \( n_\alpha \) for the flavor symmetry. This can also be computed by supersymmetric localization in the UV [Benini, Zaffaroni, 2015, 2016; CC, Kim, 2016]

Note: The operators \( \Pi_\alpha \) and \( \mathcal{H} \) are rational functions of \( x \) and \( y \).
The fibering operator

There exists another “canonical” A-model operator one can build from $\mathcal{W}$.

From the 2d point of view, the full flavor symmetry is:

$$G_{F}^{2d} = G_{F} \times U(1)_{KK}$$

We have distinguished symmetry $U(1)_{KK}$, whose conserved charge is the circle momentum. There exists a 2d background vector multiplet that couples to the KK momentum. In particular, we have the 2d twisted mass:

$$m_{KK} = \frac{1}{\beta}$$
The fibering operator

Definition: the **fibering operator** is the flux operator for $U(1)_{KK}$.

Reinstating dimensions, we find:

$$F(u, \nu) \equiv \exp \left( 2\pi i \frac{\partial}{\partial m_{KK}} \left( m_{KK} \mathcal{W}(u, \nu) \right) \right)$$

This leads to the explicit expression:

$$F(u, \nu) = \exp \left( 2\pi i (\mathcal{W} - u_\alpha \partial_{u_\alpha} \mathcal{W} - \nu_\alpha \partial_{\nu_\alpha} \mathcal{W}) \right)$$

in terms of the twisted superpotential $\mathcal{W}(u, \nu)$. 
The fibering operators

Inserting $\mathcal{F}^p$, $p \in \mathbb{Z}$, in the $A$-model, we realize a principal circle bundle over $\Sigma_g$:

$$S^1 \longrightarrow \mathcal{M}_{g,p} \xrightarrow{\pi} \Sigma_g.$$ 

$$p = \frac{1}{2\pi} \int_{\Sigma_g} da_{KK}$$

This $\mathcal{M}_{g,p}$ is the simplest example of a Seifert manifold.

Supersymmetry is preserved by a pull-back of the $A$-twist on $\Sigma_g$. 
The fibering operator

Importantly, the fibering operator is not fully gauge invariant under $G \times G_F$. Instead, we have the difference equations:

\[
F(u_a + 1, \nu) = F(u, \nu) \Pi_a(u, \nu)^{-1}
\]

\[
F(u, \nu_{\alpha} + 1) = F(u, \nu) \Pi_{\alpha}(u, \nu)^{-1}
\]

It is, however, gauge invariant (under $G$) on the Bethe vacua, where $\Pi_a(\hat{u}) = 1$.

All observables are fully $G \times G_F$ invariant.
Explicitly, in the YM-CS-matter theory:

$$\mathcal{F}(u, \nu) = e^{-\pi ik u^2 - \pi ik F \nu^2 + \frac{\pi i}{12} kg} \prod_{(\rho, \omega) \in (\mathbb{R}, \mathbb{R}_F)} \mathcal{F}^\Phi (\rho(u) + \omega(\nu))$$

in terms of the simple function:

$$\mathcal{F}^\Phi (u) = \exp \left( \frac{1}{2\pi i} \operatorname{Li}_2 \left( e^{2\pi i u} \right) + u \log \left( 1 - e^{2\pi i u} \right) \right)$$

This is a meromorphic function of $u$ with poles at $u = -1, -2, \cdots$ and zeros at $z = 1, 2, \cdots$. Note the identity:

$$\mathcal{F}_\Phi (u) \mathcal{F}_\Phi (-u) = e^{\pi i u^2 - \frac{\pi i}{6}}$$
The $\mathcal{M}_{g,p}$ partition function

We then directly find the $\mathcal{M}_{g,p}$ supersymmetric partition function:

$$Z_{\mathcal{M}_{g,p}}(\nu; n) = \sum_{\hat{u} \in S_{BE}} \mathcal{F}(\hat{u}, \nu)^p \mathcal{H}(\hat{u}, \nu)^{g-1} \Pi_\alpha(\hat{u}, \nu)^{n_\alpha}$$

We can view this as an expectation value for a defect line operator $\mathcal{F}$ at a point on $\Sigma_g$:

$$Z_{\mathcal{M}_{g,p}}(\nu) = \langle \mathcal{F}^p \rangle_{\Sigma_g \times S^1}$$

In this sense, the $\mathcal{M}_{g,p}$ partition functions is just another $A$-model observable.
The $S^3$ partition function

Special case $g = 0, p = 1$: The $S^3$ partition function.

$$Z_{S^3}(\nu) = \sum_{\hat{u} \in S_{BE}} \mathcal{F}(\hat{u}, \nu) \mathcal{H}(\hat{u}, \nu)^{-1} = \langle \mathcal{F} \rangle_{S^2 \times S^1}$$

Here, $\langle 1 \rangle_{S^2 \times S^1} = Z_{S^2 \times S^1}$ is also known as “twisted 3d index”.

