Supersymmetric indices, partition functions and the A-twist

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1701.03171 and 1707.05774 with H. Kim and B. Willett

Disclaimers

This talk will be about QFTs with four Poincaré supercharges.

The relevant supersymmetry algebras in dimensions $d = 2, 3$ and 4 are called:

$$2d\ \mathcal{N} = (2,2) \quad \xleftarrow{S^1} \quad 3d\ \mathcal{N} = 2 \quad \xleftarrow{S^1} \quad 4d\ \mathcal{N} = 1$$

We restrict ourselves to theories with a $U(1)_R$ symmetry:

$$[R, \mathcal{Q}] = -\mathcal{Q}\ , \qquad [R, \bar{\mathcal{Q}}] = \bar{\mathcal{Q}}\ ,$$

Including but not limited to: SCFTs.

The twisted chiral ring

Consider the 2d case:

$$2d\ \mathcal{N}=(2,2)\ :\qquad \{\mathcal{Q}_-,\bar{\mathcal{Q}}_-\}=P_z\ ,\quad \{\mathcal{Q}_+,\bar{\mathcal{Q}}_+\}=-P_{\bar z}\ ,$$
$$\{\mathcal{Q}_-,\bar{\mathcal{Q}}_+\}=iZ\ ,\quad \{\mathcal{Q}_+,\bar{\mathcal{Q}}_-\}=i\bar{Z}\ ,$$

with Z a complex central charge that commutes with R.

$\mathcal{N}=(2,2)$ theories contain interesting subsectors of protected local operators.

We are interested in the twisted chiral ring:

$$\omega\ :\quad [\mathcal{Q}_-,\omega]=0\ ,\quad [\bar{\mathcal{Q}}_+,\omega]\qquad (\text{ mod } \mathcal{Q} \text{ or } \bar{\mathcal{Q}}\text{-exact })$$

It is conveniently singled out by the topological A-twist.

[Witten, 1988]

Twisted chiral ring for $d=3$ and 4

In dimension $d>2$, the twisted-chiral condition breaks $so(d)$ covariance down to $so(2)$.

Twisted chiral operators are extended operators of codimension 2:

$d=3$: **half-BPS line operators** in 3d $\mathcal{N}=2$ theories, $\mathscr{L}$.
(For instance, supersymmetric Wilson loop operators.)

$d=4$: **half-BPS surface operators** in 4d $\mathcal{N}=1$ theories, $\mathcal{S}$.

Fusion algebras

Parallel twisted chiral operators have non-singular OPE.

They satisfy a fusion algebra. We must have:

$$\mathscr{L}_i \cdot \mathscr{L}_j = \mathcal{N}_{ij}{}^{k}\, \mathscr{L}_k$$

for half-BPS line operators $\mathscr{L}$ in 3d, and similarly for $\mathcal{S}$ in 4d.

These algebras have been discussed *e.g.* by:

[Kapustin, Willett, 2013; Cecotti, Gaiotto, Vafa, 2013]

A-models

Our setup will be:

- 3d $\mathcal{N}=2$ theory on $\mathbb{R}^2 \times S^1$
- 4d $\mathcal{N}=1$ theory on $\mathbb{R}^2 \times T^2$

with line operators on S^1, or surface operators on T^2.

$\mathbb{R}^2$ can be compactified to Σ_g with the A-twist. We define the "A-model" of the 3d or 4d theory as the **2d TFT on** Σ_g obtained by going to the cohomology of Q_- and $\tilde{Q}_+$.

A-model observables:

$$\langle \mathscr{L}_i \mathscr{L}_j \cdots \rangle_{\Sigma_g \times S^1} \;, \qquad\qquad \langle \mathcal{S}_i \mathcal{S}_j \cdots \rangle_{\Sigma_g \times T^2}$$

They capture the quantum ring structure constants ${\mathcal{N}_{ij}}^k$.

Another disclaimer

This talk will be exclusively concerned with ultraviolet (UV)-complete gauge theories with a UV Lagrangian: [1]

$$\langle \mathcal{O} \rangle = \int [D\mathcal{V}\, D\Phi]\, e^{-\int d^3x \sqrt{g}\, \mathcal{L}(\mathcal{V},\Phi)}\, \mathcal{O}(\mathcal{V},\Phi)$$

For instance:

- 3d $\mathcal{N}=2$ SQED
- 3d $\mathcal{N}=2^*$ quivers
- 4d $\mathcal{N}=1$ $SU(N_c)$ SQCD
- $\dots$

[1] My apologies.

Supersymmetric indices and partition functions

In the last 10 years, there has been a lot of interest in computing supersymmetric partition functions and indices:

$$Z_{\mathcal{M}_3}[\mathcal{T}_{3d\,\mathcal{N}=2}]\ , \qquad\qquad Z_{\mathcal{M}_3\times S^1}[\mathcal{T}_{4d\,\mathcal{N}=1}] = I_{\mathcal{M}_3}$$

This can be done using supersymmetric localization in many examples. [Pestun, 2007; Kapustin, Willett, Yaakov, 2010; Jafferis, 2010; Hama, Hosomichi, Lee, 2010; Benini, Eager, Hori, Tachikawa, 2013; Hori, Kim, Yi, 2014; Assel, Cassani, Martelli, 2014; ...]

For $\mathcal{M}_3 \cong \Sigma_g \times S^1$, it is a A-model observable:

$$Z_{\Sigma_g\times S^1} = \langle 1 \rangle_{\Sigma_g\times S^1}$$

Computed in [Nekrasov, Shatashvili, 2014; Benini, Zaffaroni, 2014, 2015; CC, Kim, 2015]

Can we understand more general partition functions as A-model observables?

Supersymmetric indices and partition functions

One of the best-known example is the three-sphere index:
[Romelsberger, 2005]

$$I_{S^3} = \mathrm{Tr}_{S^3}\left[(-1)^F \mathbf{p}^{J_3+J_3'+\frac{1}{2}R}\mathbf{q}^{J_3-J_3'+\frac{1}{2}R}\prod_F y_F^{Q_F}\right]$$

In 3d, this reduce to the ("squashed") S_b^3 partition function.

