Liouville quantum gravity and the Brownian map

Jason Miller

Cambridge

joint with
Scott Sheffield (MIT)

July 28, 2017

Overview

Part I: Picking surfaces at random

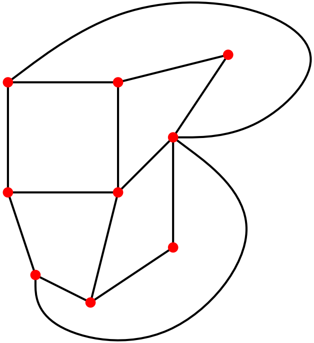
1. Discrete: random planar maps
2. Continuum: Liouville quantum gravity (LQG)
3. Relationship

Part II: Schramm-Loewner evolution

Part III: Construction of the metric on LQG

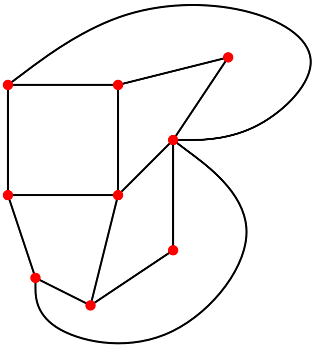
Part I: Picking surfaces at random

Random planar maps



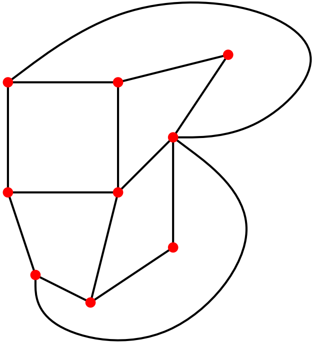
- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.

Random planar maps



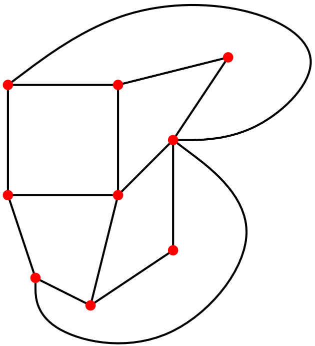
- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.
- Its **faces** are the connected components of the complement of its edges

Random planar maps



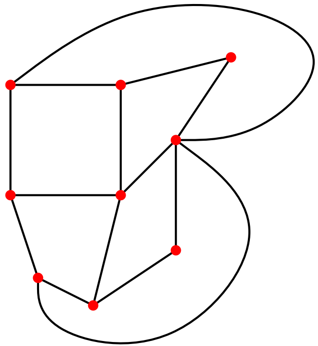
- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.
- Its **faces** are the connected components of the complement of its edges
- A map is a **quadrangulation** if each face has 4 adjacent edges

Random planar maps



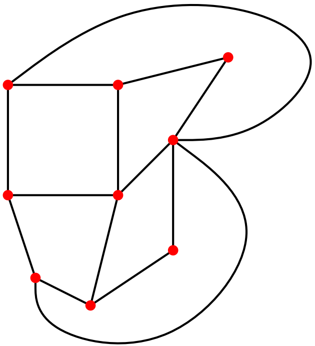
- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.
- Its **faces** are the connected components of the complement of its edges
- A map is a **quadrangulation** if each face has 4 adjacent edges
- A quadrangulation corresponds to a **metric space** when equipped with the graph distance

Random planar maps

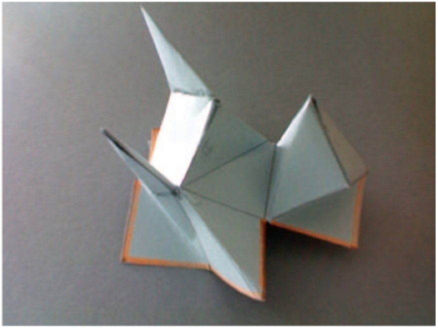
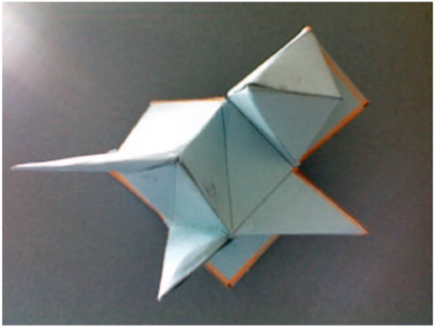


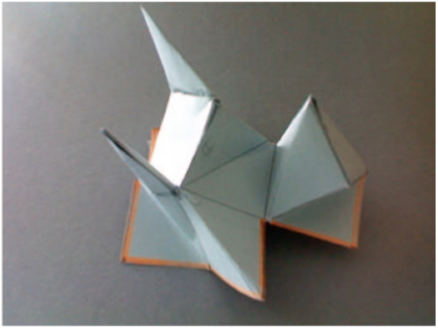
- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.
- Its **faces** are the connected components of the complement of its edges
- A map is a **quadrangulation** if each face has 4 adjacent edges
- A quadrangulation corresponds to a **metric space** when equipped with the graph distance
- Interested in **uniformly random quadrangulations** with n faces — **random planar map** (RPM).

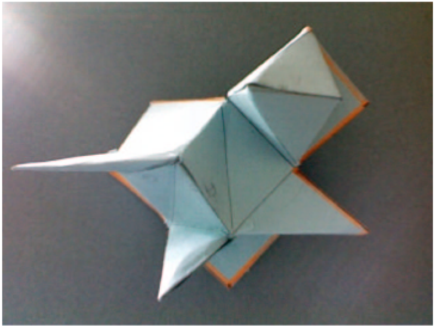
Random planar maps



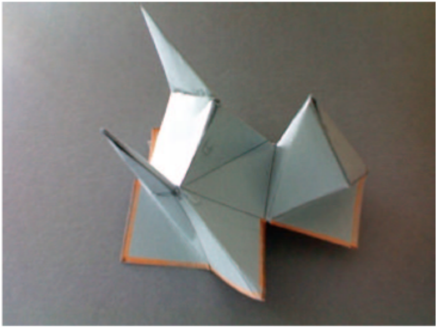
- A **planar map** is a finite graph together with an embedding in the plane so that no edges cross.
- Its **faces** are the connected components of the complement of its edges
- A map is a **quadrangulation** if each face has 4 adjacent edges
- A quadrangulation corresponds to a **metric space** when equipped with the graph distance
- Interested in **uniformly random quadrangulations** with n faces — **random planar map** (RPM).
- First studied by Tutte in 1960s while working on the four color theorem. Long history in cominbatorics, statistical physics, and probability.

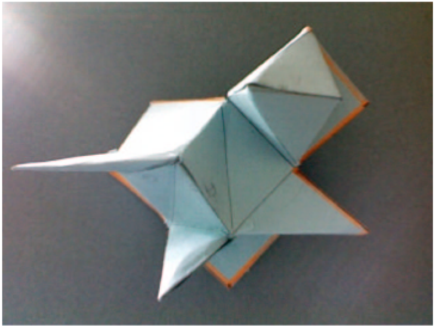







What is the structure of a typical quadrangulation when the number of faces is large?



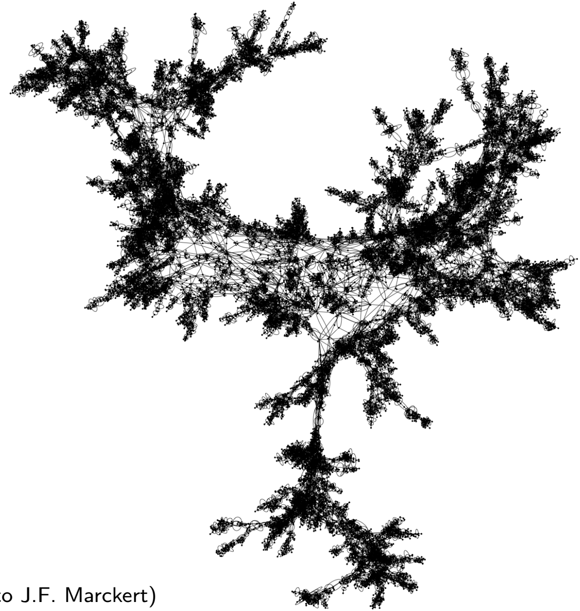


What is the structure of a typical quadrangulation when the number of faces is large?