[Benini, Zaffaroni, 2015]

So far, we chose all $R$-charges $r_i \in \mathbb{Z}$ for the chiral multiplets $\Phi_i$. This is necessary for the $A$-twist point of view.

On $S^3$, there is a canonical analytic continuation

$$Z_{S^3}(\nu) \rightarrow Z_{S^3}(\nu + (R - 1))$$

to any $r_i \in \mathbb{R}$. [CC, Dumitrescu, Festuccia, Komargodski, 2014]

In particular, this gives a nice way to compute $F_{S^3} = -\log Z_{S^3}$ for a 3d $\mathcal{N} = 2$ SCFT. [Jafferis, 2012]
4d $\mathcal{N} = 1$ theories
4d $\mathcal{N} = 1$ gauge theories

Consider an $\mathcal{N} = 1$ gauge theory. For simplicity, we take $G$ semi-simple and simply connected, and asymptotically-free theories.

We play the same game as before by compactifying on a $T^2$ with modular parameter $\tau$. Now, the $G \times G_F$ parameters $u, \nu$ are themselves valued in a torus:

$$u \sim u + 1 \sim u + \tau, \quad \nu \sim \nu + 1 \sim \nu + \tau$$

Let us introduce the convenient notation:

$$u_a = (u_a, \nu_\alpha), \quad a = (a, \alpha)$$
The twisted superpotential is given by:

\[ \mathcal{W}(u; \tau) = -\frac{A^{abc}u_au_bu_c}{6\tau} + \sum_{\rho \in (\mathbb{R}, \mathbb{R}_F)} \psi(\rho(u); \tau) \]

It is a purely quantum (one-loop) effect. We defined the “elliptic dilog”:

\[ \psi(u; \tau) \equiv -\frac{1}{2\pi i} \int_0^u dv \log \theta(v; \tau) \]

in terms of the \( \theta \)-function \( \theta(u; \tau) = \theta_1(u; \tau)/i\eta(\tau): \)

\[ \theta(u; \tau) \equiv e^{-\pi i u} q^{\frac{1}{12}} \prod_{k=0}^{\infty} (1-xq^k)(1-x^{-1}q^{k+1}), \quad x \equiv e^{2\pi i u}, q \equiv e^{2\pi i \tau} \]
In 4d, symmetries of $\mathcal{L}$ can be **anomalous**. The perturbative anomaly coefficients are:

\begin{align*}
A^{abc} &= \sum_\rho \rho^a \rho^b \rho^c \\
&\propto \text{Tr}(\mathcal{R},\mathcal{R}_F)(T^a\{T^bT^c\}) \\
A^a &= \sum_\rho \rho^a \\
&\propto \text{Tr}(\mathcal{R},\mathcal{R}_F)(T^a)
\end{align*}

We must impose the anomaly-free condition:

\begin{align*}
A^{abc} = A^a = 0, \quad A^{ab\gamma} = A^{a\beta\gamma} = 0
\end{align*}

On the other-hand, we still have non-vanishing ’t Hooft anomalies for $G_F$:

\begin{align*}
A^{\alpha\beta\gamma}, A^\alpha \neq 0
\end{align*}
't Hooft anomalies and modular transformations

Classically, we have the symmetries under the “elliptic” transformations:

\[ u \sim u + 1 \sim u + \tau, \quad \nu \sim \nu + 1 \sim \nu + \tau \]

and the full modular group \( SL(2, \mathbb{Z}) \) acting on \( T^2 \):

\[ S : \quad u_a \rightarrow \frac{u_a}{\tau}, \quad \tau \rightarrow -\frac{1}{\tau}, \quad T : \quad u_a \rightarrow u_a, \quad \tau \rightarrow \tau + 1 \]

The anomaly-free condition ensures that \( \mathcal{W} \) is fully \( G \)-invariant.