In this talk: $\mathbf{p}=\mathbf{q}\equiv q\ , \qquad \leftrightarrow \qquad b=1$

The index can be computed in terms of an elliptic hypergeometric integral:

$$I_{S^3} = q^{E_{S^3}}\,\frac{(q;q)_\infty^{2\mathrm{rk}(\mathbf{G})}}{|W_{\mathbf{G}}|}\oint\prod_{a=1}^{\mathrm{rk}(\mathbf{G})}\frac{dx_a}{2\pi i x_a}\,\frac{\prod_{\rho,\omega}\Gamma_0(x^\rho y^\omega q^{r_\omega-1};q)}{\prod_\alpha\Gamma_0(x^\alpha q^{-1};q)}\ ,$$

[Romelsberger, 2005; Dolan, Osborn, 2008]

Supersymmetric indices and partition functions

Localization computations can be very subtle.

In this talk, I'll explain a different method to compute:

$$Z_{\mathcal{M}_3}\ , \qquad\qquad Z_{\mathcal{M}_3\times S^1}\ ,$$

for 3d and 4d gauge theories, for a relatively simple family of $\mathcal{M}_3$ backgrounds allowed by supersymmetry.

This will teach us some interesting lessons about these objects, and will allow us to compute a few new interesting observables in theories with 4 supercharges.

Previous works: [Ohta, Yoshida, 2012; Nishioka, Yaakov, 2014]

Outline

3d $\mathcal{N} = 2$ theories

Outline

3d $\mathcal{N}=2$ theories

4d $\mathcal{N}=1$ theories

Outline

3d $\mathcal{N}=2$ theories

4d $\mathcal{N}=1$ theories

Applications

3d $\mathcal{N} = 2$ theories

3d $\mathcal{N}=2$ gauge theory

Consider 3d $\mathcal{N}=2$ supersymmetric
Yang-Mills-Chern-Simons-matter theories, with:

- Vector multiplet $\mathcal{V}$ for a gauge group $\mathbf{G}$, with $\mathrm{Lie}(\mathbf{G})=\mathfrak{g}$.
- Chiral multiplets Φ_i in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.
- R-symmetry-preserving superpotential $W(\Phi)$.
- A choice of CS interactions for $\mathbf{G}\times\mathbf{G}_F$:

$$S_{\rm CS}=\frac{k}{4\pi}\int_{\mathcal{M}_3} d^3x\sqrt{g}(i\epsilon^{\mu\nu\rho}\left(a_\mu\partial_\nu a_\rho-\frac{2i}{3}a_\mu a_\nu a_\rho\right)-2\sigma D+2i\tilde{\lambda}\lambda)$$

We have the CS level $k\in\mathbb{Z}$, and $\mathcal{M}_3$ must be a spin manifold.

Circle compactification

Consider the theory on $\mathbb{R}^2 \times S^1$, with S^1 a circle of radius β. Using the Kaluza-Klein (KK) expansion:

$$\phi = \sum_{n\in\mathbb{Z}} \phi_n(z,\bar{z})e^{in\psi} \ ,$$

we can consider the 3d theory as a 2d theory with an infinite number of fields, in 2d $\mathcal{N}=(2,2)$ supermultiplets.

In particular, we have a 2d vector multiplet that includes a complex scalar:

$$u = i\beta\sigma - a^{(0)} \ , \qquad a^{(0)} = \frac{1}{2\pi}\int_{S^1} a_\mu dx^\mu$$

with σ the real scalar in $\mathcal{V}$ in 3d. We take u dimensionless.

Circle compactification

Consider giving an expectation values to the scalar u in the 2d vector multiplet. This corresponds to the classical Coulomb branch of the 3d theory: [Aganagic, Hori, Karch, Tong, 2001]

$$u = \mathrm{diag}(u_a) \ , \qquad a = 1, \cdots, \mathrm{rk}(\mathbf{G})$$

Due to large gauge transformations along S^1, we have:

$$u_a \sim u_a + 1 \ , \qquad \mathfrak{M} \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathbf{G})}/W_{\mathbf{G}}$$

We will also use the variables:

$$x_a \equiv e^{2\pi i u_a}$$

Circle compactification

At a generic point on $\mathfrak{M}$, the 3d gauge group is Higgsed to:

$$\mathbf{G} \to \mathbf{H} \cong \prod_{a=1}^{\mathrm{rk}(\mathbf{G})} U(1)_a$$

We can also think in terms of *diagonalization* of the 2d vector multiplet. In the path integral language, we should still sum over topological sectors, using a functional Weyl integral formula. [Blau, Thompson, 1992, 1993].

We integrate out all massive fields and write down an effective field theory for the low-energy modes u_a and its superpartners, in twisted chiral multiplets U_a. That is our "A-model."

Mass parameters: background $\mathbf{G}_F$ vector multiplets

We are considering the theory in the presence of arbitrary supersymmetry-preserving background fields for the flavor symmetry $\mathbf{G}_F$ with maximal torus:

$$\mathbf{H}_F = \prod_\alpha U(1)_\alpha \subset \mathbf{G}_F$$

We have the flavor parameters:

$$\nu_\alpha = i\beta m_\alpha^F - a_\alpha^{(0)F} \;, \qquad\qquad y_\alpha \equiv e^{2\pi i \nu_\alpha}$$

One may call ν and y the "chemical potentials" and "fugacities", respectively.

A-model Lagrangian

Consider the A-twisted theory. This is equivalent to "curved-space supersymmetry" [Festuccia, Seiberg, 2012] on $\Sigma_g \times S^1$.

Up to Q-exact terms, the Lagrangian of the effective field theory on $\mathfrak{M}$ reads:

$$\begin{aligned} S_{\mathrm{TQFT}} &= \int_{\Sigma_g} \left(-i f_a \frac{\partial \mathcal{W}(u,\nu)}{\partial u_a} + \tilde{\Lambda}^a \Lambda^b \frac{\partial^2 \mathcal{W}(u,\nu)}{\partial u_a \partial u_b} \right) \\ &\quad + \frac{i}{2} \int_{\Sigma_g} d^2x \sqrt{g}\, \Omega(u,\nu) R \end{aligned}$$

with f_a the abelian field strength of a^a_μ and R the Ricci scalar.

[Witten, 1993; Nekrasov, Shatashvili, 2014]

Effective twisted superpotential and effective dilaton

The A-model is fully determined by the two holomorphic potentials:

$$\mathcal{W}(u,\nu)\ , \qquad \Omega(u,\nu)$$

The **effective twisted superpotential** takes the schematic form:

$$\mathcal{W} = \frac{k}{2}u(u+1) + \frac{k^F}{2}\nu(\nu+1) + \frac{k_g}{24} + \frac{1}{(2\pi i)^2}\sum_{(\rho,\omega)\in(\mathfrak{R},\mathfrak{R}_F)} \mathrm{Li}_2(x^\rho y^\omega)$$

The classical contribution is from CS terms, including background CS terms.