How many are there? **Tutte:**

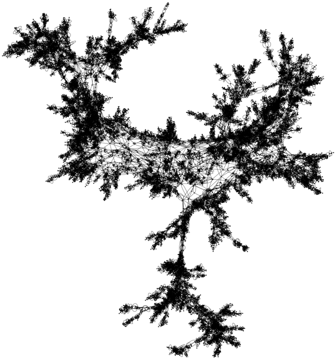
$$\frac{2 \times 3^n}{(n+1)(n+2)} \binom{2n}{n}.$$

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

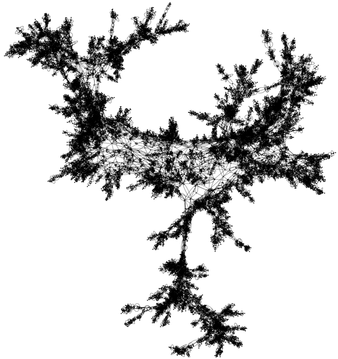
Structure of large random planar maps



(Simulation due to J.F. Marckert)

- Diameter is $n^{1/4}$ (Chaissang-Schaefer)

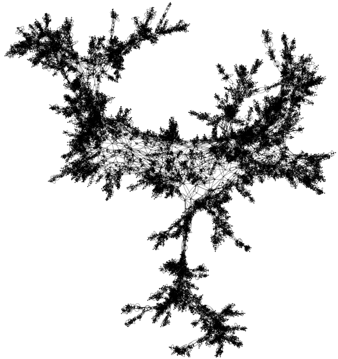
Structure of large random planar maps



(Simulation due to J.F. Marckert)

- Diameter is $n^{1/4}$ (Chaissang-Schaefer)
- Non-trivial subsequentially limiting metric spaces upon scaling distances by $n^{-1/4}$ (Le Gall)

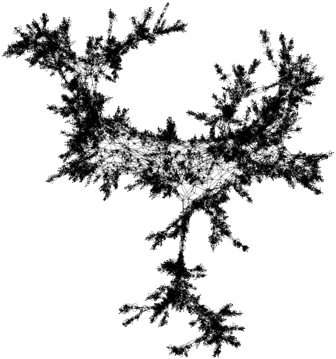
Structure of large random planar maps



(Simulation due to J.F. Marckert)

- Diameter is $n^{1/4}$ (Chaissang-Schaefer)
- Non-trivial subsequentially limiting metric spaces upon scaling distances by $n^{-1/4}$ (Le Gall)
- Subsequentially limiting space is a.s.:
  - 4-dimensional (Le Gall)
  - homeomorphic to the 2-sphere (Le Gall and Paulin, Miermont)

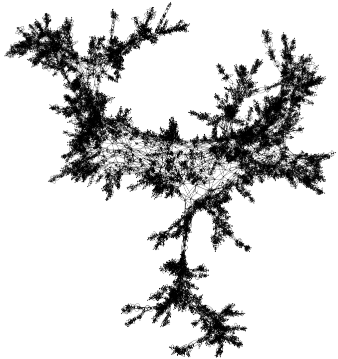
Structure of large random planar maps



(Simulation due to J.F. Marckert)

- Diameter is $n^{1/4}$ (Chaissang-Schaefer)
- Non-trivial subsequentially limiting metric spaces upon scaling distances by $n^{-1/4}$ (Le Gall)
- Subsequentially limiting space is a.s.:
  - 4-dimensional (Le Gall)
  - homeomorphic to the 2-sphere (Le Gall and Paulin, Miermont)
- There exists a unique limit in distribution: **the Brownian map** (Le Gall, Miermont)

Structure of large random planar maps



(Simulation due to J.F. Marckert)

- Diameter is $n^{1/4}$ (Chaissang-Schaefer)
- Non-trivial subsequentially limiting metric spaces upon scaling distances by $n^{-1/4}$ (Le Gall)
- Subsequentially limiting space is a.s.:
  - 4-dimensional (Le Gall)
  - homeomorphic to the 2-sphere (Le Gall and Paulin, Miermont)
- There exists a unique limit in distribution: **the Brownian map** (Le Gall, Miermont)
- The Brownian map (TBM) comes equipped with an area measure which is the limit of the rescaled measure on RPM which assigns unit mass for each face.

Liouville quantum gravity

- D planar domain

Liouville quantum gravity

- D planar domain
- Consider a surface with Riemmanian metric of the form $e^{\gamma h(z)}(dx^2 + dy^2)$ for $\gamma > 0$

Liouville quantum gravity

- D planar domain
- Consider a surface with Riemmanian metric of the form $e^{\gamma h(z)}(dx^2 + dy^2)$ for $\gamma > 0$
- Liouville quantum gravity (LQG): take h to be an instance of the Gaussian free field (GFF) on D.

Liouville quantum gravity

- D planar domain
- Consider a surface with Riemmanian metric of the form $e^{\gamma h(z)}(dx^2 + dy^2)$ for $\gamma > 0$
- Liouville quantum gravity (LQG): take h to be an instance of the Gaussian free field (GFF) on D.
- The GFF is the Gaussian process indexed by $x \in D$ with covariance $\text{cov}(h(x), h(y)) = G(x, y)$ where G is the Green's function for Δ on D

Liouville quantum gravity

- D planar domain
- Consider a surface with Riemmanian metric of the form $e^{\gamma h(z)}(dx^2 + dy^2)$ for $\gamma > 0$
- Liouville quantum gravity (LQG): take h to be an instance of the Gaussian free field (GFF) on D.
- The GFF is the Gaussian process indexed by $x \in D$ with covariance $\mathrm{cov}(h(x), h(y)) = G(x, y)$ where G is the Green's function for Δ on D
- Since $G(x, y) \sim -\log|x - y|$ for $x \sim y$, the GFF has infinite variance at points, hence is not a function. Rather, it is a distribution in the sense of Schwartz. So, the expression $e^{\gamma h}$ does not make mathematical sense.

Liouville quantum gravity

- D planar domain
- Consider a surface with Riemmanian metric of the form $e^{\gamma h(z)}(dx^2 + dy^2)$ for $\gamma > 0$
- Liouville quantum gravity (LQG): take h to be an instance of the Gaussian free field (GFF) on D.
- The GFF is the Gaussian process indexed by $x \in D$ with covariance $\mathrm{cov}(h(x), h(y)) = G(x, y)$ where G is the Green's function for Δ on D
- Since $G(x, y) \sim -\log|x - y|$ for $x \sim y$, the GFF has infinite variance at points, hence is not a function. Rather, it is a distribution in the sense of Schwartz. So, the expression $e^{\gamma h}$ does not make mathematical sense.
- Previously, only the volume form made sense of as a limit

$$\lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$$

where h_ϵ is a mollified version of h and $\epsilon^{\gamma^2/2}$ is a normalization term.

Liouville quantum gravity

- D planar domain
- Consider a surface with Riemmanian metric of the form $e^{\gamma h(z)}(dx^2 + dy^2)$ for $\gamma > 0$
- Liouville quantum gravity (LQG): take h to be an instance of the Gaussian free field (GFF) on D.
- The GFF is the Gaussian process indexed by $x \in D$ with covariance $\mathrm{cov}(h(x), h(y)) = G(x, y)$ where G is the Green's function for Δ on D
- Since $G(x, y) \sim -\log|x - y|$ for $x \sim y$, the GFF has infinite variance at points, hence is not a function. Rather, it is a distribution in the sense of Schwartz. So, the expression $e^{\gamma h}$ does not make mathematical sense.
- Previously, only the volume form made sense of as a limit

  $$\lim_{\epsilon \to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$$

  where h_ϵ is a mollified version of h and $\epsilon^{\gamma^2/2}$ is a normalization term.
- This talk is about constructing the **metric**. Not obviously possible using mollification. Will take an indirect approach.

Canonical embedding of TBM into $\mathbf{S}^2$

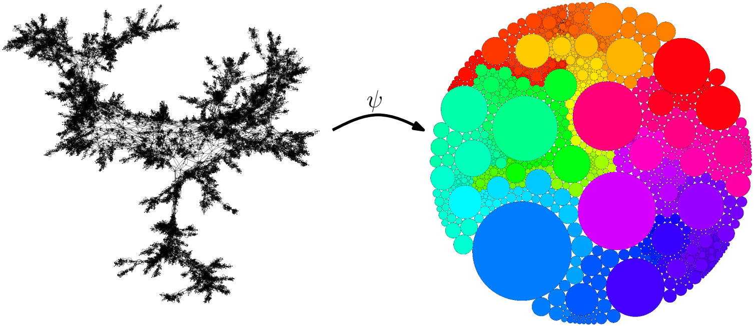
- TBM is an abstract metric measure space homeomorphic to $\mathbf{S}^2$, but it does not obviously come with a canonical embedding into $\mathbf{S}^2$

Canonical embedding of TBM into $\mathbf{S}^2$

- TBM is an abstract metric measure space homeomorphic to $\mathbf{S}^2$, but it does not obviously come with a canonical embedding into $\mathbf{S}^2$
- It is believed that there should be a "natural embedding" of TBM into $\mathbf{S}^2$ and that the embedded surface is described by a form of Liouville quantum gravity (LQG) with $\gamma = \sqrt{8/3}$

Canonical embedding of TBM into $\mathbf{S}^2$

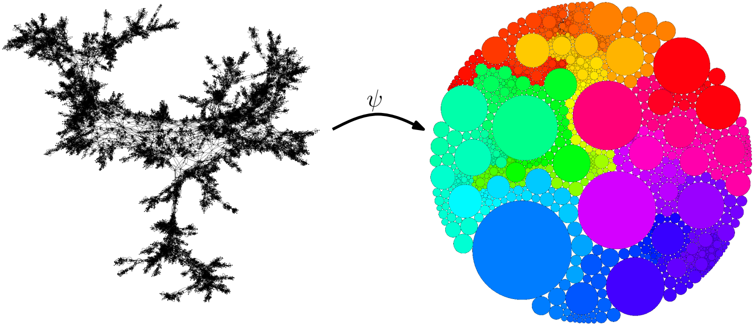
- TBM is an abstract metric measure space homeomorphic to $\mathbf{S}^2$, but it does not obviously come with a canonical embedding into $\mathbf{S}^2$
- It is believed that there should be a "natural embedding" of TBM into $\mathbf{S}^2$ and that the embedded surface is described by a form of Liouville quantum gravity (LQG) with $\gamma = \sqrt{8/3}$



- Discrete approach: take a uniformly random planar map and embed it conformally into $\mathbf{S}^2$ (circle packing, uniformization, etc...), then in the $n \to \infty$ limit it converges to a form of $\sqrt{8/3}$-LQG.

