Open-closed (little) string duality

and

Chern-Simons-Bethe/gauge correspondence

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2016-
and the project

BPS/CFT correspondence and
non-perturbative Dyson-Schwinger equations

NN, 2004-
There are two ways to realize a symmetry in quantum system.
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Start with a classical system with symmetry and quantize
Start with a classical system with symmetry and quantize

Example: geometric quantization

\[
\int_{(p,q) \in \text{coadjoint orbit}} DpDq \exp \left( i \int pdq - \int \text{Tr} A \cdot \mu(p, q) \right)
\]

\[\sim \langle v_1 | T_{\mathcal{H}} \left( P \exp \int A \right) | v_2 \rangle\]

inspiration Borel - Weil - Bott theorem, 1957

Kirillov 1961; path integral suggested in 1961 by Faddeev

Alekseev, Faddeev, Shatashvili 1988
Emergent symmetry in quantum system
Preparations:

\[ \Gamma \subset SU(2) \quad \text{finite subgroup} \]
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\[ \Gamma \subset SU(2) \quad \text{finite subgroup} \]

Irreps \( R_i \), \quad i = 0, \ldots, r
Preparations: quivers from $\Gamma$

$\Gamma \subset SU(2)$ finite subgroup

Irreps $\mathcal{R}_i \mapsto$ vertices $i = 0, \ldots, r$ of a quiver $\Gamma$

edges: $\mathcal{R}_i \otimes \mathbb{C}^2 = \bigoplus_{e \in s^{-1}(i)} \mathcal{R}_{t(e)} \bigoplus_{e \in t^{-1}(i)} \mathcal{R}_{s(e)}$
Preparations: quivers from \( \Gamma \)

\[ \Gamma \subset SU(2) \] finite subgroup

Irreps \( \mathcal{R}_i \) \( \implies \) vertices \( i = 0, \ldots, r \) of a quiver \( \Gamma \)

edges:
\[
\mathcal{R}_i \otimes \mathbb{C}^2 = \bigoplus_{e \in s^{-1}(i)} \mathcal{R}_{t(e)} \bigoplus_{e \in t^{-1}(i)} \mathcal{R}_{s(e)}
\]
Symmetry hints: McKay duality

Irreps \( \mathcal{R}_i \) \( \mapsto \) vertices \( i = 0, \ldots, r \) of a quiver \( \Gamma \)

edges: \[
\mathcal{R}_i \otimes \mathbb{C}^2 = \bigoplus_{e \in s^{-1}(i)} \mathcal{R}_{t(e)} \bigoplus_{e \in t^{-1}(i)} \mathcal{R}_{s(e)}
\]

Dynkin labels: \[
a_i = \mathcal{R}_i, \quad 2a_i = \sum_{e \in s^{-1}(i)} a_{t(e)} + \sum_{e \in t^{-1}(i)} a_{s(e)}
\]
Symmetry hints: McKay duality

Quiver $\Gamma = \text{affine Dynkin diagram of } G_\Gamma$

McKay dual simple Lie group (ADE)
Symmetry hints: Weyl group $\mathcal{W}_\Gamma$

$\text{ALE spaces} = \mathcal{C}^2/\Gamma$

Four dimensional hyperkähler manifolds, with moduli $(\mathbb{R}^r \otimes \mathbb{R}^3)/\mathcal{W}_\Gamma$

$H^2(\text{ALE}, \mathbb{Z})$ form the $\mathcal{W}_\Gamma$ - local system over the moduli space
Symmetry hints: Weyl group $\mathcal{W}_\Gamma$

ALE spaces = $\mathbb{C}^2/\Gamma$

Four dimensional hyperkähler manifolds, with moduli

$$\mathcal{M}_\Gamma = \{ \left( \zeta^\mathbb{R}_i, \zeta^\mathbb{C}_i \right) \} \in (\mathfrak{h}(G_\Gamma) \otimes \mathbb{R} \oplus \mathbb{C})/\mathcal{W}_\Gamma$$

$H^2(\text{ALE}, \mathbb{Z})$ form the $\mathcal{W}_\Gamma$ - local system over the moduli space
Emergent symmetry in quantum system

Example: Nakajima algebras

Start with the $4 + 1$ dimensional

Supersymmetric $U(w)$ gauge theory on

\[
\left( \text{ALE} = \mathbb{R}^4/\Gamma \right) \times \mathbb{R}^1
\]
In a low-energy weak-coupling adiabatic approximation \[ \Rightarrow \]

Vafa, Witten, 1994

Supersymmetric quantum mechanics on \( \mathcal{M}_{v,w}(\mathbb{R}^4/\Gamma) \)

\( U(w) \) instantons on ALE space \( \mathbb{R}^4/\Gamma \),

with topological charges \( v = (v_0, v_1, \ldots, v_r) \)

and boundary conditions at infinity

\[ U(w) \longrightarrow H_w = U(w_0) \times U(w_1) \times \ldots \times U(w_r) \]
Supersymmetric quantum mechanics on $\mathcal{M}_{v,w}(\mathbb{R}^4/\Gamma)$

$A \to$ flat connection at infinity

\[
\pi_1(S^3/\Gamma) = \Gamma \to U(w)
\]

\[
U(w) \longrightarrow H_w = U(w_0) \times U(w_1) \times \ldots \times U(w_r)
\]

\[
- \frac{1}{8\pi^2} \int_{\text{ALE}} \text{Tr} F \wedge F \sim v_0 \quad \frac{1}{2\pi i} \text{Tr} F \sim v_1[\Sigma_1] + \ldots + v_r[\Sigma_r]
\]
Supersymmetric quantum mechanics on $\mathcal{M}_{v,w}(\text{ALE})$

Ground states: cohomology $H^*(\mathcal{M}_{v,w}(\text{ALE}))$

**Nakajima:** $\mathcal{H}_{w,\Gamma} = \bigoplus_v H^*(\mathcal{M}_{v,w}(\text{ALE}))$ is an irreducible highest weight representation of Kac-Moody algebra $\hat{\mathfrak{g}}_\Gamma$ $G_\Gamma$ - McKay dual Lie group
Supersymmetric quantum mechanics on $\mathcal{M}_{v, w}(\text{ALE})$

Ground states: cohomology $H^* (\mathcal{M}_{v, w}(\text{ALE}))$

Nakajima: work $H_w \times U(1)$-equivariantly

$\mathcal{H}_{w, \Gamma} = \bigoplus_v H^* (\mathcal{M}_{v, w}(\text{ALE}))$

irrep of the Yangian $\mathcal{Y}(\widehat{g}_{\Gamma})$ of $\widehat{g}_{\Gamma}$

Ginzburg, Vasserot (finite A series); Varagnolo, 2000
More generally

$\mathcal{M}_{v,w}(\text{ALE})$ is an example of a quiver variety $\mathcal{M}_\gamma(w, v)$

Supersymmetric quantum mechanics on $\mathcal{M}_\gamma(w, v)$

Ground states: cohomology $H^*(\mathcal{M}_\gamma(w, v))$

Nakajima: work $H_w \times U(1)$-equivariantly

$\mathcal{H}_{w,\Gamma} = \bigoplus_v H^*(\mathcal{M}_\gamma(w, v))$

irrep of the Yangian $Y(g_\gamma)$ of $g_\gamma$

Varagnolo, 2000
More generally

Sigma model $\sim$ supersymmetric quantum mechanics on $L\mathcal{M}_\gamma(w, v)$

Ground states: K-theory $K(\mathcal{M}_\gamma(w, v))$

Nakajima: work $H_w \times U(1)$-equivariantly

$$\mathcal{H}_{w, \Gamma} = \bigoplus_v K(\mathcal{M}_\gamma(w, v))$$

irrep of quantum affine algebra $U_q(g_\gamma)$ of $g_\gamma$

Nakajima, 1999
SURPRISES

Need to sum over $v$:

Full symmetry is realized in a collection of quantum systems
SURPRISES

Need to sum over \( \nu \): collections of quantum systems

Natural in 4 + 1 theory but it is not a quantum field theory

No obvious realization of \( G_1 \) in the classical system
HINTS

String theory realization of the gauge theory makes the summation over $\nu$ natural

In string theory the appearance of $G_\Gamma$ comes naturally
Mental note:

String theory may provide a natural explanation
Natural habitat

for the Yangian algebras?
Natural habitat of the Yangian

Spin chains!
Natural habitat of the Yangian

Spin chains! Start with $Y(sl_2)$ for simplicity

Finite dimensional Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$
\[ H = \bigotimes_{a=1}^{L} \mathbb{C}^2 \bigotimes \mathbb{C}^2 \bigotimes \ldots \bigotimes \mathbb{C}^2 \]

Hamiltonian

\[ \hat{H} = \sum_{a=1}^{L} \sigma_a^x \otimes \sigma_{a+1}^x + \sigma_a^y \otimes \sigma_{a+1}^y + \sigma_a^z \otimes \sigma_{a+1}^z \]
Hamiltonian

\[ \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \]

\[ \hat{H} = \sum_{a=1}^{L} \sigma_a^x \otimes \sigma_{a+1}^x + \sigma_a^y \otimes \sigma_{a+1}^y + \sigma_a^z \otimes \sigma_{a+1}^z \]

\[ \vec{\sigma}_{a+L} = \vec{\sigma}_a \]

Heisenberg magnet: periodic isotropic homogeneous spin chain
Hamiltonians!

\[ \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \]

\[ \hat{H}_1 = \sum_{a=1}^{L} \sigma^x_a \otimes \sigma^x_{a+1} + \sigma^y_a \otimes \sigma^y_{a+1} + \sigma^z_a \otimes \sigma^z_{a+1} \]

\[ \hat{H}_2, \hat{H}_3, \ldots, \hat{H}_L, \ldots \]

\[ [\hat{H}_i, \hat{H}_j] = 0 \]

Quantum integrability!
Commuting Hamiltonians from Transfer Matrix

\[ \hat{T}(x) = x^L \exp \sum_{n=1}^{\infty} \frac{1}{n} x^{-n} \hat{H}_n \]

Quantum integrability \( \iff \) \([\hat{T}(x'), \hat{T}(x'')] = 0\)
Transfer matrices from the $R$-matrix

$$\hat{T}(x) = \text{Tr}_{V_{\text{aux}}} (R(x, \mu_1) R(x, \mu_2) \ldots R(x, \mu_L)) : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\mathcal{H} = V_1(\mu_1) \otimes V_2(\mu_2) \otimes \ldots \otimes V_L(\mu_L)$$
Yang-Baxter equation for the $R$-matrix

$z_0 z_2 z_1 z_3 z_6 z_5 = \bar{z}_0 \bar{z}_2 \bar{z}_1 \bar{z}_3 \bar{z}_6 \bar{z}_5$

Implies $[\hat{T}(x'), \hat{T}(x'')] = 0$ by the train argument
Yang-Baxter equation for the $R$-matrix

\[ \Rightarrow [\hat{T}(x'), \hat{T}(x'')] = 0 \text{ by the cyclicity of } \text{Tr}_{V_{aux}} \]
Transfer matrices from the $R$-matrix

$$\hat{T}(x) = \text{Tr}_{V_{\text{aux}}} \left( R(x, \mu_1) R(x, \mu_2) \ldots R(x, \mu_L) \right) : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\mathcal{H} = V_1(\mu_1) \otimes V_2(\mu_2) \otimes \ldots \otimes V_L(\mu_L)$$

$$\mu_1, \ldots, \mu_L \in \mathbb{C} \quad \text{inhomogeneities}$$

Heisenberg spin chain was homogeneous, i.e. $\mu_a = 0$
Twisted transfer matrices from the $R$-matrix

\[ \hat{T}(x; q) = \text{Tr}_{V_{\text{aux}}} g_q \left( R(x, \mu_1) R(x, \mu_2) \ldots R(x, \mu_L) \right) : \mathcal{H} \rightarrow \mathcal{H} \]

Twisted spin chain, $\vec{\sigma}_{a+L} = \text{Ad}(g_q) \vec{\sigma}_a$

For $SU(2)$: $g_q = q^{1/2} \sigma^z$
Anisotropic models from the trigonometric and elliptic $R$-matrices

Baxter, Drinfeld, Belavin, Jimbo

\[ \hat{T}(x; q) = \text{Tr}_{V_{\text{aux}}} g_q (R(x, \mu_1) R(x, \mu_2) \ldots R(x, \mu_L)) : \mathcal{H} \rightarrow \mathcal{H} \]

\[ \hat{H}_1 \rightarrow \sum_{a=1}^{L} \alpha \sigma^x_a \otimes \sigma^x_{a+1} + \beta \sigma^y_a \otimes \sigma^y_{a+1} + \gamma \sigma^z_a \otimes \sigma^z_{a+1} \]

\[
(\alpha : \beta : \gamma) = \begin{cases} 
(1 : 1 : 1) & \text{rational} & XXX \\
(1 : 1 : \Delta) & \text{trigonometric} & XXZ \\
(1 : \Delta' : \Delta'') & \text{elliptic} & XYZ 
\end{cases}
\]
Lattice model
Lattice model

Partition function via transfer matrix formalism

L. Onsager solution of the Ising model

\[ Z_{L,\tilde{L}} = \text{Tr}_{\mathcal{H}_L} \left( \hat{T}(x_1; q) \hat{T}(x_2; q) \ldots \hat{T}(x_{\tilde{L}}; q) \cdot g_\tilde{q} \right) \]
Lattice model on a torus: double trace

not in the sense of gauge theory

\[ Z_{L, L} = \text{Tr}_{\mathcal{H}_L} \left( \hat{T}(x_1; q) \hat{T}(x_2; q) \ldots \hat{T}(x_{L}; q) \cdot g_{\bar{q}} \right) \]
Lattice model on the torus

$$z_{L,\tilde{L}} = \text{Tr}_{\mathcal{H}_L} \left( \hat{T}(x_1; q) \hat{T}(x_2; q) \ldots \hat{T}(x_{\tilde{L}}; q) \cdot g_{\tilde{q}} \right)$$
Lattice model: double trace

\[ Z_{L,\tilde{L}}(q, \tilde{q}) = \sum \text{ over states on the edges of the lattice} \]

Boltzmann weights = products of $R$-matrix elements
Lattice model: modularity

Exchange $A$ and $B$ cycles

$L$ vs $\tilde{L}$

$q$ vs $\tilde{q}$
Lattice model: Hamiltonian viewpoint

\[ Z_{L,\tilde{L}} = \text{Tr}_{\mathcal{H}_L} \left( \hat{T}(x_1; q)\hat{T}(x_2; q) \ldots \hat{T}(x_{\tilde{L}}; q) \cdot g_{\tilde{q}} \right) \]

Bethe states: \( \psi_{\sigma} \in \mathcal{H} \)

\[ \hat{T}(x, q) \psi_{\sigma} = T_{\sigma}(x, q) \psi_{\sigma} \]

\[ Z_{L,\tilde{L}} = \sum \text{ over the eigenvalues of the transfer matrix} \]

\[ Z_{L,\tilde{L}}(q, \tilde{q}) = \sum_N \tilde{q}^N \sum_{\sigma_1, \ldots, \sigma_N} T_{\sigma}(x_1; q) \ldots T_{\sigma}(x_{\tilde{L}}; q) \]

Sum over the number of Bethe roots = "magnons"
Hamiltonian viewpoint: Bethe ansatz