The other (non-gauged) symmetries can be violated. For instance:

\[ S' : \quad \mathcal{W}\left(\frac{u}{\tau}, -\frac{1}{\tau}\right) = \frac{1}{\tau} \mathcal{W}(u, \tau) + \frac{1}{6\tau^2} A^{abc} u_a u_b u_c + \frac{1}{4\tau} A^a u_a \]
Flux operators and the 4d Bethe equations

One can similarly compute $\Omega(u; \tau)$. As in 3d, we may define:

$$\Pi_a(u, \nu; \tau), \quad \Pi_\alpha(u, \nu; \tau), \quad \mathcal{H}(u, \nu; \tau)$$

The Bethe equations are $\Pi_a = 1$ as before. Explicitly:

$$\prod_{(\rho, \omega) \in (\mathcal{R}, \mathcal{R}_F)} \theta(\rho(u) + \omega(\nu); \tau)^{-\rho^a} = 1, \quad \forall a,$$

and excluding any solution left invariant by (part of) the Weyl group.

For an anomaly-free theory, the LHS is modular and elliptic in all parameters, so the equations are well-defined.
Fibering operators

The 4d $A$-model is defined on $T^2 \cong S^1_{\beta_1} \times S^1_{\beta_2}$ with $\text{Im}(\tau) = \frac{\beta_2}{\beta_1}$.

In 2d, we have an $U(1)^2_{KK}$ symmetry, with mass parameters:

$$m_{KK_1} = \frac{\tau}{\beta_2}, \quad m_{KK_2} = \frac{1}{\beta_2}$$

The corresponding fibering operators are:

$$\mathcal{F}_1(u, \nu; \tau) = \exp\left(2\pi i \frac{\partial W}{\partial \tau}\right)$$

and

$$\mathcal{F}_2(u, \nu; \tau) = \exp\left(2\pi i \left(W - u_a \frac{\partial W}{\partial u_a} - \nu_\alpha \frac{\partial W}{\partial \nu_\alpha} - \tau \frac{\partial W}{\partial \tau}\right)\right)$$
Fibering operators

The apparent difference between the $\mathcal{F}_1$ and $\mathcal{F}_2$ is due to an implicit choice of modular frame. One can show that they are related by an $S$ transformation:

$$
\mathcal{F}_2 \left( \frac{u}{\tau}; -\frac{1}{\tau} \right) = e^{-\frac{\pi i}{3\tau^2} A^{abc} u_a u_b u_c} \mathcal{F}_1 (u; \tau),
$$

The insertion of $\mathcal{F}_1^{p_1} \mathcal{F}_2^{p_2}$ ($p_1, p_2 \in \mathbb{Z}$) in the $A$-model is equivalent to considering the theory on:

$$
T^2 \rightarrow \mathcal{M}_{g,p} \times S^1 \rightarrow \Sigma_g, \quad p = \gcd(p_1, p_2)
$$

This is perfectly consistent with the known classification of $\mathcal{N} = 1$ supersymmetric backgrounds. [Dumitrescu, Festuccia, Seiberg, 2012]
Fibering operators

Thus we can just choose \((p_1, p_2) = (p, 0)\) and insert \(\mathcal{F}_1^p\).

More explicitly, the fibering operator \(\mathcal{F}_1\) is given by:

\[
\mathcal{F}_1(u, \nu; \tau) = \prod_{(\rho, \omega) \in (\mathcal{R}, \mathcal{R}_F)} \Gamma_0(\rho(u) + \omega(\nu); \tau)
\]

in terms of a “reduced” elliptic \(\Gamma\)-function:

\[
\Gamma_0(u; \tau) = \Gamma_e(qx; q, q) = \prod_{k=0}^{\infty} \left( \frac{1 - x^{-1} q^{k+1}}{1 - x q^{k+1}} \right)^{k+1}
\]
The $\mathcal{M}_{g,p}$ supersymmetric index

We can directly compute the $\mathcal{M}_{g,p} \times S^1$ partition function:

$$Z_{\mathcal{M}_{g,p} \times S^1}(\nu; \tau) = \sum_{\hat{u} \in S_{BE}} \mathcal{F}_1(\hat{u}, \nu; \tau)^p \mathcal{H}(\hat{u}, \nu; \tau)^{g-1} \Pi_\alpha(\hat{u}, \nu; \tau)^{n_\alpha}$$

This computes explicitly the $\mathcal{M}_{g,p}$ index:

$$Z_{\mathcal{M}_{g,p} \times S^1} = I_{\mathcal{M}_{g,p}} = \text{Tr}_{\mathcal{M}_{g,p}} \left[ (-1)^F q^{2J_3+R} y_\alpha^Q Q_\alpha \right]$$