Canonical embedding of TBM into $\mathbf{S}^2$

- TBM is an abstract metric measure space homeomorphic to $\mathbf{S}^2$, but it does not obviously come with a canonical embedding into $\mathbf{S}^2$
- It is believed that there should be a "natural embedding" of TBM into $\mathbf{S}^2$ and that the embedded surface is described by a form of Liouville quantum gravity (LQG) with $\gamma = \sqrt{8/3}$



- Discrete approach: take a uniformly random planar map and embed it conformally into $\mathbf{S}^2$ (circle packing, uniformization, etc...), then in the $n \to \infty$ limit it converges to a form of $\sqrt{8/3}$-LQG. *Not the approach we will describe today ...*

Main result

- Liouville quantum gravity (LQG): $e^{\gamma h(z)}(dx^2 + dy^2)$, h a GFF
- The Brownian map (TBM): scaling limit of uniformly random quadrangulations

Theorem (M., Sheffield)

TBM and $\sqrt{8/3}$-LQG are equivalent. More precisely, there is a way to endow $\sqrt{8/3}$-LQG with a metric so that it is isometric to TBM.

Main result

- Liouville quantum gravity (LQG): $e^{\gamma h(z)}(dx^2 + dy^2)$, h a GFF
- The Brownian map (TBM): scaling limit of uniformly random quadrangulations

Theorem (M., Sheffield)

TBM and $\sqrt{8/3}$-LQG are equivalent. More precisely, there is a way to endow $\sqrt{8/3}$-LQG with a metric so that it is isometric to TBM.

Comments

Main result

- Liouville quantum gravity (LQG): $e^{\gamma h(z)}(dx^2 + dy^2)$, h a GFF
- The Brownian map (TBM): scaling limit of uniformly random quadrangulations

Theorem (M., Sheffield)

TBM and $\sqrt{8/3}$-LQG are equivalent. More precisely, there is a way to endow $\sqrt{8/3}$-LQG with a metric so that it is isometric to TBM.

Comments

1. Construction is purely in the continuum

Main result

- Liouville quantum gravity (LQG): $e^{\gamma h(z)}(dx^2 + dy^2)$, h a GFF
- The Brownian map (TBM): scaling limit of uniformly random quadrangulations

Theorem (M., Sheffield)

TBM and $\sqrt{8/3}$-LQG are equivalent. More precisely, there is a way to endow $\sqrt{8/3}$-LQG with a metric so that it is isometric to TBM.

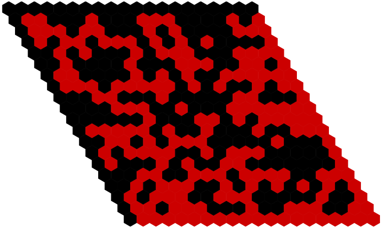
Comments

1. Construction is purely in the continuum
2. Ideas are connected to aggregation models, such as the Eden model and diffusion limited aggregation

Part II:
Schramm-Loewner evolution

Schramm-Loewner evolution (SLE)

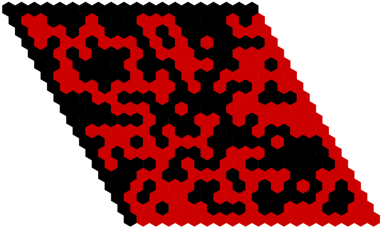
- Random fractal curve in a planar domain



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

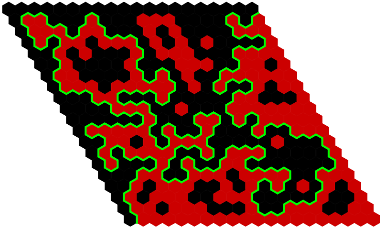
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

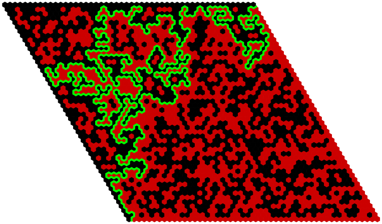
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

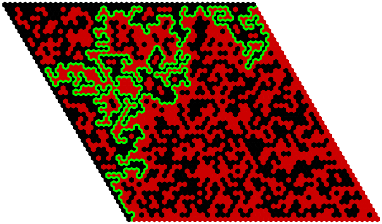
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

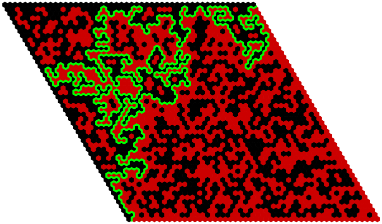
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models
- Characterized by conformal invariance and domain Markov property



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

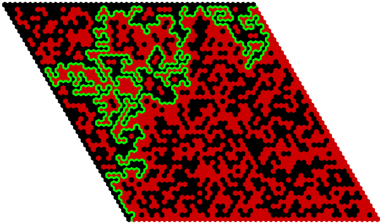
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models
- Characterized by conformal invariance and domain Markov property
- Indexed by a parameter $\kappa > 0$



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

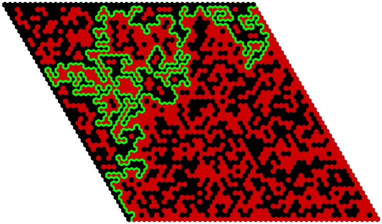
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models
- Characterized by conformal invariance and domain Markov property
- Indexed by a parameter $\kappa > 0$
- Simple for $\kappa \in (0, 4]$, self-intersecting for $\kappa \in (4, 8)$, space-filling for $\kappa \geq 8$



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

Schramm-Loewner evolution (SLE)

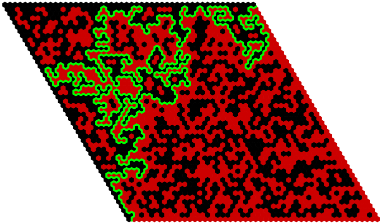
- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models
- Characterized by conformal invariance and domain Markov property
- Indexed by a parameter $\kappa > 0$
- Simple for $\kappa \in (0,4]$, self-intersecting for $\kappa \in (4,8)$, space-filling for $\kappa \geq 8$
- Dimension: $1 + \kappa/8$ for $\kappa \leq 8$



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

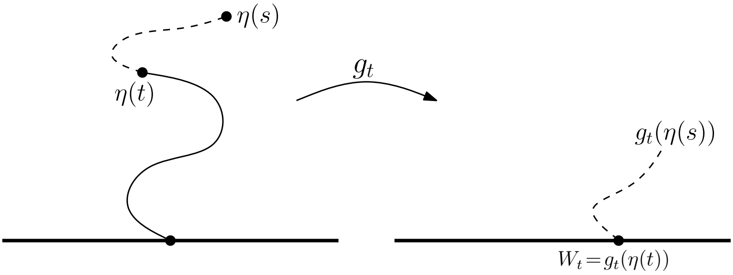
Schramm-Loewner evolution (SLE)

- Random fractal curve in a planar domain
- Introduced by Schramm in '99 to describe limits of interfaces in discrete models
- Characterized by conformal invariance and domain Markov property
- Indexed by a parameter $\kappa > 0$
- Simple for $\kappa \in (0, 4]$, self-intersecting for $\kappa \in (4, 8)$, space-filling for $\kappa \geq 8$
- Dimension: $1 + \kappa/8$ for $\kappa \leq 8$
- Some special κ values:
  - $\kappa = 2$ LERW, $\kappa = 8$ UST
  - $\kappa = 8/3$ Self-avoiding walk
  - $\kappa = 3$ Ising, $\kappa = 16/3$ FK-Ising
  - $\kappa = 4$ GFF level lines
  - $\kappa = 6$ Percolation
  - $\kappa = 12$ Bipolar orientations
  - ...