The dilog is a one-loop correction from integrating out the matter fields. This result is for the so-called $U(1)_{-\frac{1}{2}}$ quantization of a 3d Dirac fermion, which preserves gauge invariance but breaks parity. [CC, Kim, Willett, 2017]

Effective twisted superpotential and effective dilaton

Similarly, the **effective dilaton** takes the form:

$$\Omega = k^{aR} u_a + k^{\alpha R} \nu_\alpha + \frac{1}{2} k^{RR} - \frac{1}{2\pi i} \sum_{(\rho,\omega) \in (\mathfrak{R}, \mathfrak{R}_F)} (r_\omega - 1) \log(1 - x^\rho y^\omega) - \frac{1}{2\pi i} \sum_{\alpha \in \mathfrak{g}} \log(1 - x^\alpha)$$

The CS contributions are supersymmetric CS terms involving the $U(1)_R$ background gauge field [CC, Dumitrescu, Festuccia, Komargodski, Seiberg, 2012].

Note the contribution from the W-bosons.

Effective twisted superpotential and effective dilaton

The twisted superpotential is only defined modulo the ambiguity:

$$\mathcal{W} \sim \mathcal{W} + n^a u_a + n^\alpha \nu_\alpha + n^0 \ , \qquad n^a, n^\alpha, n^0 \in \mathbb{Z} \ ,$$

due to the sum over topological sectors. Similarly, we have:

$$\Omega \sim \Omega + n \ , \qquad n \in \mathbb{Z}$$

This corresponds to branch cut ambiguities in the variables u, ν. One the other hand, well-defined A-model operators will be holomorphic in u, ν.

The Bethe equations

Let us define the so-called Bethe equations:

$$\Pi_a(u,\nu) \equiv \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial u_a}\right) = 1 \ ,$$

[Nekrasov, Shatashvili, 2009]

The vacua of the A-model are two-dimensional vacua, the Bethe vacua:

$$\boxed{\mathcal{S}_{BE} = \left\{ \hat{u}_a \,\middle|\, \Pi_a(\hat{u},\nu) = 1 \ , \ \forall a \ , \quad w \cdot \hat{u} \neq \hat{u}, \ \forall w \in W_G \right\} / W_{\mathbf{G}}}$$

Importantly, we must exclude would-be “non-abelian vacua”—solutions of $\Pi_a = 1$ not acted on freely by the Weyl group—by hand.

A-model defect operators

Given $\mathcal{W}$, Ω, we can define a number of "canonical" defect operators, which are local on Σ_g.

These operators probably have an explicit construction in the three-dimensional UV theory, but we will only focus on their A-model "low energy" description.

The flavor flux operators

In the presence of flavor symmetries $U(1)_\alpha \subset \mathbf{G}_F$, we can turn on generic background vector multiplets $\mathcal{V}^F_\alpha$, as long as we preserve the A-twist supercharges.

In particular, we have chemical potentials and background fluxes

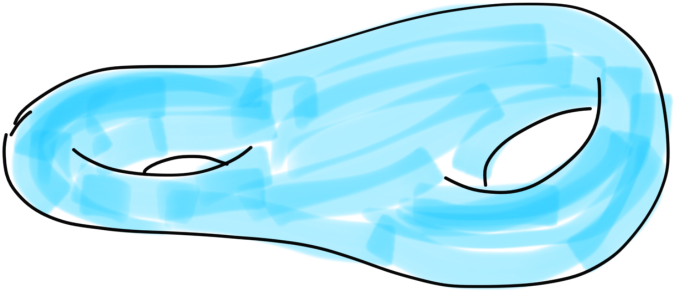
$$\nu_\alpha \;, \qquad\qquad \boxed{\frac{1}{2\pi}\int_{\Sigma_g} f_\alpha = \mathfrak{n}_\alpha \;\in \mathbb{Z}}$$

This adds a piece to the A-model action:

$$S_{\text{flux}} = \int_{\Sigma_g}\left(-i f_\alpha \frac{\partial \mathcal{W}(u,\nu)}{\partial \nu_\alpha}\right)$$

The flavor flux operators

The background field f_α can have arbitrary profile over Σ_g:



If we concentrate its flux near a point, we obtain a singularity, describable by a local operator.

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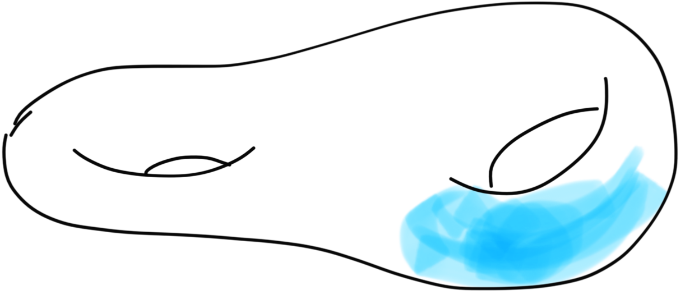

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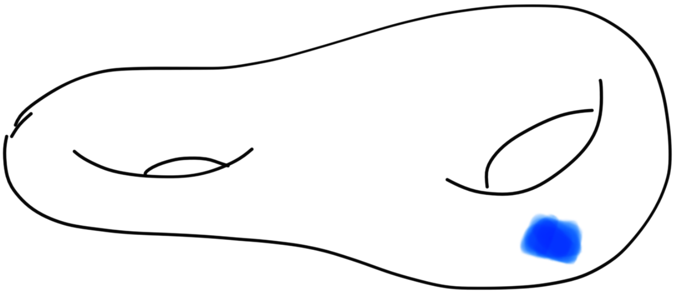

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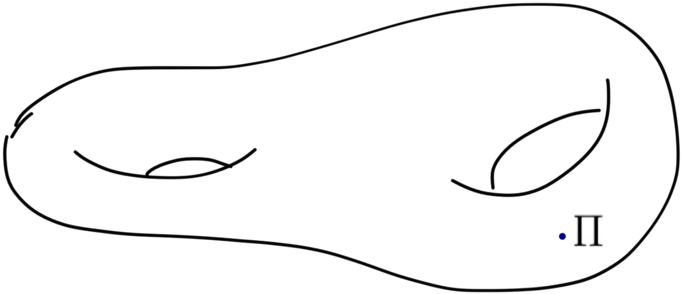

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The flavor flux operators

In particular, if we take

$$f_\alpha = 2\pi\, \mathfrak{n}_\alpha\, \delta^2(x - x_0) \ ,$$

turning on background flux is equivalent to the insertion of a local operator:

$$\Pi_\alpha(u,\nu)^{\mathfrak{n}_\alpha}$$

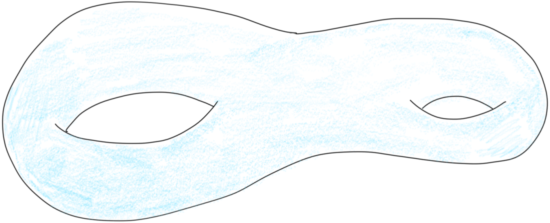
in the path integral, with

$$\boxed{\Pi_\alpha(u,\nu) = \exp\left(2\pi i \frac{\partial \mathcal{W}(u,\nu)}{\partial \nu_\alpha}\right)}$$

We call Π_α the flux operator for the flavor symmetry $U(1)_\alpha$.

