Open-closed (little) string duality

and

Chern-Simons-Bethe/gauge correspondence

NIKITA NEKRASOV

Simons Center for Geometry and Physics, Stony Brook

YITP, Stony Brook; IHES, Bures-sur-Yvette; IITP, Moscow; ITEP, Moscow

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Based on the joint work

with Mina Aganagic and Samson Shatashvili

2016-

and the project

BPS/CFT correspondence and
non-perturbative Dyson-Schwinger equations

NN, 2004-

There are two ways to realize a symmetry in quantum system

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Start with a classical system with symmetry and quantize

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Example: geometric quantization

$$\int_{(p,q)\in \text{coadjoint orbit}} DpDq \exp\left(\mathrm{i} \int pdq - \int \mathrm{Tr} A \cdot \mu(p,q) \right)$$

$$\sim \qquad \langle v_1 \,|\, T_{\mathcal{H}} \left(P \exp \int A \right) \,|\, v_2 \rangle$$

inspiration Borel - Weil - Bott theorem, 1957

Kirillov 1961; path integral suggested in 1961 by Faddeev

Alekseev, Faddeev, Shatashvili 1988

Emergent symmetry in quantum system

Preparations:

$\Gamma \subset SU(2)$ finite subgroup

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Irreps $\mathcal{R}_i$, $i = 0, \ldots, r$

Preparations: quivers from Γ

$$\Gamma \subset SU(2) \qquad \text{finite subgroup}$$

Irreps $\mathcal{R}_i \Longrightarrow$ vertices $i = 0, \ldots, r$ of a quiver Γ

edges: $$\mathcal{R}_i \otimes \mathbb{C}^2 = \bigoplus_{e \in s^{-1}(i)} \mathcal{R}_{t(e)} \bigoplus_{e \in t^{-1}(i)} \mathcal{R}_{s(e)}$$

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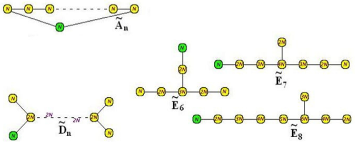
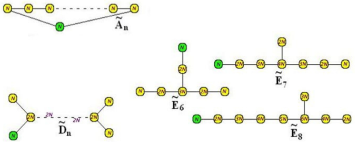


Symmetry hints: McKay duality

Irreps $\mathcal{R}_i \Longrightarrow$ vertices $i = 0, \ldots, r$ of a quiver Γ

edges: $$\mathcal{R}_i \otimes \mathbb{C}^2 = \bigoplus_{e \in s^{-1}(i)} \mathcal{R}_{t(e)} \bigoplus_{e \in t^{-1}(i)} \mathcal{R}_{s(e)}$$

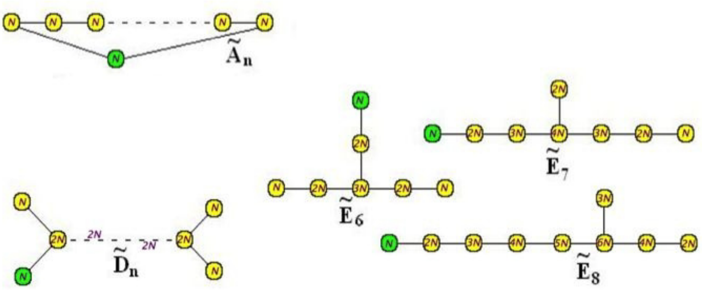
Dynkin labels: $$a_i = \mathcal{R}_i, \qquad 2a_i = \sum_{e \in s^{-1}(i)} a_{t(e)} + \sum_{e \in t^{-1}(i)} a_{s(e)}$$



Symmetry hints: McKay duality

Quiver Γ = affine Dynkin diagram of G_Γ

McKay dual simple Lie group (ADE)

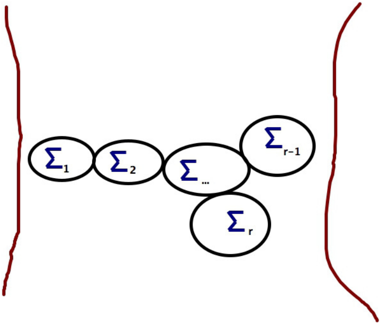


Symmetry hints: Weyl group $\mathcal{W}_\Gamma$

ALE spaces = $\widetilde{\mathbb{C}^2/\Gamma}$

Four dimensional hyperkähler manifolds, with moduli $(\mathbb{R}^r \otimes \mathbb{R}^3)/\mathcal{W}_\Gamma$