In particular (after analytic continuation in the $R$-charges), this gives a new evaluation formula for the $\mathcal{N} = 1$ superconformal index in the “round” limit $q = p = q$. 
Modular properties

The $\mathit{SL}(2, \mathbb{Z})$ generators:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{T} = S^3TS = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

act on the $A$-model operators as:

$$S[\Phi_1] = e^{\frac{\pi i}{3\tau}} A^{abc}u_a u_b u_c \, \Phi_2^{-1}, \quad \tilde{T}[\Phi_1] = \Phi_1 \Phi_2,$$

$$S[\Phi_2] = e^{-\frac{\pi i}{3\tau^2}} A^{abc}u_a u_b u_c \, \Phi_1, \quad \tilde{T}[\Phi_2] = \Phi_2,$$

$$S[\Pi_a] = e^{\frac{\pi i}{2}} A^a e^{\frac{\pi i}{\tau}} A^{abc}u_b u_c \, \Pi_a, \quad \tilde{T}[\Pi_a] = e^{-\frac{\pi i}{6}} A^a \, \Pi_a,$$

$$S[\mathcal{H}] = e^{\frac{\pi i}{2}} A^R e^{\frac{\pi i}{\tau}} A^{Rbc}u_b u_c \, \mathcal{H}, \quad \tilde{T}[\mathcal{H}] = e^{-\frac{\pi i}{6}} A^R \, \mathcal{H}$$

It all follows from the properties of $\mathcal{W}$ and $\Omega$. 
For $p \neq 0$, the $\mathcal{M}_{g,p} \times S^1$ background breaks $SL(2, \mathbb{Z})$ explicitly.

Some hitherto mysterious modular action on the $S^3$ index (e.g. in [Spiridonov, Vartanov, 2012]) are simply explained.

For $p = 0$, the $\Sigma_g \times T^2$ partition function transforms simply:

$$S[Z_{\Sigma_g \times T^2}] = e^{\frac{\pi i}{2} \left(n_{\alpha} A^\alpha + (g-1) A^R\right)} e^{\frac{\pi i}{\tau} \left(n_{\alpha} A^{\alpha \beta \gamma} + (g-1) A^{R \beta \gamma}\right) \nu_\beta \nu_\gamma} Z_{\Sigma_g \times T^2}$$

$$\tilde{T}[Z_{\Sigma_g \times T^2}] = e^{-\frac{\pi i \tau}{6} \left(n_{\alpha} A^\alpha + (g-1) A^R\right)} Z_{\Sigma_g \times T^2}$$

Note that it transforms as an $\mathcal{N} = (0, 2)$ elliptic genus. Indeed, there is (formally) a 2d $\mathcal{N} = (0, 2)$ theory on $T^2$ obtained by dimensional reduction on $\Sigma_g$. 

Modular properties
Some consistency checks

This sum-over-Bethe-vacua formula reproduces (and generalizes) a number of previous results obtained by different methods. In particular:

• The limit $\beta \to 0$ on the $S^1$ factor is governed by the trace anomalies—“Cardy-like formula.” [di Pietro, Komargodski, 2014]

• The $\beta \to \infty$ limit is governed by a so-called “supersymmetric Casimir energy.” [Assel, Cassani, Martelli, 2014; Assel, Cassani, di Pietro, Komargodski, Lorenzen, Martelli, 2015; Bobev, Bullimore, Kim, 2015]

• We relate the $S^3 \times S^1$ partition function [Romelsberger, 2007; Assel, Cassani, Martelli, 2014] to the $S^2 \times T^2$ partition function [Benini, Zaffaroni, 2015; Honda, Yoshida, 2015]:

$$Z_{S^3 \times S^1} = \langle \mathcal{F}_1 \rangle_{S^2 \times T^2}$$
<table>
<thead>
<tr>
<th>Introduction</th>
<th>3d $\mathcal{N} = 2$ theories</th>
<th>4d $\mathcal{N} = 1$ theories</th>
<th>Applications</th>
<th>Outlook</th>
</tr>
</thead>
</table>

**Applications**
3d $\mathcal{N} = 2$ Wilson loop algebras

Supersymmetric Wilson loops are realized on the Coulomb branch $\mathcal{M}$ as:

$$W_{\mathcal{R}} = \text{Tr}_{\mathcal{R}} (x) = \sum_{\rho \in \mathcal{R}} x^\rho$$

for $\mathcal{R}$ a representation of $\mathbf{G}$. The Wilson loop algebra is of the form:

$$\mathcal{A} \cong R[x_a, x_a^{-1}]^{W_G} / I_{BE}, \quad R = \mathbb{Q}(y_{\alpha}, y_{\alpha}^{-1})$$