The handle-gluing operator

Any 2d TQFT has a "handle-gluing operator" $\mathcal{H}$:



The explicit form of $\mathcal{H}$ for simple LG models is given by [Vafa, 1990]. The generalization to 2d *gauge* theories was investigated more recently. [Melnikov, Plesser, 2005; Nekrasov, Shatashvili, 2014]

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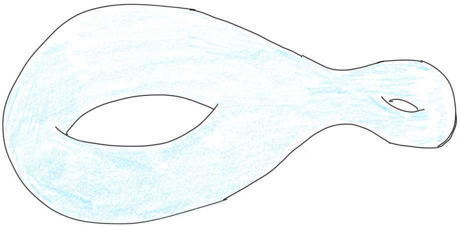

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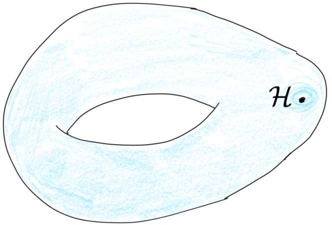

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The handle-gluing operator

In the A-twisted gauge theory, the handle-gluing operator can be seen as a "flux operator for $U(1)_R$".

It is given explicitly by: [Nekrasov, Shatashvili, 2014]

$$\boxed{\mathcal{H}(u,\nu) = e^{2\pi i \Omega(u,\nu)} \det_{ab}\left(\frac{\partial^2 \mathcal{W}(u,\nu)}{\partial u_a \partial u_b}\right)}$$

The appearance of the Hessian determinant of $\mathcal{W}$ is due to the presence of fermionic zero modes from the gauginos.

$\Sigma_g \times S^1$ correlators and the 3d twisted index

In any 2d TQFT, we have:

$$\langle \mathcal{O} \rangle_{\Sigma_g} = \langle \mathcal{O}\mathcal{H}^g \rangle_{\mathbb{C}P^1} = \mathrm{Tr}_V \left(\mathcal{H}^{g-1} \mathcal{O} \right)$$

where V is the TQFT Hilbert space. For us, $V \cong \mathcal{S}_{\mathrm{BE}}$.

For the 3d A-model, we then find: [Nekrasov, Shatashvili, 2014]

$$\boxed{\Big\langle \mathscr{L} \Big\rangle_{\Sigma_g \times S^1} = \sum_{\hat{x} \in \mathcal{S}_{\mathrm{BE}}} \mathscr{L}(\hat{x})\, \mathcal{H}(\hat{x}, y)^{g-1}\, \Pi_\alpha(\hat{x}, y)^{\mathfrak{n}_\alpha}}$$

for any line operator $\mathscr{L}$ on $\Sigma_g \times S^1$, in the presence of background fluxes $\mathfrak{n}_\alpha$ for the flavor symmetry. This can also be computed by supersymmetric localization in the UV [Benini, Zaffaroni, 2015, 2016; CC, Kim, 2016]

Note: The operators Π_α and $\mathcal{H}$ are *rational functions of* x *and* y.

The fibering operator

There exists another "canonical" A-model operator one can build from $\mathcal{W}$.

From the 2d point of view, the full flavor symmetry is:

$$\mathbf{G}_F^{\rm 2d} = \mathbf{G}_F \times U(1)_{KK}$$

We have distinguished symmetry $U(1)_{KK}$, whose conserved charge is the circle momentum. There exists a 2d background vector multiplet that couples to the KK momentum. In particular, we have the 2d twisted mass:

$$m_{KK} = \frac{1}{\beta}$$

The fibering operator

Definition: the fibering operator is the flux operator for $U(1)_{KK}$.

Reinstating dimensions, we find:

$$\mathcal{F}(u,\nu) \equiv \exp\left(2\pi i \frac{\partial}{\partial m_{KK}}\Big(m_{KK}\mathcal{W}(u,\nu)\Big)\right)$$

This leads to the explicit expression:

$$\boxed{\mathcal{F}(u,\nu) = \exp\Big(2\pi i(\mathcal{W} - u_a\partial_{u_a}\mathcal{W} - \nu_\alpha\partial_{\nu_\alpha}\mathcal{W})\Big)}$$

in terms of the twisted superpotential $\mathcal{W}(u,\nu)$.

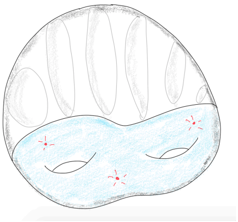
The fibering operators

Inserting $\mathcal{F}^p$, $p \in \mathbb{Z}$, in the A-model, we realize a principal circle bundle over Σ_g:

$$S^1 \longrightarrow \mathcal{M}_{g,p} \xrightarrow{\pi} \Sigma_g \ .$$

$$p = \frac{1}{2\pi} \int_{\Sigma_g} da_{KK}$$

This $\mathcal{M}_{g,p}$ is the simplest example of a Seifert manifold.



Supersymmetry is preserved by a pull-back of the A-twist on Σ_g.

The fibering operator

Importantly, the fibering operator is not fully gauge invariant under $\mathbf{G}\times\mathbf{G}_F$. Instead, we have the difference equations:

$$\mathcal{F}(u_a+1,\nu)=\mathcal{F}(u,\nu)\,\Pi_a(u,\nu)^{-1}$$

$$\mathcal{F}(u,\nu_\alpha+1)=\mathcal{F}(u,\nu)\,\Pi_\alpha(u,\nu)^{-1}$$

It is, however, gauge invariant (under $\mathbf{G}$) on the Bethe vacua, where $\Pi_a(\hat{u})=1$.

All observables are fully $\mathbf{G}\times\mathbf{G}_F$ invariant.