Bethe states: $\psi_\sigma \in \mathcal{H}$

$$\hat{T}(x, q) \psi_\sigma = T_\sigma(x, q) \psi_\sigma$$

for all $x$
Lightning review of Bethe ansatz

Faddeev, Sklyanin, Takhtajan
Kulish, Reshetikhin
Isergin, Korepin
Drinfeld, Jimbo, Miwa

Monodromy matrix

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = R(x, \mu_1) \ldots R(x, \mu_L) : V_{aux} \otimes \mathcal{H} \to V_{aux} \otimes \mathcal{H}$$
Lightning review of Bethe ansatz

**Monodromy matrix**

\[
\begin{pmatrix}
A(x) & B(x) \\
C(x) & D(x)
\end{pmatrix} : V_{aux} \otimes \mathcal{H} \rightarrow V_{aux} \otimes \mathcal{H}
\]

**Yangian \( Y(sl_2) \) generators**

\[A(x), B(x), C(x), D(x) : \mathcal{H} \rightarrow \mathcal{H}\]
Lightning review of Bethe ansatz

Monodromy matrix

\[
\begin{pmatrix}
A(x) & B(x) \\
C(x) & D(x)
\end{pmatrix} = R(x, \mu_1) \ldots R(x, \mu_L) : V_{\text{aux}} \otimes \mathcal{H} \rightarrow V_{\text{aux}} \otimes \mathcal{H}
\]

Bethe state

\[
\psi_\sigma = B(\sigma_1)B(\sigma_2) \ldots B(\sigma_N)| \downarrow \ldots \downarrow \rangle
\]
Lightning review of Bethe ansatz

Bethe state (algebraic Bethe ansatz)

$$\psi_\sigma = B(\sigma_1)B(\sigma_2) \ldots B(\sigma_N) | \downarrow \downarrow \ldots \downarrow$$

Bethe roots $\sigma_1, \ldots, \sigma_N$
Lightning review of Bethe ansatz

Bethe equations

\[
q \prod_{a=1}^{L} \frac{\sigma_i - \mu_a + u}{\sigma_i - \mu_a - u} = \prod_{j \neq i} \frac{\sigma_i - \sigma_j + 2u}{\sigma_i - \sigma_j - 2u}
\]

Solutions = Bethe roots \(\sigma_1, \ldots, \sigma_N\)

Planck constant \(\approx u\)
Lightning review of Bethe ansatz

Functional Bethe Ansatz: $T - Q$ relation

Baxter, Sklyanin

\[ P(x - u)Q_\sigma(x + 2u) + q P(x + u)Q_\sigma(x - 2u) = T_\sigma(x; q)Q_\sigma(x) \]

\[ Q_\sigma(x) = \prod_{i=1}^{N} (x - \sigma_i), \quad P(x) = \prod_{a=1}^{L} (x - \mu_a) \]

The content of this equation: $T_\sigma(x; q)$ has no singularities in $x$
Lightning review of Bethe ansatz

\[ Q_\sigma(x) = \prod_{i=1}^{N} (x - \sigma_i) = \text{eigenvalue of Baxter operator } \hat{Q}(x) \]

\[ P(x) = \prod_{a=1}^{L} (x - \mu_a) = \text{Drinfeld polynomial} \]
Lightning review of Bethe ansatz

$q$-character form of Bethe equations

E. Frenkel, Reshetikhin

\[ Y_\sigma(x + 2u) + q \ell(x) Y_\sigma(x)^{-1} = \frac{T_\sigma(x; q)}{P(x - u)} \]

\[ T_\sigma(x; q) \] is a polynomial in \( x \)

\[ Y_\sigma(x) = \frac{Q_\sigma(x)}{Q_\sigma(x - 2u)} \]

\[ \ell(x) = \frac{P(x + u)}{P(x - u)} \]
Lightning review of Bethe ansatz

$q$-character form of Bethe equations

\[ Y_\sigma(x + 2u) + q\ell(x)Y_\sigma(x)^{-1} = \frac{T_\sigma(x; q)}{P(x - u)} \]

\[ Y_\sigma(x) = \frac{Q_\sigma(x)}{Q_\sigma(x - 2u)} = \]

eigenvalue of the operator $\hat{Y}(x)$
\[ \widehat{Y}(x + 2u) + q \ell(x) \widehat{Y}(x)^{-1} = \]

the fundamental $q$-character of $Y(sl_2)$
$q$-characters for general quivers

\[ \hat{Y}_i(x+2u) + \]

\[ + q_i \ell_i(x) \hat{Y}_i(x)^{-1} \prod_{e \in s^{-1}(i)} \hat{Y}_{t(e)}(x+\mu_e+u) \prod_{e \in t^{-1}(i)} \hat{Y}_{s(e)}(x-\mu_e+u) + \ldots \]

\[ = \text{the fundamental } q\text{-character of } Y(g_\Gamma) \]

\[ \ell_i(x) = \frac{P_i(x + u)}{P_i(x - u)} \]

the $\ell$-weight
**q-character for** $\widehat{A}_0$

NN, Pestun, Shatashvili, 2013


**Additional parameter** $\varepsilon = \mu_e$

\[
\sum_{\lambda} q^{\lfloor \lambda \rfloor} \prod_{\square \in \lambda} \ell(x + c_{\square}) \frac{\prod_{\square \in \partial_+ \lambda} \hat{Y}(x + 2u + c_{\square})}{\prod_{\square \in \partial_- \lambda} \hat{Y}(x + c_{\square})} = \text{the fundamental } q\text{-character of } Y(\hat{u}(1))
\]

$c_{\square} = \varepsilon(i - j) - u(i + j - 2), \quad \square = (i, j)$

\[
\ell(x) = \frac{P(x+u)}{P(x-u)}
\]
Bethe/gauge correspondence
Bethe/gauge correspondence

NN, Shatashvili 2007
Bethe/gauge correspondence

Prior work: Moore, NN, Shatashvili, 1997

Givental, 1993

Gorsky, NN, 1992-1994

Gerasimov, Shatashvili, 2006
$\mathcal{N} = (2, 2), d = 2$ super-Poincare invariant gauge theory

Bethe/gauge correspondence

Quantum integrable system
Supersymmetric vacua (in finite volume)

Bethe/gauge correspondence

Stationary states = joint eigenvectors of quantum integrals of motion
Twisted chiral ring, e.g. \( \mathcal{O}_n = \frac{1}{(2\pi i)^n n!} \text{Tr} \sigma^n \)

\[ \uparrow \]

Bethe/gauge correspondence

\[ \downarrow \]

Quantum integrals of motion \( \hat{H}_n \), e.g. \( \hat{\text{Tr}} L^n \) for Lax operator \( L \)
Effective twisted superpotential $\mathcal{W}(\sigma_1, \ldots, \sigma_N)$

\[\uparrow\]

Bethe/gauge correspondence

\[\downarrow\]

The Yang-Yang functional $\mathcal{Y}(\sigma_1, \ldots, \sigma_N)$
$\mathcal{N} = (2, 2), d = 2$ super-Poincare invariant gauge theory

$\iff$ ? $\implies$

Quantum integrable system
$\mathcal{N} = 4$, $d = 2$ $U(N)$ gauge theory with $L$ hypermultiplets in the fundamental representation