Critical percolation, hexagonal lattice
Each hexagon is colored red or black with prob. $\frac{1}{2}$

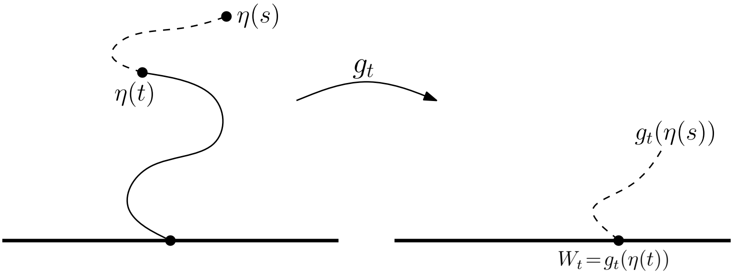
SLE_κ



Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\star)$$

SLE_κ

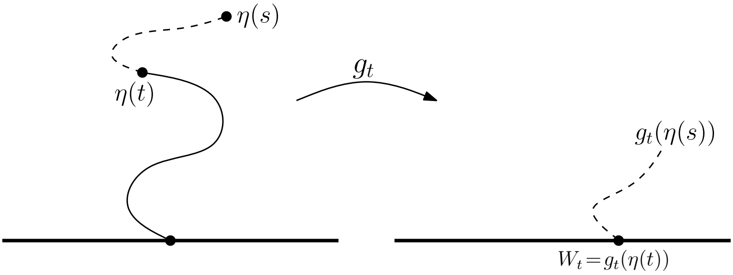


Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\bigstar)$$

SLE_κ **in H**: The random curve associated with $(\bigstar)$ with $W_t = \sqrt{\kappa} B_t$, B a standard Brownian motion.

SLE_κ

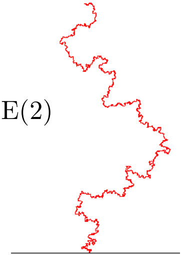


Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

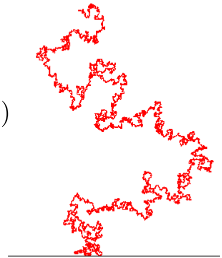
$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\star)$$

SLE_κ in $\mathbf{H}$: The random curve associated with $(\star)$ with $W_t = \sqrt{\kappa} B_t$, B a standard Brownian motion. Other domains: apply conformal mapping.

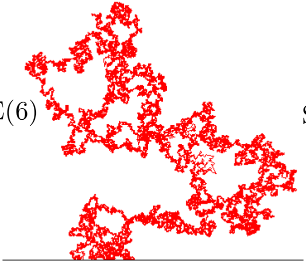
SLE(2)



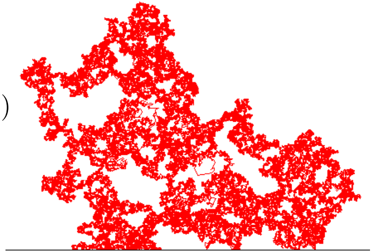
SLE(4)



SLE(6)



SLE(8)

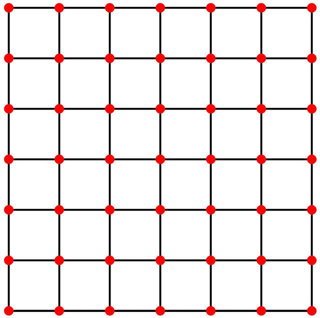


Simulations due to Tom Kennedy.

Part III:
Construction of the metric

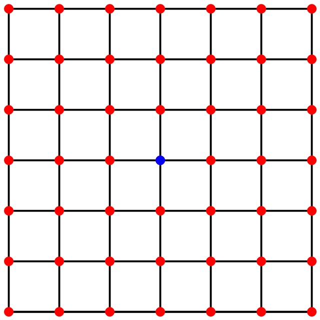
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



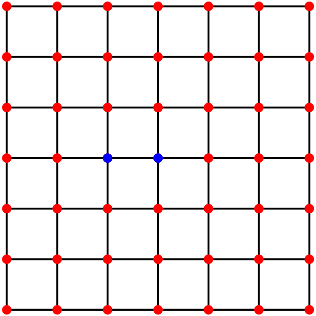
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



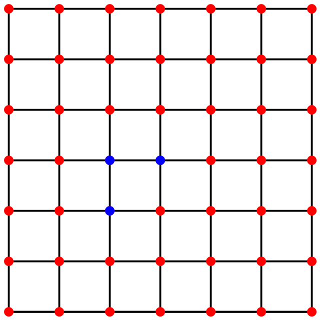
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



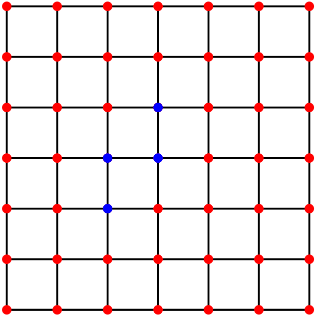
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



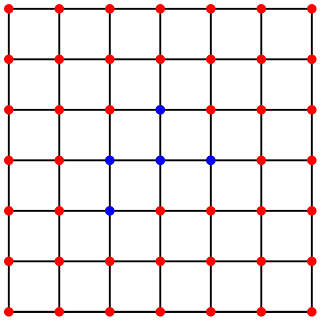
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



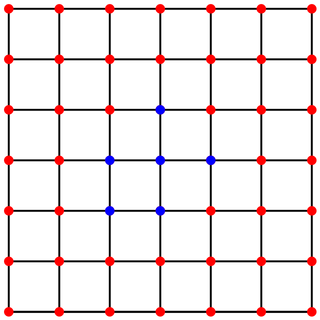
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



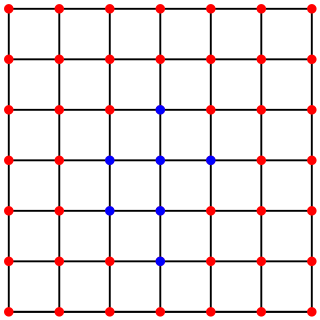
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



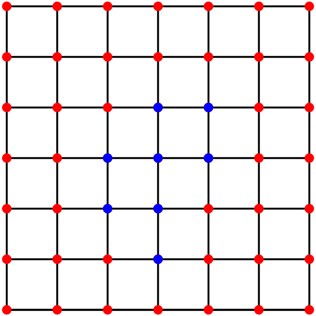
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



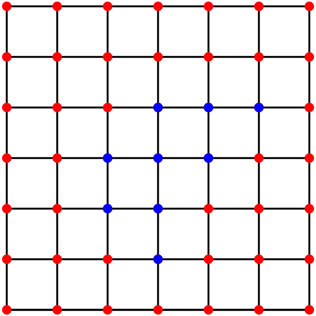
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



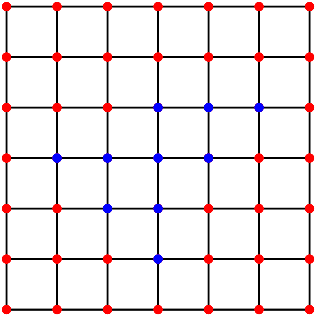
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



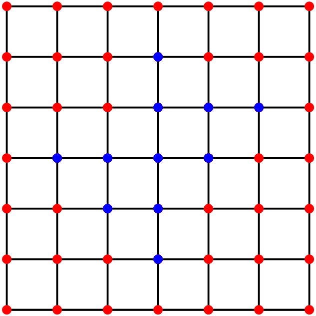
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



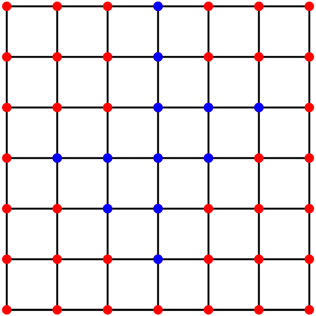
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



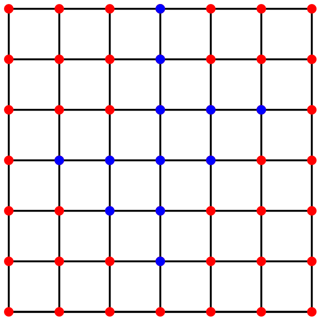
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.



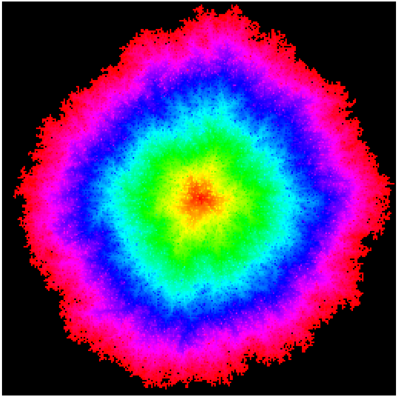
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.
- **Question:** Large scale behavior of the growth?



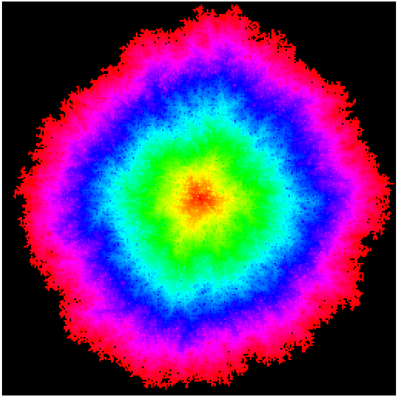
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.
- **Question:** Large scale behavior of the growth?