The fibering operator

Explicitly, in the YM-CS-matter theory:

$$\mathcal{F}(u,\nu) = e^{-\pi i k\, u^2 - \pi i k^F \nu^2 + \frac{\pi i}{12} k_g} \prod_{(\rho,\omega)\in(\mathfrak{R},\mathfrak{R}_F)} \mathcal{F}^{\Phi}(\rho(u)+\omega(\nu))$$

in terms of the simple function:

$$\mathcal{F}^{\Phi}(u) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2\left(e^{2\pi i u}\right) + u\log\left(1-e^{2\pi i u}\right)\right)$$

This is a meromorphic function of u with poles at $u=-1,-2,\cdots$ and zeros at $z=1,2,\cdots$. Note the identity:

$$\mathcal{F}_{\Phi}(u)\mathcal{F}_{\Phi}(-u) = e^{\pi i u^2 - \frac{\pi i}{6}}$$

The $\mathcal{M}_{g,p}$ partition function

We then directly find the $\mathcal{M}_{g,p}$ supersymmetric partition function:

$$Z_{\mathcal{M}_{g,p}}(\nu;\mathfrak{n}) = \sum_{\hat{u}\in\mathcal{S}_{\mathrm{BE}}} \mathcal{F}(\hat{u},\nu)^p\,\mathcal{H}(\hat{u},\nu)^{g-1}\,\Pi_\alpha(\hat{u},\nu)^{\mathfrak{n}_\alpha}$$

We can view this as an expectation value for a defect line operator $\mathcal{F}$ at a point on Σ_g:

$$\boxed{Z_{\mathcal{M}_{g,p}}(\nu) = \langle\mathcal{F}^p\rangle_{\Sigma_g\times S^1}}$$

In this sense, the $\mathcal{M}_{g,p}$ partition functions is just another A-model observable.

The S^3 partition function

Special case $g=0, p=1$: The S^3 partition function.

$$Z_{S^3}(\nu) = \sum_{\hat{u}\in\mathcal{S}_{\rm BE}} \mathcal{F}(\hat{u},\nu)\,\mathcal{H}(\hat{u},\nu)^{-1} = \langle\mathcal{F}\rangle_{S^2\times S^1}$$

Here, $\langle 1\rangle_{S^2\times S^1} = Z_{S^2\times S^1}$ is also known as "twisted 3d index".
[Benini, Zaffaroni, 2015]

So far, we chose all R-charges $r_i \in \mathbb{Z}$ for the chiral multiplets Φ_i.
This is necessary for the A-twist point of view.

On S^3, there is a canonical analytic continuation

$$Z_{S^3}(\nu) \to Z_{S^3}(\nu + (R-1))$$

to any $r_i \in \mathbb{R}$. [CC, Dumitrescu, Festuccia, Komargodski, 2014]

In particular, this gives a nice way to compute $F_{S^3} = -\log Z_{S^3}$ for a 3d $\mathcal{N}=2$ SCFT. [Jafferis, 2012]

4d $\mathcal{N}=1$ theories

4d $\mathcal{N} = 1$ gauge theories

Consider an $\mathcal{N} = 1$ gauge theory. For simplicity, we take $\mathbf{G}$ semi-simple and simply connected, and asymptotically-free theories.

We play the same game as before by compactifying on a T^2 with modular parameter τ. Now, the $\mathbf{G} \times \mathbf{G}_F$ parameters u, ν are themselves valued in a torus:

$$u \sim u + 1 \sim u + \tau, \qquad \nu \sim \nu + 1 \sim \nu + \tau$$

Let us introduce the convenient notation:

$$\mathbf{u_a} = (u_a, \nu_\alpha) \ , \qquad \mathbf{a} = (a, \alpha)$$

4d $\mathcal{N}=1$ gauge theories

The twisted superpotential is given by:

$$\mathcal{W}(\mathbf{u};\tau) = -\frac{\mathcal{A}^{\mathbf{abc}}\mathbf{u_a}\mathbf{u_b}\mathbf{u_c}}{6\tau} + \sum_{\rho\in(\mathfrak{R},\mathfrak{R}_F)} \boldsymbol{\psi}(\rho(\mathbf{u});\tau)$$

It is a purely quantum (one-loop) effect. We defined the "elliptic dilog":

$$\boldsymbol{\psi}(u;\tau) \equiv -\frac{1}{2\pi i}\int_0^u dv \log\theta(v;\tau)$$

in terms of the θ-function $\theta(u;\tau) = \theta_1(u;\tau)/i\eta(\tau)$:

$$\theta(u;\tau) \equiv e^{-\pi i u} q^{\frac{1}{12}} \prod_{k=0}^{\infty}(1-xq^k)(1-x^{-1}q^{k+1}) \,, \quad x \equiv e^{2\pi i u}\,, q \equiv e^{2\pi i\tau}$$

Anomalies

In 4d, symmetries of $\mathcal{L}$ can be anomalous. The perturbative anomaly coefficients are:

$$\mathcal{A}^{\mathbf{abc}} = \sum_\rho \rho^{\mathbf{a}}\rho^{\mathbf{b}}\rho^{\mathbf{c}} \qquad \propto \mathrm{Tr}_{(\mathfrak{R},\mathfrak{R}_F)}(T^{\mathbf{a}}\{T^{\mathbf{b}}T^{\mathbf{c}}\})$$

$$\mathcal{A}^{\mathbf{a}} = \sum_\rho \rho^{\mathbf{a}} \qquad \propto \mathrm{Tr}_{(\mathfrak{R},\mathfrak{R}_F)}(T^{\mathbf{a}})$$

We must impose the anomaly-free condition:

$$\mathcal{A}^{abc} = \mathcal{A}^{a} = 0 \;, \qquad \mathcal{A}^{ab\gamma} = \mathcal{A}^{a\beta\gamma} = 0$$

On the other-hand, we still have non-vanishing 't Hooft anomalies for $\mathbf{G}_F$:

$$\mathcal{A}^{\alpha\beta\gamma}\,, \mathcal{A}^{\alpha} \neq 0$$

't Hooft anomalies and modular transformations

Classically, we have the symmetries under the "elliptic" transformations:

$$u \sim u+1 \sim u+\tau, \qquad \nu \sim \nu+1 \sim \nu+\tau$$

and the full modular group $SL(2,\mathbb{Z})$ acting on T^2:

$$S \ : \ \ \mathbf{u_a} \to \frac{\mathbf{u_a}}{\tau}\ , \ \ \tau \to -\frac{1}{\tau}\ , \qquad T \ : \ \ \mathbf{u_a} \to \mathbf{u_a}\ , \ \ \tau \to \tau+1$$

The anomaly-free condition ensures that $\mathcal{W}$ is fully $\mathbf{G}$-invariant.