Example of Bethe/gauge correspondence

Inhomogeneous twisted length $L \, SU(2)$ spin $\frac{1}{2}$ chain in the sector with $N$ spins up
Softly broken $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$, $d = 2$ $U(N)$ gauge theory by the twisted mass $u$, corresponding to the $U(1)$ symmetry

$$Q, \tilde{Q} \mapsto e^{iu} Q, e^{iu} \tilde{Q}$$

$$\Phi \mapsto e^{-2iu} \Phi$$

Inhomogeneities $\mu_a = \text{twisted masses}$

$\leftrightarrow U(L)$ flavor symmetry of $\mathcal{N} = 4$ theory

the twist parameter $q = \text{Kähler modulus}$

$$q = e^{2\pi i t} = e^{i\vartheta - 2\pi r}$$
Bethe equations

= quantum cohomology (twisted chiral ring) relations

\[ q \prod_{a=1}^{L} \frac{\sigma_i - \mu_a + u}{\sigma_i - \mu_a - u} = \prod_{j \neq i} \frac{\sigma_i - \sigma_j + 2u}{\sigma_i - \sigma_j - 2u} \]

Solutions =

eigenvalues of the complex scalar in the \( U(N) \) vector multiplet:

\[ \sigma \sim \text{diag}(\sigma_1, \ldots, \sigma_N) \]

up to permutations of \( \sigma_i \)'s –

the remainder of the \( U(N) \) gauge symmetry
Bethe equations

= quantum cohomology (twisted chiral ring) relations

\[ 1 = q \prod_{a=1}^{L} \frac{\sigma_i - \mu_a + u}{\sigma_i - \mu_a - u} \prod_{j \neq i} \frac{\sigma_i - \sigma_j - 2u}{\sigma_i - \sigma_j + 2u} = \exp \left( \frac{\partial \tilde{W}}{\partial \sigma_i} \right) \]

\[ \tilde{W}(\sigma_1, \ldots, \sigma_N) = \text{effective twisted superpotential} \]

one-loop exact computation!
Baxter $\mathcal{Q}$-operator

$= \text{characteristic polynomial of the adjoint Higgs}$

$Q(x) = \text{Det}(x - \sigma)$
Gauged linear sigma model on $T^*\text{Gr}(N, L)$

low energy description of our gauge theory for $r \gg 0$

\[ Q(x)[p] = c_x(\mathcal{E}_p) = \text{Chern polynomial of the tautological bundle} \]

\[ Q(x)[p] = x^N - c_1(\mathcal{E}_p)x^{N-1} + c_2(\mathcal{E}_p)x^{N-2} - \ldots \]

local operator $Q(x)[p], p \in \Sigma$

in the sigma model with worldsheet $\Sigma$, roughly:

\[ \mathcal{E}_p \to M, \quad \mathcal{E}_p = ev_p^*E \]

$E = \text{rk } \mathbb{N} \text{ tautological bundle over } T^*\text{Gr}(N, L)$

$ev : \Sigma \times M \longrightarrow T^*\text{Gr}(N, L)$ evaluation map
Lift to three dimensions

\[ \Sigma \longrightarrow S^1 \times \Sigma \]

Twisted masses \(\rightarrow\) Wilson loops + real masses

\[ \text{XXX} \rightarrow \text{XXZ} = \text{trigonometric case} \]
Lift to four dimensions

\[ \Sigma \longrightarrow E \times \Sigma \]

Elliptic curve \( E \)

Twisted masses \( \rightarrow \) Holomorphic \( GL(L) \times \mathbb{C}^\times \) bundle on \( E \)

\[ XXX \rightarrow XYZ = \text{elliptic case} \]
Lift to four dimensions

\[ \Sigma \longrightarrow E \times \Sigma \]

Elliptic curve \( E \)

Twisted masses \( \rightarrow \) Holomorphic \( GL(L) \times \mathbb{C}^\times \) bundle on \( E \)

\( XYZ = \text{elliptic case} \) — anomalous when \( L \neq 2N \)
What is the meaning of $T_\sigma(x)$?

What is the meaning of $T - Q$ relations?
Quiver gauge theory
$\mathcal{N} = (4, 4)$ quiver gauge theory

$\mathcal{N} = 4$ softly broken down to $\mathcal{N} = 2$

Quiver $\gamma$ with the set $\text{Vert}_{\gamma}$ of vertices and the set $\text{Edge}_{\gamma}$ of edges
$\mathcal{N} = (4, 4)$ quiver gauge theory

$\mathcal{N} = 4$ softly broken down to $\mathcal{N} = 2$

Quiver $\gamma$ with the set $\text{Vert}_\gamma$ of vertices and the set $\text{Edge}_\gamma$ of edges

$e \in \text{Edge}_\gamma, \quad s(e), t(e) \in \text{Vert}_\gamma$

source and target
Examples of quivers

$A_3$

$\hat{A}_0$

$E_6$
Apologies for notations

\[ N_i, L_i \]

Stand both for vector spaces (colors \( \mathbb{C}^{N_i} \) and flavors \( \mathbb{C}^{L_i} \)), their dimensions, sometimes characters

\[ N_i \sim \sum_{\alpha \in [N_i]} e^{\sigma_i,\alpha} \]

\[ L_i \sim \sum_{f \in [L_i]} e^{\mu_i,f} \]

\[ [p] := \{1, 2, \ldots, p\} \]
Gauge group

\[ G = \times_{i \in \text{Vert}_\gamma} U(N_i) \]
Vector multiplet scalars

\[ \Phi_i, \sigma_i \in \text{LieGL}(N_i) \]
Matter hypermultiplets

Fundamentals \( Q_i \in \text{Hom}(L_i, N_i), \tilde{Q}_i \in \text{Hom}(N_i, L_i) \)
Matter hypermultiplets

Fundamentals $Q_i \in \text{Hom}(L_i, N_i), \tilde{Q}_i \in \text{Hom}(N_i, L_i)$

Bi-fundamentals $Q_{e} \in \text{Hom}(N_{s(e)}, N_{t(e)}), \tilde{Q}_{e} \in \text{Hom}(N_{t(e)}, N_{s(e)})$
Matter superpotential

\[ W = \sum_{i \in \text{Vert}_\gamma} \text{Tr}_{L_i} \left( \tilde{Q}_i \Phi_i Q_i \right) + \]
\[ + \sum_{e \in \text{Edge}_\gamma} \text{Tr}_{N_{s(e)}} \left( \tilde{Q}_e \Phi_{t(e)} Q_e \right) - \text{Tr}_{N_{t(e)}} \left( Q_e \Phi_{s(e)} \tilde{Q}_e \right) \]
Matter masses, compatible with $\mathcal{N} = 4$

$$\mathcal{M}_i \in \text{End}(L_i), \quad \mu_e \in \mathbb{C}$$

Twisted masses of the fundamental and the bi-fundamental hypermultiplets, respectively

$$\left(Q_i, \tilde{Q}_i, Q_e, \tilde{Q}_e\right) \rightarrow \left(Q_i e^{-i\mathcal{M}_i}, e^{i\mathcal{M}_i} \tilde{Q}_i, e^{i\mu_e} Q_e, e^{-i\mu_e} \tilde{Q}_e\right)$$
Susy breaking by the twisted mass $u$

$$W = \sum_{i \in \text{Vert}_\gamma} \text{Tr}_{M_i} \left( \tilde{Q}_i \Phi_i Q_i \right) +$$

$$+ \sum_{e \in \text{Edge}_\gamma} \text{Tr}_{N_{s(e)}} \left( \tilde{Q}_e \Phi_{t(e)} Q_e \right) - \text{Tr}_{N_{t(e)}} \left( Q_e \Phi_{s(e)} \tilde{Q}_e \right)$$