$H^2(\mathrm{ALE}, \mathbb{Z})$ form the $\mathcal{W}_\Gamma$ - local system over the moduli space



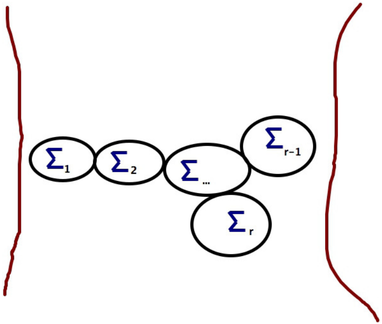
Symmetry hints: Weyl group $\mathcal{W}_\Gamma$

ALE spaces = $\widetilde{\mathbb{C}^2/\Gamma}$

Four dimensional hyperkähler manifolds, with moduli

$$\mathcal{M}_\Gamma = \{ \left(\zeta_i^{\mathbb{R}}, \zeta_i^{\mathbb{C}}\right) \} \in (\mathfrak{h}(G_\Gamma) \otimes \mathbb{R} \oplus \mathbb{C})/\mathcal{W}_\Gamma$$

$H^2(\mathrm{ALE}, \mathbb{Z})$ form the $\mathcal{W}_\Gamma$ - local system over the moduli space



Emergent symmetry in quantum system

Example: Nakajima algebras

Start with the $4+1$ dimensional

Supersymmetric $U(w)$ gauge theory on

$$\left(\text{ALE} = \widetilde{\mathbb{R}^4/\Gamma}\right) \times \mathbb{R}^1$$

In a low-energy weak-coupling adiabatic approximation $\Longrightarrow$

Vafa, Witten, 1994

Supersymmetric quantum mechanics on $\mathcal{M}_{\mathbf{v},\mathbf{w}}(\widetilde{\mathbb{R}^4/\Gamma})$

$U(w)$ instantons on ALE space $\widetilde{\mathbb{R}^4/\Gamma}$,
with topological charges $\mathbf{v} = (v_0, v_1, \ldots, v_r)$
and boundary conditions at infinity
$U(w) \longrightarrow H_{\mathbf{w}} = U(w_0) \times U(w_1) \times \ldots \times U(w_r)$

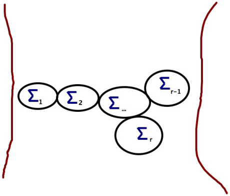
Supersymmetric quantum mechanics on $\mathcal{M}_{\mathbf{v},\mathbf{w}}(\widetilde{\mathbb{R}^4/\Gamma})$

$A \to$ flat connection at infinity

$$\pi_1(\mathbf{S}^3/\Gamma) = \Gamma \to U(w)$$

$$U(w) \longrightarrow H_{\mathbf{w}} = U(w_0) \times U(w_1) \times \ldots \times U(w_r)$$

$$-\frac{1}{8\pi^2}\int_{\mathrm{ALE}} \mathrm{Tr} F \wedge F \sim v_0 \qquad \frac{1}{2\pi \mathrm{i}} \mathrm{Tr} F \sim v_1[\Sigma_1] + \ldots + v_r[\Sigma_r]$$

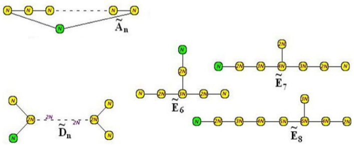


Supersymmetric quantum mechanics on $\mathcal{M}_{\mathbf{v},\mathbf{w}}(\mathrm{ALE})$

Ground states: cohomology $H^*\left(\mathcal{M}_{\mathbf{v},\mathbf{w}}(\mathrm{ALE})\right)$

Nakajima: $\mathcal{H}_{\mathbf{w},\Gamma} = \bigoplus_{\mathbf{v}} H^*\left(\mathcal{M}_{\mathbf{v},\mathbf{w}}\left(\mathrm{ALE}\right)\right)$ is

an irreducible
highest weight
representation
of Kac-Moody algebra $\widehat{\mathfrak{g}}_\Gamma$
G_Γ - McKay dual Lie group



Supersymmetric quantum mechanics on $\mathcal{M}_{\mathbf{v},\mathbf{w}}(\mathrm{ALE})$

Ground states: cohomology $H^*\left(\mathcal{M}_{\mathbf{v},\mathbf{w}}(\mathrm{ALE})\right)$