The other (non-gauged) symmetries can be violated. For instance:

$$S \ : \ \ \mathcal{W}\Big(\frac{\mathbf{u}}{\tau}, -\frac{1}{\tau}\Big) \ = \ \frac{1}{\tau}\mathcal{W}(\mathbf{u},\tau) + \frac{1}{6\tau^2}\mathcal{A}^{\mathbf{abc}}\mathbf{u_a}\mathbf{u_b}\mathbf{u_c} + \frac{1}{4\tau}\mathcal{A}^{\mathbf{a}}\mathbf{u_a}$$

Flux operators and the 4d Bethe equations

One can similarly compute $\Omega(\mathbf{u};\tau)$. As in 3d, we may define:

$$\Pi_a(u,\nu;\tau)\ , \qquad \Pi_\alpha(u,\nu;\tau)\ , \qquad \mathcal{H}(u,\nu;\tau)$$

The Bethe equations are $\Pi_a = 1$ as before. Explicitly:

$$\boxed{\prod_{(\rho,\omega)\in(\mathfrak{R},\mathfrak{R}_F)} \theta(\rho(u)+\omega(\nu);\tau)^{-\rho^a} = 1\ ,\ \ \forall a\ ,}$$

and excluding any solution left invariant by (part of) the Weyl group.

For an anomaly-free theory, the LHS is modular and elliptic in all parameters, so the equations are well-defined.

Fibering operators

The 4d A-model is defined on $T^2 \cong S^1_{\beta_1} \times S^1_{\beta_2}$ with $\mathrm{Im}(\tau) = \frac{\beta_2}{\beta_1}$.

In 2d, we have an $U(1)^2_{KK}$ symmetry, with mass parameters:

$$m_{\mathrm{KK}_1} = \frac{\tau}{\beta_2}\,, \qquad m_{\mathrm{KK}_2} = \frac{1}{\beta_2}$$

The corresponding fibering operators are:

$$\boxed{\mathcal{F}_1(u,\nu;\tau) = \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial \tau}\right)}$$

and

$$\boxed{\mathcal{F}_2(u,\nu;\tau) = \exp\left(2\pi i\left(\mathcal{W} - u_a \frac{\partial \mathcal{W}}{\partial u_a} - \nu_\alpha \frac{\partial \mathcal{W}}{\partial \nu_\alpha} - \tau \frac{\partial \mathcal{W}}{\partial \tau}\right)\right)}$$

Fibering operators

The apparent difference between the $\mathcal{F}_1$ and $\mathcal{F}_2$ is due to an implicit choice of modular frame. One can show that they are related by an S transformation:

$$\mathcal{F}_2\left(\frac{\mathbf{u}}{\tau};-\frac{1}{\tau}\right)=e^{-\frac{\pi i}{3\tau^2}\mathcal{A}^{\mathbf{abc}}\mathbf{u_a u_b u_c}}\,\mathcal{F}_1(\mathbf{u};\tau)\ ,$$

The insertion of $\mathcal{F}_1^{p_1}\mathcal{F}_2^{p_2}$ $(p_1,p_2\in\mathbb{Z})$ in the A-model is equivalent to considering the theory on:

$$T^2\longrightarrow\mathcal{M}_{g,p}\times S^1\longrightarrow\Sigma_g\ ,\qquad\qquad p=\gcd(p_1,p_2)$$

This is perfectly consistent with the known classification of $\mathcal{N}=1$ supersymmetric backgrounds. [Dumitrescu, Festuccia, Seiberg, 2012]

Fibering operators

Thus we can just choose $(p_1, p_2) = (p, 0)$ and insert $\mathcal{F}_1^p$.

More explicitly, the fibering operator $\mathcal{F}_1$ is given by:

$$\boxed{\mathcal{F}_1(u,\nu;\tau) = \prod_{(\rho,\omega)\in(\mathfrak{R},\mathfrak{R}_F)} \Gamma_0(\rho(u)+\omega(\nu);\tau)}$$

in terms of a "reduced" elliptic Γ-function:

$$\Gamma_0(u;\tau) = \Gamma_e(qx;q,q) = \prod_{k=0}^{\infty}\left(\frac{1-x^{-1}q^{k+1}}{1-xq^{k+1}}\right)^{k+1}$$

The $\mathcal{M}_{g,p}$ supersymmetric index

We can directly compute the $\mathcal{M}_{g,p} \times S^1$ partition function:

$$\boxed{Z_{\mathcal{M}_{g,p}\times S^1}(\nu;\tau) = \sum_{\hat{u}\in\mathcal{S}_{BE}} \mathcal{F}_1(\hat{u},\nu;\tau)^p\, \mathcal{H}(\hat{u},\nu;\tau)^{g-1}\, \Pi_\alpha(\hat{u},\nu;\tau)^{\mathfrak{n}_\alpha}}$$

This computes explicitly the $\mathcal{M}_{g,p}$ index:

$$Z_{\mathcal{M}_{g,p}\times S^1} = I_{\mathcal{M}_{g,p}} = \mathrm{Tr}_{\mathcal{M}_{g,p}}\left[(-1)^F q^{2J_3+R}\, y_\alpha^{Q_\alpha}\right]$$

In particular (after analytic continuation in the R-charges), this gives a new evaluation formula for the $\mathcal{N}=1$ superconformal index in the "round" limit $\mathbf{q}=\mathbf{p}=q$.

Modular properties

The $SL(2,\mathbb{Z})$ generators:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \tilde{T} = S^3 T S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

act on the A-model operators as:

$$\begin{aligned} S[\mathcal{F}_1] &= e^{\frac{\pi i}{3\tau}\mathcal{A}^{\mathbf{abc}}\mathbf{u_a u_b u_c}}\ \mathcal{F}_2{}^{-1} , \qquad & \tilde{T}[\mathcal{F}_1] &= \mathcal{F}_1 \mathcal{F}_2 , \\ S[\mathcal{F}_2] &= e^{-\frac{\pi i}{3\tau^2}\mathcal{A}^{\mathbf{abc}}\mathbf{u_a u_b u_c}}\ \mathcal{F}_1 , \qquad & \tilde{T}[\mathcal{F}_2] &= \mathcal{F}_2 , \\ S[\Pi_\mathbf{a}] &= e^{\frac{\pi i}{2}\mathcal{A}^{\mathbf{a}}} e^{\frac{\pi i}{\tau}\mathcal{A}^{\mathbf{abc}}\mathbf{u_b u_c}}\ \Pi_\mathbf{a} , \qquad & \tilde{T}[\Pi_\mathbf{a}] &= e^{-\frac{\pi i}{6}\mathcal{A}^{\mathbf{a}}}\ \Pi_\mathbf{a} , \\ S[\mathcal{H}] &= e^{\frac{\pi i}{2}\mathcal{A}^{R}} e^{\frac{\pi i}{\tau}\mathcal{A}^{R\mathbf{bc}}\mathbf{u_b u_c}}\ \mathcal{H} , \qquad & \tilde{T}[\mathcal{H}] &= e^{-\frac{\pi i}{6}\mathcal{A}^{R}}\ \mathcal{H} \end{aligned}$$

It all follows from the properties of $\mathcal{W}$ and Ω.