The most important $U(1)$ symmetry

$$\left( Q_i, \tilde{Q}_i, Q_e, \tilde{Q}_e, \Phi_i \right) \rightarrow \left( e^{iu} Q_i, e^{iu} \tilde{Q}_i, e^{iu} Q_e, e^{iu} \tilde{Q}_e, e^{-2iu} \Phi_i \right)$$
Integrate out massive matter

\[ \widetilde{W}(\sigma_{i,\alpha}) = \]

\[ \sum_{i \in \text{Vert}_\gamma} \sum_{\alpha \in [N_i]} \left( \log(q_i) \sigma_{i,\alpha} + \sum_{\beta \in [N_i]} \varpi (-2u + \sigma_{i,\alpha} - \sigma_{i,\beta}) + \right. \]

\[ + \sum_{f \in [L_i]} \left( \varpi (u + \sigma_{i,\alpha} - \mu_{i,f}) + \varpi (u - \sigma_{i,\alpha} + \mu_{i,f}) \right) \]

\[ + \sum_{e \in \text{Edge}_\gamma} \sum_{\alpha \in [N_{t(e)}]} \sum_{\beta \in [N_{s(e)}]} \left( \varpi (u + \mu_e + \sigma_{t(e),\alpha} - \sigma_{s(e),\beta}) + \right. \]

\[ \left. \varpi (u - \mu_e + \sigma_{s(e),\beta} - \sigma_{t(e),\alpha}) \right) \]
Rational case

$$\varpi(z) = z \left( \log(z) - 1 \right), \quad \exp \varpi'(z) = z$$
Trigonometric case

$$\varpi_R(z) =$$

$$= R \frac{z^2}{2} - \log(2R)z - \frac{1}{2R} \text{Li}_2\left( e^{-2Rz} \right) - \frac{\pi^2}{12R},$$

$$\exp \varpi_R'(z) = \frac{\sinh(Rz)}{R}$$
Elliptic case

\[ \varpi_{R, \rho}(z) = R \frac{z^2}{2} - \log(2R)z - \frac{\pi^2}{12R} + \]

\[ + \sum_{n=0}^{\infty} \frac{1}{2R} \left( \text{Li}_2 \left( e^{2\pi in \rho} e^{-2Rz} \right) - \text{Li}_2 \left( e^{2\pi i(n+1) \rho} e^{2Rz} \right) \right), \]

\[ \exp \varpi'_{R, \rho}(z) = \frac{1}{2iR} \frac{\theta_{11}(2iRz; \rho)}{\theta'_{11}(0; \rho)}. \]
Supersymmetric vacua of the quiver gauge theory

\[
\exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1, \quad i \in \text{Vert}_\gamma, \alpha \in [N_i]
\]
Supersymmetric vacua of the quiver gauge theory

\[ \exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1, \quad i \in \text{Vert}_\gamma, \alpha \in [N_i] \]

Correspond to Bethe equations of a spin chain with \( Y(g_\gamma) \) symmetry
Supersymmetric vacua of the quiver gauge theory

$$\exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1, \quad i \in \text{Vert}_\gamma, \alpha \in [N_i]$$
q-character formulation

\[ \exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1, \quad i \in \text{Vert}_\gamma, \alpha \in [N_i] \]

Can be reformulated as the system of conditions for the \( q \)-characters

\[ T_i(x) := y_i(x + 2u) + \]

\[ + q_i \ell_i(x) \frac{\prod_{e \in s^{-1}(i)} y_{t(e)}(x + \mu_e + u) \prod_{e \in t^{-1}(i)} y_{s(e)}(x - \mu_e + u)}{y_i(x)} + \ldots \]

to have no singularities in \( x \)

except for the poles coming from \( \ell_i(x) \)'s
Partition function on $T^2$

$$\text{Tr}_{\mathcal{H}_{\text{susy}}[(N_i)]}(-1)^F \exp\left(-\sum_k t_k \Omega_k^{(0)}\right)$$

↑

Bethe/gauge correspondence

↓

Gibbs ensemble partition function in the weight $\vec{N}$ subspace

$$\text{Tr}_{\mathcal{H}_{\text{QIS}}[(N_i)]} \exp\left(-\sum_k t_k \hat{H}_k\right)$$
Partition function on $T^2$

$$\text{Tr}_{\mathcal{H}_{\text{susy}}[(N_i)]}(-1)^F \exp - \sum_k t_k \theta_k^{(0)} \sim \sum_{\text{vac}} e^{-\sum_k t_k \langle \theta_k \rangle_{\text{vac}}}$$

assuming all vacua are bosonic

$\uparrow$

Bethe/gauge correspondence

$\downarrow$

Gibbs ensemble partition function in the weight $\vec{N}$ subspace

$$\text{Tr}_{\mathcal{H}_{\text{QIS}}[(N_i)]} \exp - \sum_k t_k \hat{H}_k$$
Partition function on $T^2$ of the ensemble of gauge theories

\[ Z = \sum_{(N_i)} \prod_i \tilde{q}_i^{N_i} \text{Tr}_{\mathcal{H}_{\text{susy}}[(N_i)]}(-1)^F \exp - \sum_k t_k \mathcal{O}_k^{(0)} \]

\[ \uparrow \quad \text{Bethe/gauge correspondence} \quad \downarrow \]

Toroidal Lattice model Partition function

\[ Z = \text{Tr}_{\mathcal{H}_{\text{QIS}}} \prod_i \tilde{q}_i^{\tilde{N}_i} \exp - \sum_k t_k \hat{H}_k \]
Questions

- Why sum over $\vec{N}$?

- Why choose $t_k \mathcal{O}_k$ in such a way, that

$$\exp - \sum_k t_k \hat{H}_k = \hat{T}(x_1; q) \ldots \hat{T}(x_L; q)?$$
Questions

• Why choose $t_k O_k$ in such a way, that

$$\exp - \sum_k t_k \hat{H}_k = \hat{T}(x_1; q) \ldots \hat{T}(x_{\tilde{L}}; q)$$

\(\hat{T}(x; q)\) turns out to be a natural observable within the twisted chiral ring
$q$-characters

$\hat{T}(x; q)$ turns out to be a natural observable within the twisted chiral ring

Well-behaved with respect to the non-perturbative Dyson-Schwinger relations
Non-perturbative Dyson-Schwinger relations
Non-perturbative Dyson-Schwinger relations

Contributions of topologically distinct sectors to the path integral are related to each other

⇔

Analytic properties of \( \langle \hat{T}(x; q) \rangle \), e.g. no poles in \( x \)
Remarks

• When the quiver $\gamma$ is one of the affine Dynkin diagrams

• Bethe equations correspond to the spin chains with Kac-Moody spin groups

• There are gauge theories corresponding to the super-Lie algebras

• For general $\gamma$ — a wild Lie algebra $g_\gamma$
Remarks

What has changed compared to the old results of Nakajima et al.

Unlike simple Lie groups, Yangians, quantum affine algebras, etc. have inequivalent maximal commutative subalgebras.

To see them all, we need

the \( q \)-parameters: Kähler moduli

of the two dimensional theory

not visible at the level of supersymmetric quantum mechanics!
Remarks

Max commutative = Bethe subalgebras

at the level of supersymmetric quantum mechanics, \( q \to 0 \)

become Gelfand-Zetlin subalgebras

Nazarov, 1995
Remarks

The original formulation of Bethe/gauge correspondence

mostly concerned with the commutative (quantum integrals) subalgebra

The non-abelian structure provides rigidity and offers an exciting perspective

on the string landscape of vacua
Remarks

The original formulation of Bethe/gauge correspondence:

The non-abelian structure comes from domain walls

viewed as operators in the spirit of S-branes

Gutperle, Strominger, 2002
Remarks

Recent progress:

The non-abelian structure, i.e. $R$-matrices can be understood mathematically using the stable envelope basis

Maulik, Okounkov 2012; Aganagic, Okounkov 2016
If one replaces the $R$-matrices with spectral parameters by the $R$-matrices without (finite quantum group $U_q(g)$), one can describe the lattice model using Chern-Simons theory with gauge group $G$ in three dimensions.

$$S_{CS} = \frac{k}{4\pi} \int \text{Tr} \left( AdA + \frac{2}{3} A^3 \right)$$

Witten 1989

- How to introduce the spectral parameter into Chern-Simons theory?
Cohomological field theory perspective