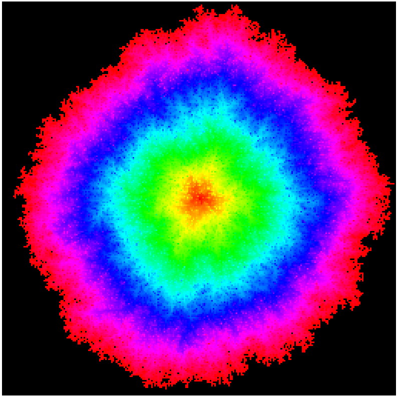
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.
- **Question:** Large scale behavior of the growth?
- Cox and Durrett (1981) showed that the macroscopic shape is convex



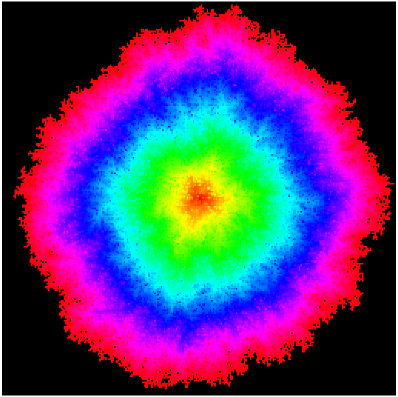
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.
- **Question:** Large scale behavior of the growth?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk



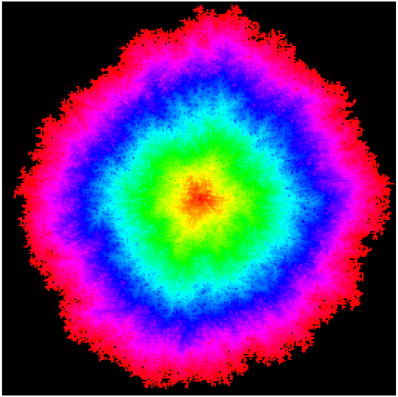
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.
- **Question:** Large scale behavior of the growth?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
- $\mathbf{Z}^2$ has preferential directions



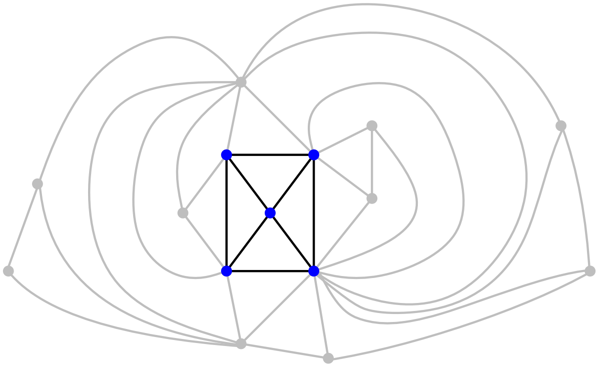
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.
- **Question:** Large scale behavior of the growth?
- Cox and Durrett (1981) showed that the macroscopic shape is convex
- Computer simulations show that it is not a Euclidean disk
- $\mathbf{Z}^2$ has preferential directions
- But a random planar map does not ...



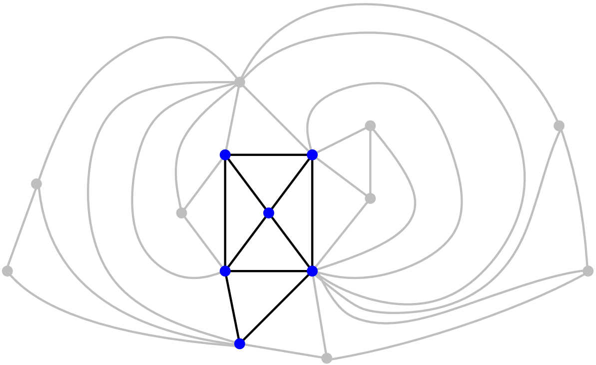
Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



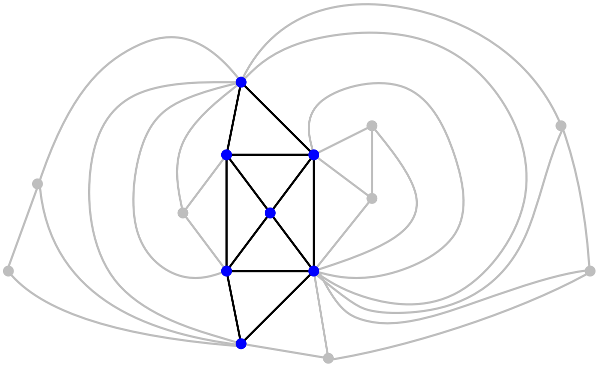
Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



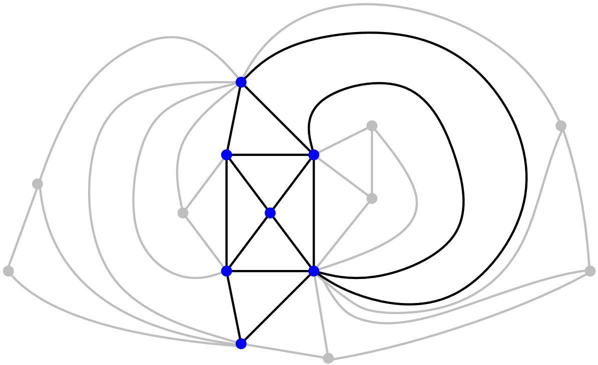
Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



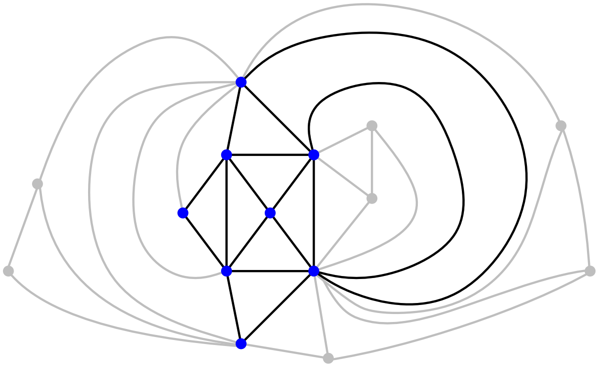
Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



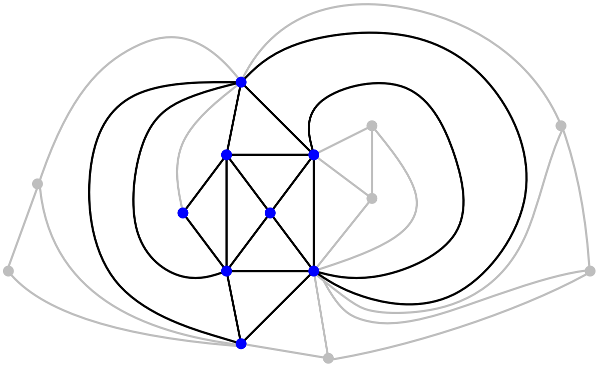
Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



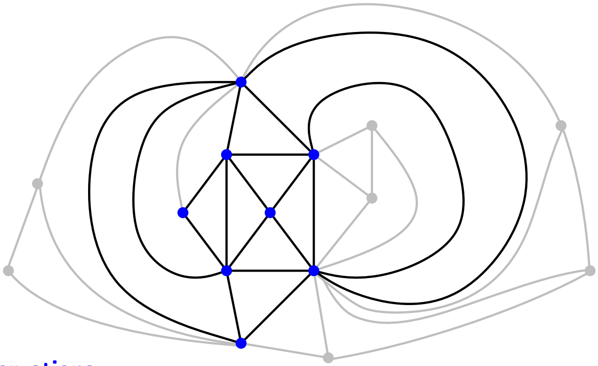
Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



Eden model on random planar maps I

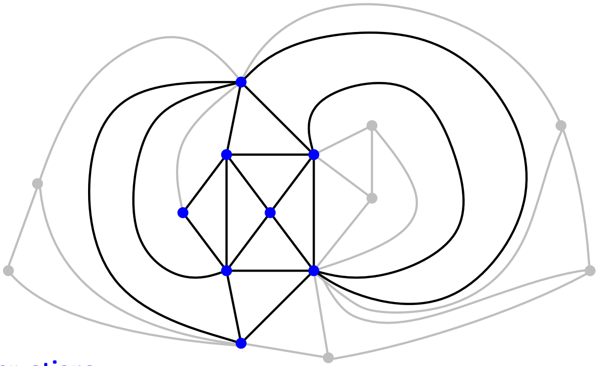
- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



Important observations:

Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).

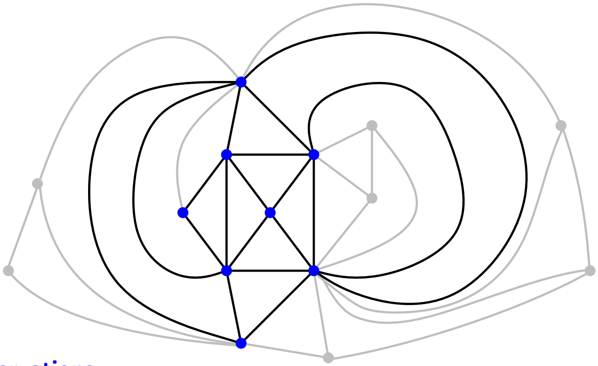


Important observations:

- Conditional law of map given growth at time n only depends on the boundary lengths of the outside components.

Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).

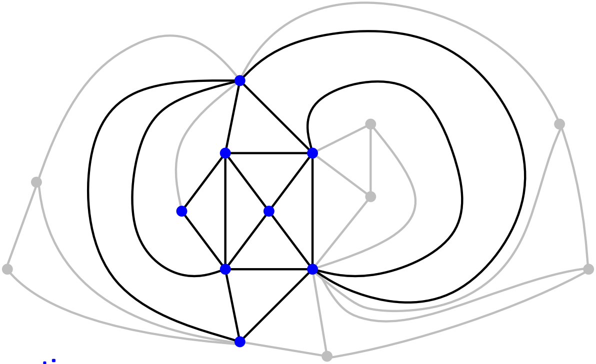


Important observations:

- Conditional law of map given growth at time n only depends on the boundary lengths of the outside components. *Exploration respects the Markovian structure of the map.*

Eden model on random planar maps I

- RPM, random vertex x. Grow Eden cluster from x (Angel's peeling process).



Important observations:

- Conditional law of map given growth at time n only depends on the boundary lengths of the outside components. *Exploration respects the Markovian structure of the map.*

Belief: At large scales this is close to a ball in the graph metric (now **proved** by Curien and Le Gall)

Eden model on random planar maps II

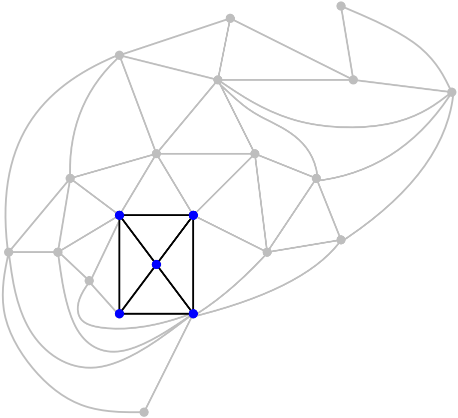
Goal: Make sense of the Eden model in the continuum on top of a LQG surface

- Explain a discrete variant of the Eden model that involves two operations that we do know how to perform in the continuum:
  - Sample random points according to boundary length
  - Draw (scaling limits of) critical percolation interfaces (SLE_6)

Eden model on random planar maps II

Variant:

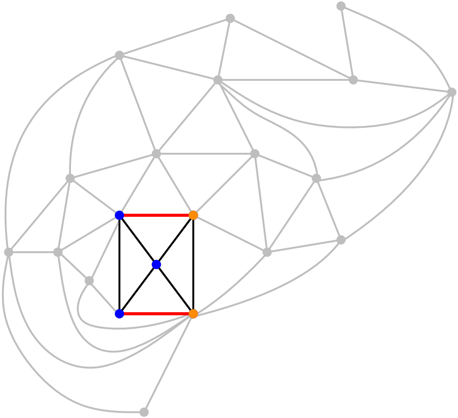
- Pick two edges on outer boundary of cluster



Eden model on random planar maps II

Variant:

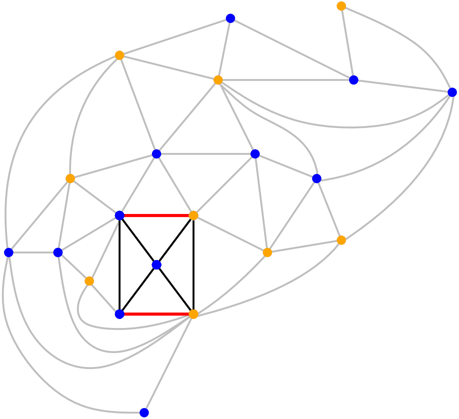
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow



Eden model on random planar maps II

Variant:

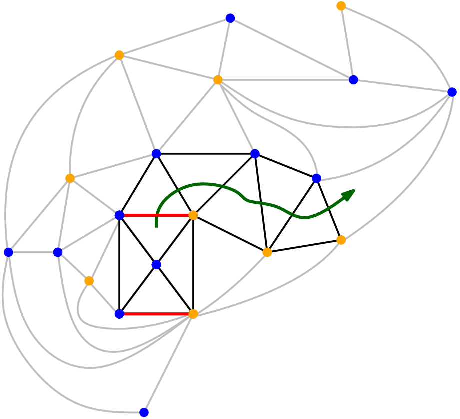
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$



Eden model on random planar maps II

Variant:

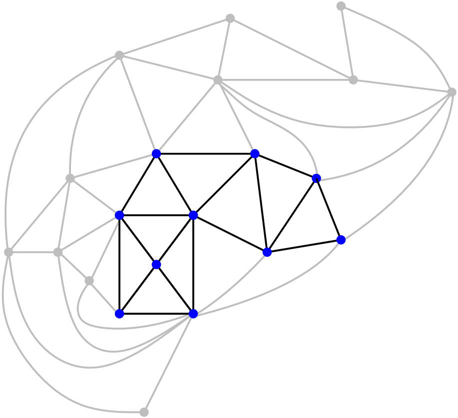
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface



Eden model on random planar maps II

Variant:

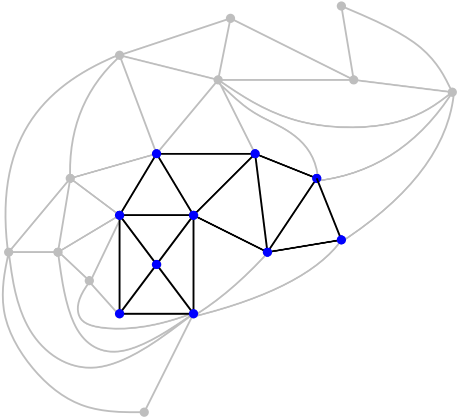
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors



Eden model on random planar maps II

Variant:

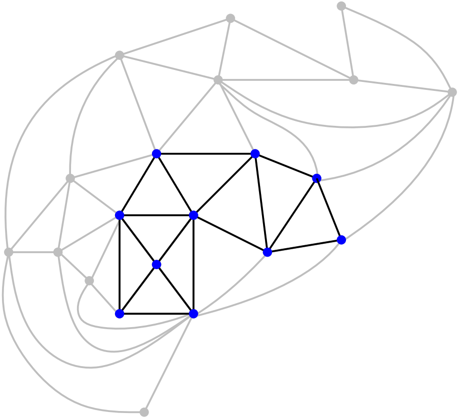
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

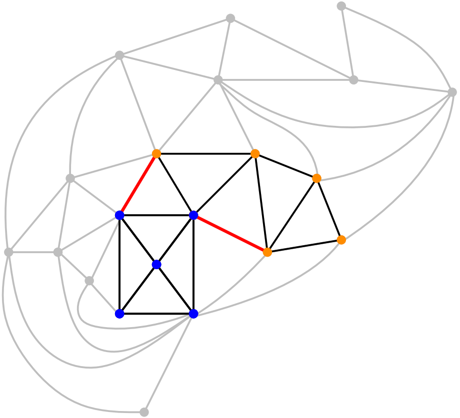
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

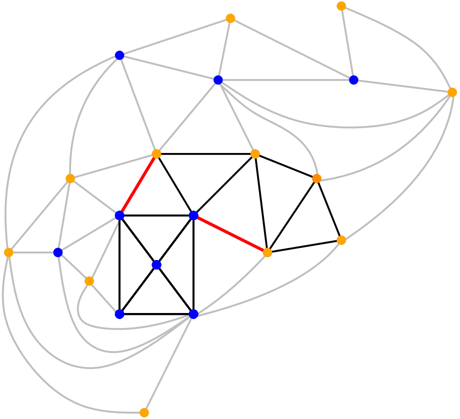
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

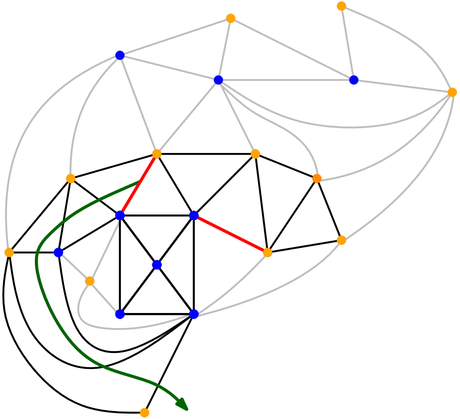
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

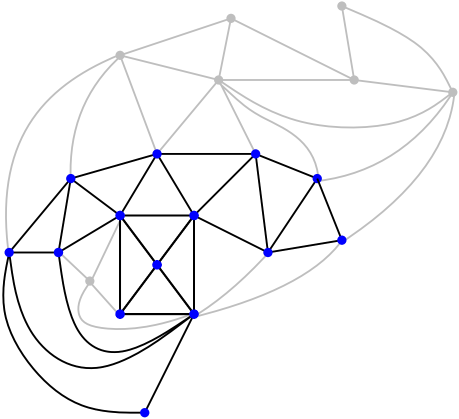
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