Modular properties

For $p \neq 0$, the $\mathcal{M}_{g,p} \times S^1$ background breaks $SL(2,\mathbb{Z})$ explicitly.

Some hitherto mysterious modular action on the S^3 index (*e.g.* in [Spiridonov, Vartanov, 2012]) are simply explained.

For $p = 0$, the $\Sigma_g \times T^2$ partition function transforms simply:

$$\begin{aligned} S[Z_{\Sigma_g \times T^2}] &= e^{\frac{\pi i}{2}\left(\mathfrak{n}_\alpha \mathcal{A}^\alpha + (g-1)\mathcal{A}^R\right)}\, e^{\frac{\pi i}{\tau}\left(\mathfrak{n}_\alpha \mathcal{A}^{\alpha\beta\gamma} + (g-1)\mathcal{A}^{R\beta\gamma}\right)\nu_\beta \nu_\gamma}\; Z_{\Sigma_g \times T^2} \\ \tilde{T}[Z_{\Sigma_g \times T^2}] &= e^{-\frac{\pi i}{6}\left(\mathfrak{n}_\alpha \mathcal{A}^\alpha + (g-1)\mathcal{A}^R\right)}\; Z_{\Sigma_g \times T^2} \end{aligned}$$

Note that it transforms as an $\mathcal{N} = (0,2)$ elliptic genus. Indeed, there is (formally) a 2d $\mathcal{N} = (0,2)$ theory on T^2 obtained by dimensional reduction on Σ_g.

Some consistency checks

This sum-over-Bethe-vacua formula reproduces (and generalizes) a number of previous results obtained by different methods. In particular:

- The limit $\beta \to 0$ on the S^1 factor is governed by the trace anomalies—"Cardy-like formula." [di Pietro, Komargodski, 2014]
- The $\beta \to \infty$ limit is governed by a so-called "supersymmetric Casimir energy." [Assel, Cassani, Martelli, 2014; Assel, Cassani, di Pietro, Komargodski, Lorenzen, Martelli, 2015; Bobev, Bullimore, Kim, 2015]
- We relate the $S^3 \times S^1$ partition function [Romelsberger, 2007; Assel, Cassani, Martelli, 2014] to the $S^2 \times T^2$ partition function [Benini, Zaffaroni, 2015; Honda, Yoshida, 2015]:

$$\boxed{Z_{S^3 \times S^1} = \langle \mathcal{F}_1 \rangle_{S^2 \times T^2}}$$

Applications

3d $\mathcal{N}=2$ Wilson loop algebras

Supersymmetric Wilson loops are realized on the Coulomb branch $\mathfrak{M}$ as:

$$W_{\mathfrak{R}} = \mathrm{Tr}_{\mathfrak{R}}\left(x\right) = \sum_{\rho\in\mathfrak{R}} x^{\rho}$$

for $\mathfrak{R}$ a representation of $\mathbf{G}$. The Wilson loop algebra is of the form:

$$\mathcal{A} \cong \boldsymbol{R}[x_a, x_a^{-1}]^{W_{\mathbf{G}}}/I_{\mathrm{BE}}\ , \qquad \boldsymbol{R} = \mathbb{Q}(y_\alpha, y_\alpha^{-1})$$

Example: $U(N)_k$ $\mathcal{N}=2$ CS theory:

$$\mathcal{A} \cong \mathbb{Z}[x_a, x_a^{-1}]^{S_N}/I\ , \qquad I = \Big((-x_a)^k\Big)$$

This is the Verlinde algebra for pure $U(N)_{\hat{k}}$ CS at level $\hat{k} = k - \mathrm{sign}(k)N$.

3d $\mathcal{N}=2$ Wilson loop algebras

Another very interesting example is $U(N)$ YM theory with N_f fundamental and N_f antifundamental chiral multiplets. The theory has $\mathbf{G}_F = SU(N_f)\times SU(N_f)\times U(1)_A\times U(1)_T$. Consider the fugacities:

$$y_i, \qquad \tilde{y}_i\ , \qquad y_A\ , \qquad z\ , \qquad \tilde{z}\equiv z\, y_A^{-N_f}$$

such that $\prod_{i=1}^{N_f} y_i^{-1} = \prod_{i=1}^{N_f}\tilde{y}_i = y_A^{-N_f}$. The Bethe equations are:

$$P(x_a)=0\ , \quad a=1,\cdots,N_c, \qquad x_a\neq x_b \ \text{ if } \ a\neq b$$

$$P(x)=\prod_{i=1}^{N_f}(x-y_i)-\tilde{z}\prod_{j=1}^{N_f}(x-\tilde{y}_j)$$

3d $\mathcal{N}=2$ Wilson loop algebras

We can easily write down an explicit presentation of the algebra. Let us denote the Wilson loop $W_{\mathfrak{R}}$ by the Young tableau of the $U(N_c)$ rep. $\mathfrak{R}$.

For instance, consider the case of $U(3)$, $N_f=5$:

$$
\begin{aligned}
\ydiagram{3} &= R_3 - R_2\, \ydiagram{1} + R_1\, \ydiagram{2}\,, \\
\ydiagram{3,1} &= R_4 - R_2\, \ydiagram{1,1} + R_1\, \ydiagram{2,1}\,, \\
\ydiagram{3,1,1} &= R_5 - R_2\, \ydiagram{1,1,1} + R_1\, \ydiagram{2,1,1}\,, \qquad R_n \equiv \frac{1}{1-\tilde{z}}\left(s_n(y) - \tilde{z}\, s_n(\tilde{y})\right)
\end{aligned}
$$

This theory has a $U(2)$, $N_f=5$ Aharony dual. [Aharony, 1997] The Bethe equations also encode the duality relations for Wilson loops:

$$
\ydiagram{1}^{D} = R_1 - \ydiagram{1}\,, \qquad \ydiagram{1,1}^{D} = R_2 - R_1\, \ydiagram{1} + \ydiagram{2}\,,
$$

In the limit $R_n \to 0$, one recovers the Verlinde algebra for $U(3)_{\hat{k}=2}$ and level-rank duality.