Start with CohFT with the moduli space $\mathcal{M}$ of solutions

Fields/Equations/Symmetries paradigm:

$d$-dimensional fields

$$Q^2 = 0$$

Correlations functions: integrals of products of cohomology classes of $\mathcal{M}$

$$\langle Q_1 \ldots Q_p \rangle^d \sim \int_{\mathcal{M}} \Omega_1 \wedge \ldots \wedge \Omega_p$$

$$\{Q, Q_i\} = 0, \quad Q_i \leftrightarrow \Omega_i, \quad d\Omega_i = 0$$
Oxidation of cohomological field theory:
make fields \( t \)-dependent

\( K \)-theory of \( \mathcal{M} \)

Fields/Equations/Symmetries paradigm:
loop space, i.e. \( d + 1 \)-dimensional fields

Correlations functions: pushforwards of \( K \)-theory classes of \( \mathcal{M} \)

\[
Q^2 = \partial_t
\]

\[
\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle^{d+1} \sim \int_{\mathcal{M}} \hat{A}(\mathcal{M}) \wedge \Omega_1 \wedge \ldots \wedge \Omega_p
\]
Double Loop upgrade

Baulieu, NN, Losev, 1997
Costello, 2013

Oxidation of cohomological field theory:
make fields $z, \bar{z}$-dependent

$Ell$-cohomology of $\mathcal{M}$

Fields/Equations/Symmetries paradigm:
double loop space, i.e. $d+2$-dimensional fields

Correlations functions: pushforwards in elliptic cohomology of $\mathcal{M}$

$$Q^2 = \partial \bar{z}$$

$$\langle \theta_1 \ldots \theta_p \rangle^{d+2} \sim \int_{\mathcal{M}} \widehat{Ell}(\mathcal{M}) \wedge \Omega_1 \wedge \ldots \wedge \Omega_p$$
3d CS = loop upgrade of 2d YM

\[ \mathcal{M} = \text{moduli space of } G - \text{ flat connections on } \Sigma \]

\[ \mathcal{N} = 2d = 2 \text{ super-Yang-Mills theory, twisted version} \]

\[ Q A = \Psi , \; Q \Psi = D A \sigma , \; Q \sigma = 0 \]

\[ Q \chi = H , \; Q H = [\sigma , \chi] , \; Q \bar{\sigma} = \eta , \; Q \eta = [\sigma , \bar{\sigma}] \]
Review of the cohomological field theory on

\[ M = \text{moduli space of } G - \text{ flat connections on } \Sigma \]

\[ N = 2 \ d = 2 \text{ super-Yang-Mills theory, twisted version} \]

\[ QA = \psi, \ Q\psi = D_A\sigma, \ Q\sigma = 0 \]

\[ Q\chi = H, \ QH = [\sigma, \chi], \ Q\bar{\sigma} = \eta, \ Q\eta = [\sigma, \bar{\sigma}] \]

\[ S_0 = Q \int_\Sigma \text{Tr} \left( \chi \left( iF_A - g_{\text{YM}}^2 \ast H \right) + \psi \wedge \ast D_A\bar{\sigma} + \eta[\sigma, \bar{\sigma}] \right) \]

\[ \uparrow \]

Bare action
Review of 2d YM as deformation of SYM

\[ \mathcal{M} = \text{moduli space of } G - \text{ flat connections on } \Sigma \]

\[ \mathcal{N} = 2 d = 2 \text{ super-Yang-Mills theory, twisted version} \]

\[ S_0 + i\kappa \int_{\Sigma} \text{Tr} \left( \sigma F_A + \frac{1}{2} \psi \wedge \psi \right) \]

↑

2-observable, viewed as deformation of the action
Twisted $F$-term in the physical theory, \[ \widetilde{W} = \frac{\kappa}{2} \text{Tr}\sigma^2 \]
Review of 2d YM

$N = 2 \, d = 2$ super-Yang-Mills theory, twisted version

\[ S_0 + i\kappa \int_\Sigma \text{Tr} \left( \sigma F_A + \frac{1}{2} \psi \wedge \psi \right) \]

↑

2-observable, viewed as deformation of the action

Twisted $F$-term in the physical theory,

\[ \tilde{W} = \frac{\kappa}{2} \text{Tr} \sigma^2 \]

Twisted $\bar{F}$-term,

\[ \tilde{W}^* = \frac{\bar{\kappa}}{2} \text{Tr} \bar{\sigma}^2 \]

shifts the action by the $Q$-exact term

\[ +i\bar{\kappa} \int_\Sigma \text{Tr} (\bar{\sigma} H + \eta \chi) \]
Review of 2d YM

Take the limit $\bar{\kappa} \to \infty$

$$i\bar{\kappa} \int_\Sigma \text{Tr} (\bar{\sigma} H + \eta \chi)$$

The quartet $\bar{\sigma}, H, \eta, \chi$ decouples: and we are left with $A, \psi, \sigma$

Witten 1992

$$Q A = \psi, Q \psi = D_A \sigma, Q \sigma = 0$$

$$S = i\kappa \int_\Sigma \text{Tr} \left( \sigma F_A + \frac{1}{2} \psi \wedge \psi \right)$$

↑

2-observable, becomes the action

Add 0-observable $t \text{Tr} \sigma^2 \implies 2$d Yang-Mills theory
Loop upgrade

\[ S = i\kappa \int_{\Sigma} \text{Tr} \left( \sigma F_A + \frac{1}{2} \psi \wedge \psi \right) \]

\[ \Downarrow \]

\[ S_{CS} = \frac{k}{4\pi} \int_{\Sigma \times S^1} \text{Tr} \left( \text{Ad}A + \frac{2}{3} A^3 + \psi \psi \right) \]
Double Loop upgrade

NN, PhD. thesis 1996, proposed to explain the representation theory of quantum affine algebras

\[ S = i\kappa \int_{\Sigma} \text{Tr} \left( \sigma F_A + \frac{1}{2} \psi \wedge \psi \right) \]

\[ \downarrow \]

\[ S_{AdCS} = \kappa \int_{\Sigma \times E} dz \wedge \text{Tr} \left( \text{Ad} A + \frac{2}{3} A^3 + \psi \psi \right) \]

where \( dz \) is a holomorphic one-differential on \( E \):

an elliptic curve, a cylinder, or a plane

Recent revival, Costello 2013
Susy of the Double Loop upgrade

\[ S_{4\text{dCS}} = \kappa \int_{\Sigma \times E} dz \wedge \text{Tr} \left( \text{Ad}A + \frac{2}{3} A^3 + \psi \psi \right) \]

is \( \mathcal{Q} \)-invariant, with

\[ \mathcal{Q}A_m = \psi_m, \quad \mathcal{Q}\psi_m = F_{m\bar{z}}, \quad \mathcal{Q}A_{\bar{z}} = 0 \]

\[ \mathcal{Q}A_z = \eta, \quad \mathcal{Q}\eta = F_{z\bar{z}} \]

\[ \mathcal{Q}\chi = H, \quad \mathcal{Q}H = D_{\bar{z}}\chi \]

\[ m = 1, 2 \rightarrow \text{coordinates on } \Sigma \]
Anomaly of the Double Loop upgrade

\[ S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \text{Tr} \left( \text{Ad}A + \frac{2}{3}A^3 + \psi\psi \right) \]

when \( E \) is an elliptic curve, is not gauge invariant

\[ S_{4dCS} \rightarrow S_{4dCS} + \kappa \int_{\Sigma \times E} dz \wedge \text{integral } 3 \text{ - form} \]

under large gauge transformations
Anomaly of the Double Loop upgrade

\[ S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \text{Tr} \left( \text{Ad}A + \frac{2}{3}A^3 + \psi\psi \right) \]

when \( E \) is an elliptic curve, is not gauge invariant

\[ S_{4dCS} \rightarrow S_{4dCS} + \kappa \int_{\Sigma \times E} dz \wedge \text{integral 3-form} \]

under large gauge transformations, incommensurate periods...
Double Loop upgrade of $\mathcal{N} = 4 \ d = 4$ theory

$$S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \text{Tr} \left( \text{Ad}A + \frac{2}{3} A^3 + \psi \psi \right)$$

when $E$ is an elliptic curve, is not gauge invariant

$$S_{4dCS} \rightarrow S_{4dCS} + \kappa \int_{\Sigma \times E} dz \wedge \text{integral } 3 - \text{form}$$

under large gauge transformations: incommensurate periods...
String theory realization
String theory realization