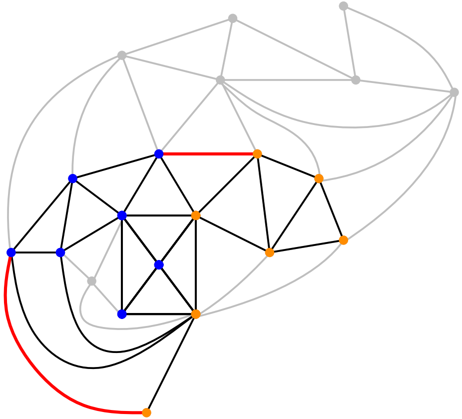
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

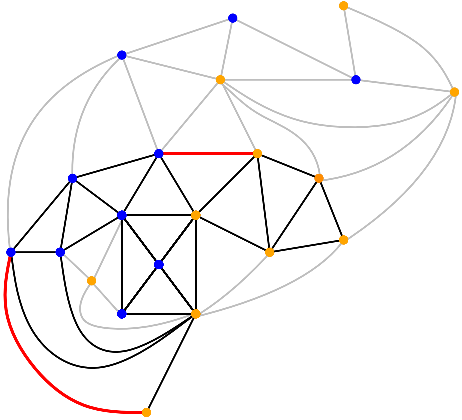
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

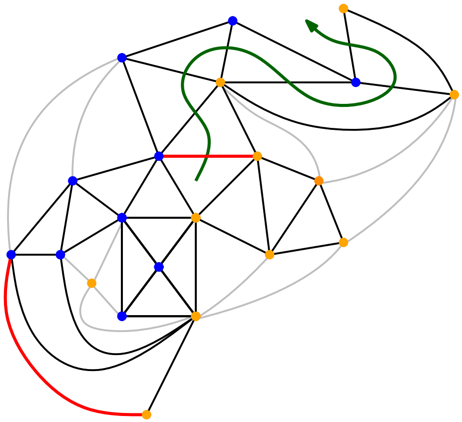
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

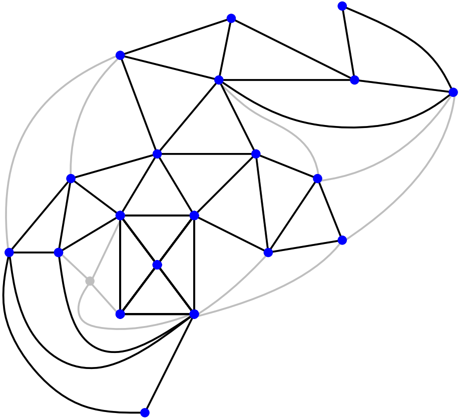
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

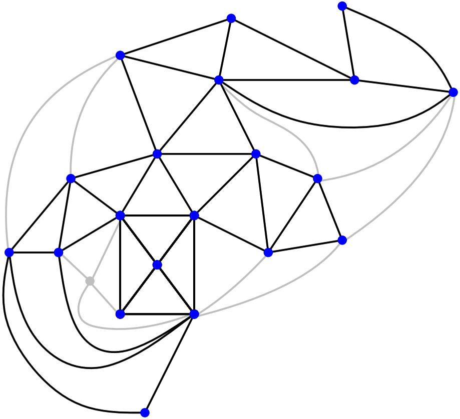
- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



Eden model on random planar maps II

Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat

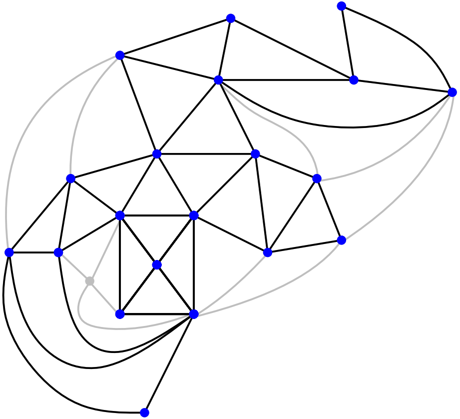


- *This exploration also respects the Markovian structure of the map.*

Eden model on random planar maps II

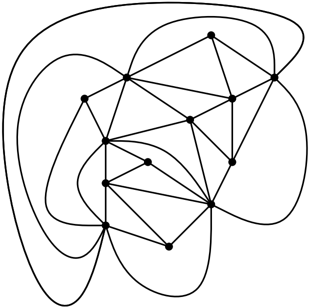
Variant:

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
- Color vertices on rest of map blue or yellow with prob. $\frac{1}{2}$
- Explore percolation (blue/yellow) interface
- Forget colors
- Repeat



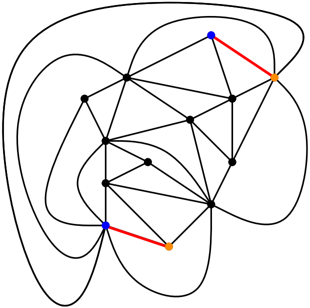
- *This exploration also respects the Markovian structure of the map.*
- Expect that at large scales this growth process looks the same as the Eden model, hence the same as the graph metric ball

Continuum limit ansatz



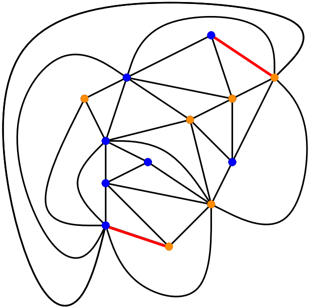
- Sample a random planar map

Continuum limit ansatz



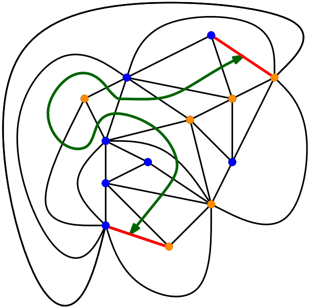
- Sample a random planar map and two edges uniformly at random

Continuum limit ansatz



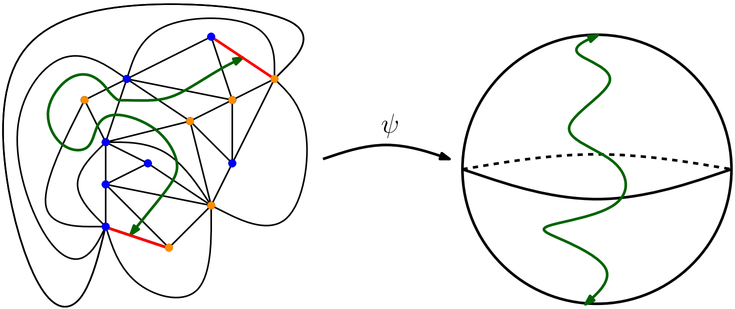
- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$

Continuum limit ansatz



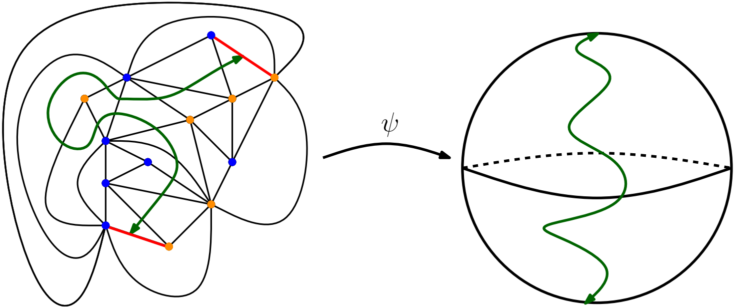
- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface

Continuum limit ansatz



- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface
- Conformally map to the sphere

Continuum limit ansatz



- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface
- Conformally map to the sphere

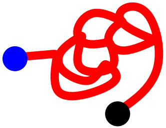
Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x

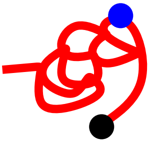
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6



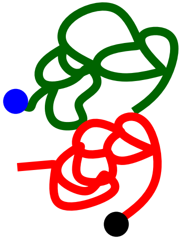
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length



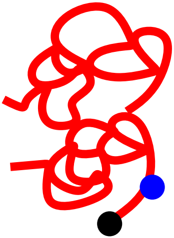
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat



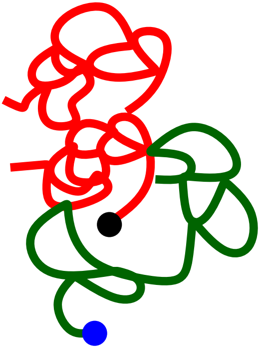
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat



Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat

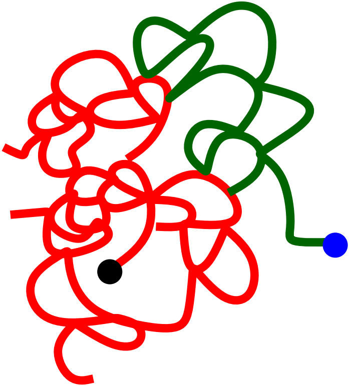


Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat



Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat
- Know the conditional law of the LQG surface at each stage

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat
- Know the conditional law of the LQG surface at each stage

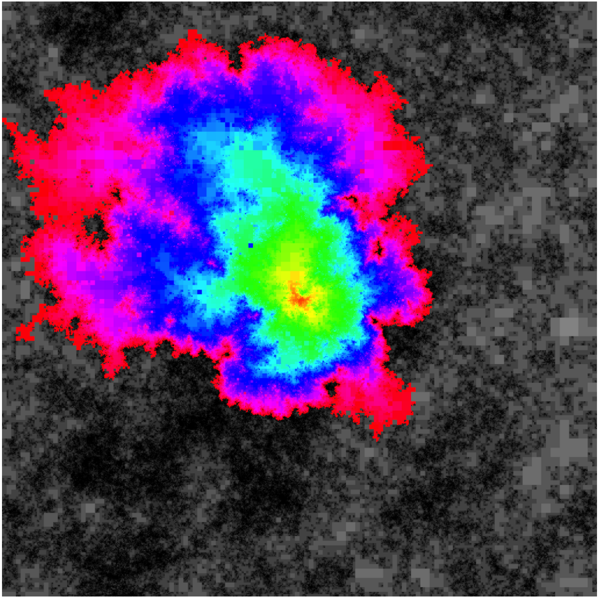
$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process.

Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point x
- Draw δ units of SLE_6
- Resample the tip according to boundary length
- Repeat
- Know the conditional law of the LQG surface at each stage

$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process.

In the limit, this describes the growth of a metric ball in a metric space which is isometric to TBM.



Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
- η determines the manner in which it grows

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
- η determines the manner in which it grows

Let μ_{HARM} (resp. μ_{LEN}) be harmonic (resp. length) measure on a γ-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left(\frac{d\mu_{\mathrm{HARM}}}{d\mu_{\mathrm{LEN}}} \right)^{\eta} d\mu_{\mathrm{LEN}}.$$

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
- η determines the manner in which it grows

Let μ_{HARM} (resp. μ_{LEN}) be harmonic (resp. length) measure on a γ-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left(\frac{d\mu_{\mathrm{HARM}}}{d\mu_{\mathrm{LEN}}}\right)^{\eta} d\mu_{\mathrm{LEN}}.$$

- **Eden model:** $\eta = 0$

What is $\mathrm{QLE}(\gamma^2, \eta)$?

$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
- η determines the manner in which it grows

Let μ_{HARM} (resp. μ_{LEN}) be harmonic (resp. length) measure on a γ-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left(\frac{d\mu_{\mathrm{HARM}}}{d\mu_{\mathrm{LEN}}}\right)^{\eta} d\mu_{\mathrm{LEN}}.$$

- **Eden model:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$

What is $\mathrm{QLE}(\gamma^2, \eta)$?

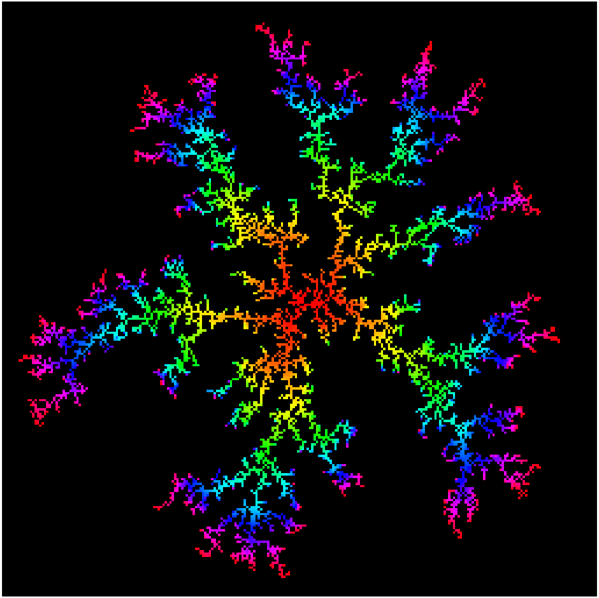
$\mathrm{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\mathrm{QLE}(\gamma^2, \eta)$

- γ is the type of LQG surface on which the process grows
- η determines the manner in which it grows

Let μ_{HARM} (resp. μ_{LEN}) be harmonic (resp. length) measure on a γ-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

$$\left(\frac{d\mu_{\mathrm{HARM}}}{d\mu_{\mathrm{LEN}}}\right)^{\eta} d\mu_{\mathrm{LEN}}.$$

- **Eden model:** $\eta = 0$
- **Diffusion limited aggregation:** $\eta = 1$
- **η-dieletric breakdown model:** general values of η



Simulation of Euclidean DLA

DLA in math?

DLA in math?

Not a lot of progress.

DLA in math?

Not a lot of progress.

Open questions

DLA in math?

Not a lot of progress.

Open questions

- Does DLA have a “scaling limit”?

DLA in math?

Not a lot of progress.

Open questions

- Does DLA have a "scaling limit"?
- Is the shape random at large scales?

DLA in math?

Not a lot of progress.

Open questions

- Does DLA have a "scaling limit"?
- Is the shape random at large scales?
- Does the macroscopic shape look like a tree (i.e., does it make macroscopic loops)?

DLA in math?

Not a lot of progress.

Open questions

- Does DLA have a "scaling limit"?
- Is the shape random at large scales?
- Does the macroscopic shape look like a tree (i.e., does it make macroscopic loops)?
- What is its asymptotic dimension? Simulation prediction: ≈ 1.71 on $\mathbf{Z}^2$

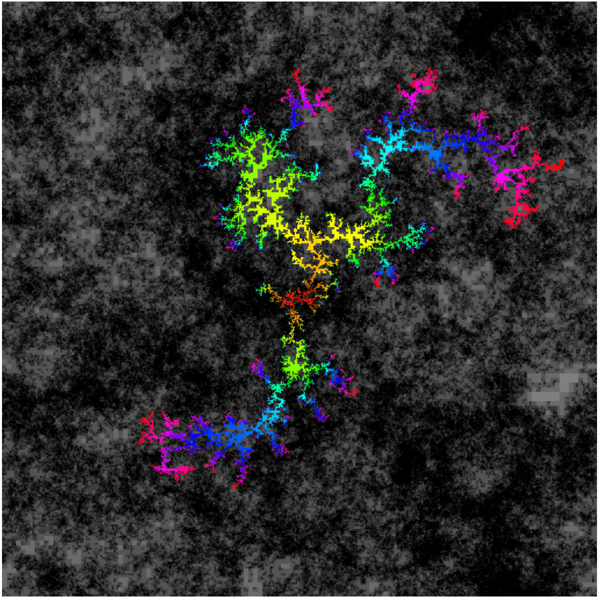
DLA in math?

Not a lot of progress.

Open questions

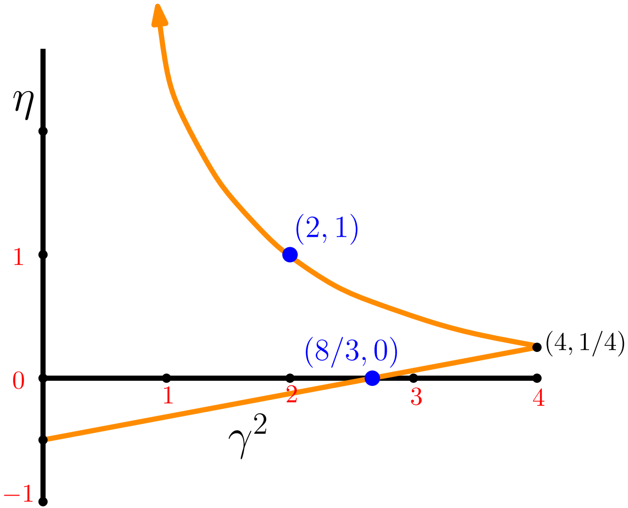
- Does DLA have a "scaling limit"?
- Is the shape random at large scales?
- Does the macroscopic shape look like a tree (i.e., does it make macroscopic loops)?
- What is its asymptotic dimension? Simulation prediction: ≈ 1.71 on $\mathbf{Z}^2$

What about DLA on random planar maps and Liouville quantum gravity surfaces?



Discrete approximation of $\mathrm{QLE}(2,1)$. DLA on a $\sqrt{2}$-LQG

$\mathrm{QLE}(\gamma^2, \eta)$ processes we can construct



Each of the $\mathrm{QLE}(\gamma^2, \eta)$ processes with (γ^2, η) on the orange curves is built from an SLE_κ process using tip re-randomization.

Other γ?

- Other γ values correspond to random planar maps which are decorated by a statistical physics model (e.g., the Ising model).

Other γ?

- Other γ values correspond to random planar maps which are decorated by a statistical physics model (e.g., the Ising model).
- Very little is understood about how the metric should behave or how to construct it for $\gamma \neq \sqrt{8/3}$.

Other γ?

- Other γ values correspond to random planar maps which are decorated by a statistical physics model (e.g., the Ising model).
- Very little is understood about how the metric should behave or how to construct it for $\gamma \neq \sqrt{8/3}$.
- For example, the Hausdorff dimension of γ-LQG for $\gamma \neq \sqrt{8/3}$ is not known.