Example: $U(N)_k \mathcal{N} = 2$ CS theory:

$$\mathcal{A} \cong \mathbb{Z}[x_a, x_a^{-1}]^{S_N} / I, \quad I = \left((-x_a)^k\right)$$

This is the Verlinde algebra for pure $U(N)_{\hat{k}}$ CS at level $\hat{k} = k - \text{sign}(k)N$. 

3d $\mathcal{N} = 2$ Wilson loop algebras

Another very interesting example is $U(N)$ YM theory with $N_f$ fundamental and $N_f$ antifundamental chiral multiplets. The theory has $G_F = SU(N_f) \times SU(N_f) \times U(1)_A \times U(1)_T$. Consider the fugacities:

$$y_i, \quad \tilde{y}_i, \quad y_A, \quad z, \quad \tilde{z} \equiv z y_A^{-N_f}$$

such that $\prod_{i=1}^{N_f} y_i^{-1} = \prod_{i=1}^{N_f} \tilde{y}_i = y_A^{-N_f}$. The Bethe equations are:

$$P(x_a) = 0, \quad a = 1, \cdots, N_c, \quad x_a \neq x_b \text{ if } a \neq b$$

$$P(x) = \prod_{i=1}^{N_f} (x - y_i) - \tilde{z} \prod_{j=1}^{N_f} (x - \tilde{y}_j)$$
3d $\mathcal{N} = 2$ Wilson loop algebras

We can easily write down an explicit presentation of the algebra. Let us denote the Wilson loop $W_{\mathcal{R}}$ by the Young tableau of the $U(N_c)$ rep. $\mathcal{R}$.

For instance, consider the case of $U(3), N_f = 5$:

$$
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array} &= R_3 - R_2 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array} + R_1 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array}, \\
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\filldraw (1,0) rectangle (2,1);
\end{tikzpicture}
\end{array} &= R_4 - R_2 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array} + R_1 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array}, \\
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\filldraw (1,0) rectangle (2,1);
\filldraw (2,0) rectangle (3,1);
\end{tikzpicture}
\end{array} &= R_5 - R_2 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array} + R_1 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array}, \\
R_n &\equiv \frac{1}{1 - \tilde{z}} (s_n(y) - \tilde{z}s_n(\tilde{y})).
\end{align*}
$$

This theory has a $U(2), N_f = 5$ Aharony dual. [Aharony, 1997] The Bethe equations also encode the duality relations for Wilson loops:

$$
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array}^D &= R_1 - \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array}, \\
\begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\filldraw (1,0) rectangle (2,1);
\end{tikzpicture}
\end{array}^D &= R_2 - R_1 \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
\filldraw (0,0) rectangle (1,1);
\end{tikzpicture}
\end{array},
\end{align*}
$$

In the limit $R_n \to 0$, one recovers the Verlinde algebra for $U(3)_{k=2}$ and level-rank duality.
Witten index

By taking the chemical potentials $\nu_\alpha$ generic enough, we have a discrete number of Bethe vacua—the 2d theory is fully massive. (We only consider theories where $\nu$ can be taken “generic enough.”)

The quantity:

$$ |S_{BE}| \in \mathbb{N} $$

is the simplest $A$-model observable. It is simply the Witten index:

$$ Z_{Td} = |S_{BE}|, \quad d = 3 \text{ or } 4 $$

This is a regulated Witten index in the presence of generic masses for the matter fields. In 3d, it is known to be invariant as we change the mass parameters. [Intriligator, Seiberg, 2013] Similar in 4d.
The Witten index of SQCD

Consider $\mathcal{N} = 1$ SQCD: $SU(N_c)$ with $N_f$ flavors.

The Bethe equations are:

$$e^{2\pi i \lambda} \prod_{i=1}^{N_f} \frac{\theta(-v_a + \tilde{\nu}_i)}{\theta(v_a + \nu_i)} = 1, \quad a = 1, \ldots, N_c,$$

$$\sum_{a=1}^{N_c} v_a = \mu_B$$

One can compute:

$$Z_{T^4} = |S_{BE}| = \binom{N_f - 2}{N_c - 1}$$

This result is nicely consistent with Seiberg duality.
Infrared dualities of supersymmetric QFTs

The $A$-model is itself a TFT. In particular, it is RG invariant. This leads to new tests of infrared dualities. The full $A$-models $A$ and $A^D$ of infrared-dual theories $T$ and $T^D$ should match.