Various puzzles will be resolved

Different approaches will be connected
String theory realization

IIB string on $ALE \times \mathbb{R}^2 \times E \times T^2$
String theory realization

IIB string on \((ALE \times \mathbb{R}^2) \tilde{\times}_u E \times T^2\)
String theory realization

IIB string on \( (\text{ALE} \times \mathbb{R}^2) \sim_u E \times T^2 \)

ALE with \( \zeta_i^C = 0 \implies U(1) \)-isometry

ALE is twisted with a line bundle \( \mathcal{L}_u \) over \( E \)

\( \mathbb{R}^2 \) is twisted with a line bundle \( \mathcal{L}_{u^{-2}} \) over \( E \)
String theory realization of our gauge theory

IIB string on \((ALE \times R^2) \sim_u E \times T^2\) with fractional \(D\)-branes
String theory realization of our gauge theory

IIB string on \((\text{ALE} \times \mathbb{R}^2) \tilde{\times}_u E \times T^2\)

with \textit{LD7}-branes on \text{ALE} \times E \times T^2

with \textit{ND3}-branes on E \times T^2

with \textit{KD1}-branes on E
String theory realization of our gauge theory

IIB string on \((\text{ALE} \times \mathbb{R}^2) \times_u E \times T^2\)
with \(L D_7\)-branes on \(\text{ALE} \times E \times T^2\)
with \(N D_3\)-branes on \(E \times T^2\)
with \(K D_1\)-branes on \(E\)

Fractionalization: \((L, N, K) \longrightarrow (L_i, N_i, K_i)\)
Compact branes to be summed over
String theory realization of the $q$-character

IIB string on $(\text{ALE} \times \mathbb{R}^2) \tilde{\times}_u E \times T^2$

with fractional $D7, D3, D1$-branes and $D5$-branes in addition
String theory realization of the $q$-character

IIB string on $(\text{ALE} \times \mathbb{R}^2) \tilde{\times} \ E \times T^2$
with $LD7$-branes on $\text{ALE} \times E \times T^2$
with $ND3$-branes on $E \times T^2$
with $KD1$-branes on $E$
with $\tilde{LD5}$-branes on $\text{ALE} \times E$
String theory realization of the $q$-character

with $ND3$-branes on $E \times T^2$

with $\tilde{LD}5$-branes on $ALE \times E$

This is similar to the construction of crossed instantons

$qq$-character and $q$-character observables

in 4d and 5d supersymmetric gauge theories

NN, Pestun 2012; NN, Pestun Shatashvili 2013; NN 2015-
String theory realization

Now we shall get a 6d-ish version

of Chern-Simons theory, dual

to the collection of quiver gauge theories
String theory realization

Now we shall get a 6d-ish version of CS theory

using T-dual string background(s)
**$T$-dual description: electric frame**

$T$-duality along $E$ and one of the circles in $T^2$

IIA string on $A\text{LE} \times \mathbb{R}^2 \times \hat{E} \times \hat{S}^1 \times S^1$

with fractional $D$-branes
$T$-dual description: magnetic frame

T-dualize one of the circles in $T^2$

IIA string on $\text{ALE} \times \mathbb{R}^2 \times E \times \hat{S}^1 \times S^1$

with fractional $D$-branes
Six dimensional super-Yang-Mills

Witten; Strominger; Greene, Morrison, Strominger; Bershadsky, Sadov, Vafa, 1995

IIA string on ALE
Six dimensional super-Yang-Mills: electric frame

IIA string on $\text{ALE} \times \mathbb{R}^2 \times \tilde{E} \times \tilde{S}^1 \times S^1$

with fractional $D$-branes $\Longrightarrow$ electric sources

more details below
Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\text{ALE} \times \mathbb{R}^2 \times E \times \tilde{S}^1 \times S^1$

with fractional $D$-branes $\implies$ magnetic sources

Twist by $\mathcal{L}_u \implies 6d \Omega$-deformation of SYM
Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\text{ALE} \times \mathbb{R}^2 \times E \times \tilde{S}^1 \times S^1$

with fractional $D$-branes $\implies$ magnetic sources

Twist by $\mathcal{L}_u \implies 6d \ \Omega$-deformation of SYM
preserving $\mathcal{N} = (2, 2) \ d = 2$ super-Poincare invariance
with 2 out of 4 scalars remaining massless: root of a Higgs branch
Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\text{ALE} \times \mathbb{R}^2 \times E \times \tilde{S}^1 \times S^1$

with fractional $D$-branes $\Rightarrow$ magnetic sources

Twist by $L_u \Rightarrow 6d \, \Omega$-deformation of SYM

In the limit of vanishing size $E \Rightarrow \mathcal{N} = 2^*$ theory in 4d
with special $\Omega$-deformation, $m = -\epsilon \Rightarrow$ massless chiral in 2d
Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\text{ALE} \times \mathbb{R}^2 \times E \times \tilde{S}^1 \times S^1$

with fractional $D$-branes $\implies$ magnetic sources

Twist by $\mathcal{L}_u \implies 6d \ \Omega$-deformation of SYM

In the limit of vanishing size $E \implies \mathcal{N} = 2^*$ theory in 4d

with special $\Omega$-deformation, $m = -\varepsilon \implies$ massless chiral in 2d

magnetic membranes reduce to

susy 't Hooft operators wrapped on $A$ and $B$ cycles on $\tilde{S}^1 \times S^1$
Six dimensional super-Yang-Mills: electric frame

IIA string on \( \text{ALE} \times \mathbb{R}^2 \times \tilde{E} \times \tilde{S}^1 \times S^1 \)

with fractional \( D \)-branes \( \rightarrow \) electric sources

Twist by \( \mathcal{L}_u \) upon \( T \)-duality on \( E \)

produces the Neveu-Schwarz \( B \)-field, with \( H = dB \neq 0 \)
Six dimensional Chern-Simons theory

IIA string on $\text{ALE} \times \mathbb{R}^2 \times \tilde{E} \times \tilde{S}^1 \times S^1$

with electric sources

with the Neveu-Schwarz $B$-field, with $H = dB \neq 0 \implies$

$$\int_{\mathbb{R}^2 \times \tilde{E} \times \tilde{S}^1 \times S^1} H \wedge CS(A)$$

from the $\int C \wedge G \wedge G$ Chern-Simons term in 11d
Four dimensional Chern-Simons theory

IIA string on $\text{ALE} \times \mathbb{R}^2 \times \tilde{E} \times \tilde{S}^1 \times S^1$

Neveu-Schwarz $B$-field, so that $H = dB \neq 0$

$$\int_{\mathbb{R}^2} H \sim \text{Re} \frac{dz}{u}$$

Supersymmetric localization $\implies$

$$\int_{\mathbb{R}^2 \times \tilde{E} \times \tilde{S}^1 \times S^1} H \wedge CS(A) = \frac{1}{u} \int_{\tilde{E} \times \tilde{S}^1 \times S^1} dz \wedge CS(A)$$

Up to $\mathcal{Q}$-exact terms, $\mathcal{A} = A + i(\ldots)$
Open-closed string duality
Open-closed string duality
Open-closed string duality

open string 1-loop

3-brane b.c.
Open-closed string duality

closed string exchange = tree level

3-brane b.c.
Open-closed string duality

closed string exchange = tree level

D3' b.c.