Witten index

By taking the chemical potentials ν_α generic enough, we have a discrete number of Bethe vacua—the 2d theory is fully massive. (We *only* consider theories where ν can be taken "generic enough.")

The quantity:

$$|\mathcal{S}_{BE}| \in \mathbb{N}$$

is the simplest A-model observable. It is simply the Witten index:

$$Z_{T^d} = |\mathcal{S}_{BE}| \;, \qquad d = 3 \text{ or } 4$$

This is a regulated Witten index in the presence of generic masses for the matter fields. In 3d, it is known to be invariant as we change the mass parameters. [Intriligator, Seiberg, 2013] Similar in 4d.

The Witten index of SQCD

Consider $\mathcal{N}=1$ SQCD: $SU(N_c)$ with N_f flavors.

The Bethe equations are:

$$e^{2\pi i\lambda}\prod_{i=1}^{N_f}\frac{\theta(-v_a+\tilde{\nu}_i)}{\theta(v_a+\nu_i)}=1\ ,\qquad a=1,\cdots,N_c\ ,\qquad\qquad \sum_{a=1}^{N_c}v_a=\mu_B$$

One can compute:

$$\boxed{\mathcal{Z}_{T^4}=|\mathcal{S}_{BE}|=\binom{N_f-2}{N_c-1}}$$

This result is nicely consistent with Seiberg duality.

Infrared dualities of supersymmetric QFTs

The A-model is itself a TFT. In particular, it is RG invariant. This leads to new tests of infrared dualities. The full A-models $\mathbf{A}$ and $\mathbf{A}^D$ of infrared-dual theories $\mathcal{T}$ and $\mathcal{T}^D$ should match.

I.e. there exists an isomorphism:

$$\mathcal{D} : \mathbf{A} \to \mathbf{A}^D$$

In particular, on Bethe vacua:

$$\mathcal{D} : \mathcal{S}_{\mathrm{BE}} \to \mathcal{S}_{\mathrm{BE}} \;:\; \hat{u} \mapsto \hat{u}^D$$

Two operators $\mathcal{O} \in \mathbf{A}$ and $\mathcal{O}^D \in \mathbf{A}^D$ are dual if and only if:

$$\boxed{\mathcal{O}(\hat{u}) = \mathcal{O}_D(\hat{u}^D)}$$

The existence of $\mathcal{D}$ implies that $Z_{\mathcal{M}_{g,p}}$ or $Z_{\mathcal{M}_{g,p}\times S^1}$ (etc.) match.

4d $\mathcal{N}=1$ Seiberg duality

Seiberg duality: $SU(N_c)$ with N_f flavors Q_i, $\tilde{Q}_j$ **dual to** $SU(N_f-N_c)$ with N_f flavors q^i, $\tilde{q}^j$, N_f^2 singlets M_{ij} and $W=Mq\tilde{q}$. [Seiberg, 1994]

Bethe equations:

$$\boxed{\Pi_0(v_a,\lambda)=1\ ,\quad a=1,\cdots,N_c\ ,\qquad \sum_{a=1}^{N_c} v_a=\mu_B}$$

$$\Pi_0(v,\lambda)\equiv e^{2\pi i\lambda}\prod_{i=1}^{N_f}\frac{\theta(-v+\tilde{\nu}_i)}{\theta(v+\nu_i)}\ .$$

Let $\tilde{v}_k$ denote the N_f solutions to $\Pi_0(v,\lambda)=1$ at arbitrary λ.
Bethe vacuum:

$$\{\hat{v}_a,\lambda_0\}\ \mid\ \{\hat{v}_a\}_{a=1}^{N_c}\equiv A\subset\{\tilde{v}_k\}_{k=1}^{N_f}\ ,\quad \lambda=\lambda_0\ \text{such that}\ \sum_a \hat{v}_a=\mu_B$$

4d $\mathcal{N}=1$ Seiberg duality

Duality map:

$$\boxed{\mathcal{D}:\{\hat{v}_a,\lambda_0\}\mapsto\{\hat{v}^D_{\bar{a}},\lambda^D_0\}\ ,}\qquad \{\hat{v}^D_{\bar{a}}\}_{\bar{a}=1}^{N_f-N_c}=A^c\subset\{\tilde{v}_k\}_{k=1}^{N_f}\ ,\quad \lambda^D_0=-\lambda_0$$

Equality of fibering operators of dual theories is equivalent to the identity:

$$\boxed{\prod_{k=1}^{N_f}\prod_{i=1}^{N_f}\Gamma_0(\tilde{v}_k+\nu_i)\Gamma_0(-\tilde{v}_k+\tilde{\nu}_i)=\prod_{i,j=1}^{N_f}\Gamma_0(\nu_i+\tilde{\nu}_j)}$$

We don't have a proof, but we can check it numerically. On the other hand, some simple manipulations on θ-functions implies that $\mathcal{H}(\hat{u})=\mathcal{H}^D(\hat{u}^D)$.

That then implies the equality of all $\mathcal{M}_{g,p}$ indices:

$$Z_{\mathcal{M}_{g,p}\times S^1}[\mathcal{T}]=Z_{\mathcal{M}_{g,p}\times S^1}[\mathcal{T}^D]$$

Summary and outlook

We described simple TFT for (abelianized) 2d gauge fields—the A-models—that compute supersymmetric partition functions—and, more generally, expectation values of codimension-2 operators— in 3d and 4d gauge theories with 4 supercharges.

What's next?

- Describe any allowed (Seifert) $\mathcal{M}_3$ in this language.
- Extend to "non-Lagrangian" theories.
- Describe the algebra of half-BPS surface operators in 4d $\mathcal{N}=1$ gauge theories. [in progress] (Mathematical interpretation?)
- Describe and interpret the Ω-deformation at genus $g=0$. "Quantization" of twisted chiral ring? Connect to 3d holomorphic blocks [Beem, Dimofte, Pasquetti, 2012]
- 3d/3d correspondence: M5-branes on $\mathcal{M}_{g,p}\times\mathcal{M}_3^{\mathrm{TFT}}$?