I.e. there exists an isomorphism:

$$D : A \rightarrow A^D$$

In particular, on Bethe vacua:

$$D : S_{BE} \rightarrow S_{BE} : \hat{u} \mapsto \hat{u}^D$$

Two operators $\mathcal{O} \in A$ and $\mathcal{O}^D \in A^D$ are dual if and only if:

$$\mathcal{O}(\hat{u}) = \mathcal{O}_D(\hat{u}^D)$$

The existence of $D$ implies that $Z_{\mathcal{M}_{g,p}}$ or $Z_{\mathcal{M}_{g,p} \times S^1}$ (etc.) match.
4d $\mathcal{N} = 1$ Seiberg duality

**Seiberg duality:** $SU(N_c)$ with $N_f$ flavors $Q_i, \tilde{Q}_j$ dual to $SU(N_f - N_c)$ with $N_f$ flavors $q^i, \tilde{q}^j$, $N_f^2$ singlets $M_{ij}$ and $W = Mq\tilde{q}$. [Seiberg, 1994]

Bethe equations:

$$\Pi_0(v_a, \lambda) = 1, \quad a = 1, \ldots, N_c, \quad \sum_{a=1}^{N_c} v_a = \mu_B$$

$$\Pi_0(v, \lambda) \equiv e^{2\pi i \lambda} \prod_{i=1}^{N_f} \frac{\theta(-v + \tilde{v}_i)}{\theta(v + \nu_i)}.$$

Let $\tilde{v}_k$ denote the $N_f$ solutions to $\Pi_0(v, \lambda) = 1$ at arbitrary $\lambda$.

Bethe vacuum:

$$\{\hat{v}_a, \lambda_0\} \mid \{\hat{v}_a\}_{a=1}^{N_c} \equiv A \subset \{\tilde{v}_k\}_{k=1}^{N_f}, \quad \lambda = \lambda_0 \text{ such that } \sum_{a} \hat{v}_a = \mu_B$$
4d $\mathcal{N} = 1$ Seiberg duality

Duality map:

\[
\mathcal{D} : \{ \hat{v}_a, \lambda_0 \} \mapsto \{ \hat{v}_a^D, \lambda_0^D \} , \quad \{ \hat{v}_a^D \}_{a=1}^{N_f-N_c} = A_c \subset \{ \tilde{v}_k \}_{k=1}^{N_f}, \quad \lambda_0^D = -\lambda_0
\]

Equality of fibering operators of dual theories is equivalent to the identity:

\[
\prod_{k=1}^{N_f} \prod_{i=1}^{N_f} \Gamma_0(\tilde{v}_k + \nu_i) \Gamma_0(-\tilde{v}_k + \tilde{v}_i) = \prod_{i,j=1}^{N_f} \Gamma_0(\nu_i + \tilde{v}_j)
\]

We don’t have a proof, but we can check it numerically. On the other hand, some simple manipulations on $\theta$-functions implies that $\mathcal{H}(\hat{u}) = \mathcal{H}^D(\hat{u}^D)$.

That then implies the equality of all $\mathcal{M}_{g,p}$ indices:

\[
Z_{\mathcal{M}_{g,p} \times S^1}[\mathcal{T}] = Z_{\mathcal{M}_{g,p} \times S^1}[\mathcal{T}^D]
\]
Summary and outlook

We described simple TFT for (abelianized) 2d gauge fields—the $A\text{-}models$—that compute supersymmetric partition functions—and, more generally, expectation values of codimension-2 operators—in 3d and 4d gauge theories with 4 supercharges.

What’s next?

- Describe any allowed (Seifert) $\mathcal{M}_3$ in this language.
- Extend to “non-Lagrangian” theories.
- Describe the algebra of half-BPS surface operators in 4d $\mathcal{N} = 1$ gauge theories. [in progress] (Mathematical interpretation?)
- Describe and interpret the $\Omega$-deformation at genus $g = 0$. “Quantization” of twisted chiral ring? Connect to 3d holomorphic blocks [Beem, Dimofte, Pasquetti, 2012]
- 3d/3d correspondence: M5-branes on $\mathcal{M}_{g,p} \times \mathcal{M}_3^{\text{TFT}}$?