"gluons" of $G_T$ gauge group

D3'' b.c.

D3' D3''
Line operators

D4 brane on ALE $\times S^1$

$\mathbb{R}^4/\Gamma$

D0 branes running around $S^1$

Line operator
Line operators and instanton moduli

\[ L \text{ D4 branes on ALE } \times S^1 \]

\[ R^4/\Gamma \]

\[ N \text{ D0 branes running around } S^1 \]
Line operators and instanton moduli

$D4$ branes on $\text{ALE} \times S^1$

supersymmetric quantum mechanics on instanton moduli

$M_N[U(L)](\mathbb{R}^4/\Gamma)$

$D0$ branes running around $S^1$

Line operator
Two kinds of line operators

\[ \text{D4 branes on } \text{ALE} \times S^1_B \]
\[ \text{D4 branes on } \text{ALE} \times S^1_A \]

\[ \mathcal{M}_n [U(L)] (R^4/\Gamma) \]

Line operator

Another line operator

\[ N \text{ D0 branes running around } S^1_A \]

\[ N \text{ D0 branes running around } S^1_B \]
Two kinds of line operators and instanton moduli
String exchanges between line operators
String exchanges between line operators

- D4 branes on $\text{ALE} \times S^1_A$
- $\text{N} \text{D}0$ branes running around $S^1_A$
- $\text{N} \text{D}0$ branes running around $S^1_B$
- Supersymmetric quantum mechanics on instanton moduli $\mathcal{M}_N[U(L)](\mathbb{R}^4/\Gamma)$
- Closed string exchange, formerly known as one-loop open string diagram
Gluon exchanges between Wilson loops

D4 branes on ALE x $S^1_A$

$R^3/\Gamma$

$\hat{G}_\Gamma$ representation, highest weight $L$

$\hat{G}_\Gamma$ representation, highest weight $\tilde{L}$

$G_\Gamma$ gluon exchange, formerly known as one-loop quiver matter diagram

$S^1_A$

$S^1_B$
Wilson loop = Wilson loop with a hat

\[ \hat{W}_{R_\hat{\lambda}}[C, q] = \text{Tr}_{R_\hat{\lambda}} \left( q^{L_0} P \exp \oint_C A \right), \]

for the highest weight representation \( R_\hat{\lambda} \) of \( \hat{G}_\Gamma \) Kac-Moody group.
Wilson loop = Wilson loop with a hat

\[ \hat{W}_{R\hat{\chi}}[C, q] = \text{Tr}_{R\hat{\chi}} \left( q^{L_0} P \exp \oint_C A \right), \]

Can be expanded in ordinary $G_\Gamma$ - Wilson loops,

with higher spin representations suppressed by powers of $q$. 
\[ \hat{W}_{R_\lambda}[C, q] = \text{Tr}_{R_\lambda} \left( q^{L_0} P \exp \oint C A \right), \]

In our story, the two kinds of line operators we encounter correspond to

\[ \hat{\lambda} = \sum_{i=0}^{r} L_i \varpi_i, \quad \text{and} \quad \hat{\lambda} = \sum_{i=0}^{r} \tilde{L}_i \varpi_i \]

respectively, with \( \varpi_i \) being the fundamental weights of \( \hat{g}_\Gamma \).
Dictionary: weight subspaces

\[ R_{\hat{\lambda}} = \bigoplus_{\hat{w}} R_{\hat{\lambda}}^{\hat{w}}, \]

In our story, the two weight subspaces we encounter correspond to

\[ \hat{w} = \sum_{i=0}^{r} L_i \varpi_i - N_i \alpha_i, \quad \text{and} \quad \hat{w} = \sum_{i=0}^{r} \tilde{L}_i \varpi_i - \tilde{N}_i \alpha_i \]

respectively, with \( \tilde{N}_i = K_i \), and \( \alpha_i \) being the simple roots of \( \hat{g}_\Gamma \).
Twist parameters

The parameters $q_i$ and $\tilde{q}_i$

$\leftrightarrow$

background $G^\mathbb{C}_\Gamma$-flat connection on $\tilde{S}^1 \times S^1$
Twist parameters

The parameters $q_i$ and $\tilde{q}_i$

↔

background $G_C^\Gamma$-flat connection on $\tilde{S}^1 \times S^1$

Can be fixed in the six-dimensional setup (in 4d problematic)
Virasoro

$q_i$ and $\tilde{q}_i \leftrightarrow$ flat $G^C_\Gamma$-connection on $\tilde{S}^1 \times S^1$

and the parameters $q$ and $\tilde{q}$ of the Wilson loop operators

$$q = \prod_{i \in \text{Vert}_\gamma} q^{\alpha_i}_i, \quad \tilde{q} = \prod_{i \in \text{Vert}_\gamma} \tilde{q}^{\alpha_i}_i$$

In string theory:

$$q = \exp - \frac{L_{\tilde{S}^1} M_s}{g_s} + i \int_{\tilde{S}^1} C_{(1)} \, , \quad \tilde{q} = \exp - \frac{L_{S^1} M_s}{g_s} + i \int_{S^1} C_{(1)}$$

$C_{(1)} = \text{Background IIA Ramond-Ramond } U(1) \text{ flat gauge field on } \tilde{S}^1 \times S^1$
Virasoro and M-theory

$q_i$ and $\tilde{q}_i \leftrightarrow$ flat $G^C_T$-connection on $\tilde{S}^1 \times S^1$

and the parameters $q$ and $\tilde{q}$ of the Wilson loop operators

$$q = \prod_{i \in \text{Vert}_\gamma} q_{i}^{a_{i}} , \quad \tilde{q} = \prod_{i \in \text{Vert}_\gamma} \tilde{q}_{i}^{a_{i}}$$

Lift to M-theory: line operators become $M5$ branes wrapped on $\text{ALE} \times (\tilde{S}^1 \text{ or } S^1) \times S^1_{10}$, respectively

$q, \tilde{q}$ – elliptic curve nodes for $(\tilde{S}^1 \text{ or } S^1) \times S^1_{10}$
Take the limit $g_s \rightarrow 0$, keeping $M_s$ finite: $q, \tilde{q} \rightarrow 0$

The *Wilson loop* operator becomes the ordinary one

Decouple one of the nodes, e.g. the affine one

$\hat{g}_\Gamma \rightarrow g_\Gamma$, $q_0 \rightarrow 0$, $\tilde{q}_0 \rightarrow 0$

$L_0 = K_0 = 0 = \tilde{L}_0 = N_0$
In lieu of conclusions

The Chern-Simons $\int dz \wedge CS(A)$ approach to quantum groups has the advantage of making the group whose quantum deformation one is seeking, visible in the structure of sources.

Seems problematic for groups outside the ADE (BCFG) classification.

Bethe/gauge correspondence does not have the explicit $G_\Gamma$ symmetry but is more general (covers all quivers and also super-algebras).

Lots of things to learn and understand better...
One wild speculation

Naively, to describe affine quiver theories

One would attempt to study the $LG_\Gamma$, or $\hat{G}_\Gamma$ gauge theory

One additional dimension: 7d theory? natural in M-theory on ALE

However, we learned: $U(1)_{L_0} \subset \hat{G}_\Gamma$ is the 10d RR $U(1)$ gauge field

Perhaps we’ll learn about the origin of the (12d?) $E_8$ gauge field

Horava, Witten 1996; Witten 1997; Diaconescu, Moore, Witten 2000

Lots of things to learn and understand better...
THANK YOU