Nakajima: work $H_{\mathbf{w}} \times U(1)$-equivariantly

$$\mathcal{H}_{\mathbf{w},\Gamma} = \bigoplus_{\mathbf{v}} H^*\left(\mathcal{M}_{\mathbf{v},\mathbf{w}}(\mathrm{ALE})\right)$$

irrep of the Yangian $Y(\widehat{\mathfrak{g}}_\Gamma)$ of $\widehat{\mathfrak{g}}_\Gamma$

Ginzburg,Vasserot (finite A series); Varagnolo, 2000

More generally

$\mathcal{M}_{\mathbf{v},\mathbf{w}}(\mathrm{ALE})$ is an example of a quiver variety $\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})$

Supersymmetric quantum mechanics on $\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})$

Ground states: cohomology $H^*\left(\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})\right)$

Nakajima: work $H_{\mathbf{w}} \times U(1)$-equivariantly

$$\mathcal{H}_{\mathbf{w},\Gamma} = \bigoplus_{\mathbf{v}} H^*\left(\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})\right)$$

irrep of the Yangian $Y(\mathfrak{g}_\gamma)$ of $\mathfrak{g}_\gamma$

Varagnolo, 2000

More generally

Sigma model $\sim$ supersymmetric quantum mechanics on $L\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})$

Ground states: K-theory $K\left(\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})\right)$

Nakajima: work $H_{\mathbf{w}} \times U(1)$-equivariantly

$$\mathcal{H}_{\mathbf{w},\Gamma} = \bigoplus_{\mathbf{v}} K\left(\mathfrak{M}_\gamma(\mathbf{w}, \mathbf{v})\right)$$

irrep of quantum affine algebra $U_q(\mathfrak{g}_\gamma)$ of $\mathfrak{g}_\gamma$

Nakajima, 1999

SURPRISES

Need to sum over $\mathbf{v}$:

Full symmetry is realized in a collection of quantum systems

SURPRISES

Need to sum over $\mathbf{v}$: collections of quantum systems

Natural in $4+1$ theory but it is not a quantum field theory

No obvious realization of G_Γ in the classical system

HINTS

String theory realization of the gauge theory

makes the summation over $\mathbf{v}$ natural

In string theory the appearence of G_Γ comes naturally

Mental note:

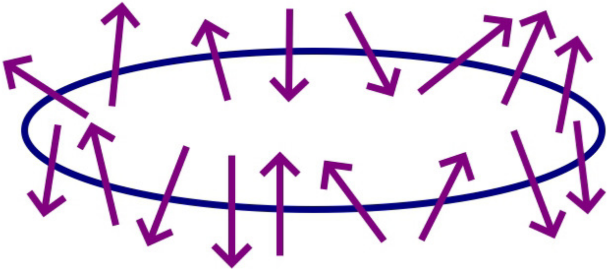
String theory may provide a natural explanation

Natural habitat

for the Yangian algebras?

Natural habitat of the Yangian

Spin chains!



Natural habitat of the Yangian

Spin chains! Start with $Y(sl_2)$ for simplicity

Finite dimensional Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$$

$$\mathcal{H} = \overbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}^{L\,\text{times}}$$

Hamiltonian

$$\widehat{H} = \sum_{a=1}^{L} \sigma_a^x \otimes \sigma_{a+1}^x + \sigma_a^y \otimes \sigma_{a+1}^y + \sigma_a^z \otimes \sigma_{a+1}^z$$

Hamiltonian

$$\mathcal{H} = \overbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}^{L \text{ times}}$$

$$\widehat{H} = \sum_{a=1}^{L} \sigma_a^x \otimes \sigma_{a+1}^x + \sigma_a^y \otimes \sigma_{a+1}^y + \sigma_a^z \otimes \sigma_{a+1}^z$$

$$\vec{\sigma}_{a+L} = \vec{\sigma}_a$$

Heisenberg magnet: periodic isotropic homogeneous spin chain

Hamiltonians!

$$\mathcal{H} = \overbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}^{L\,\text{times}}$$

$$\widehat{H}_1 = \sum_{a=1}^{L} \sigma_a^x \otimes \sigma_{a+1}^x + \sigma_a^y \otimes \sigma_{a+1}^y + \sigma_a^z \otimes \sigma_{a+1}^z$$

$$\widehat{H}_2, \widehat{H}_3, \ldots, \widehat{H}_L, \ldots$$

$$[\widehat{H}_i, \widehat{H}_j] = 0$$

Quantum integrability!

Commuting Hamiltonians from Transfer Matrix

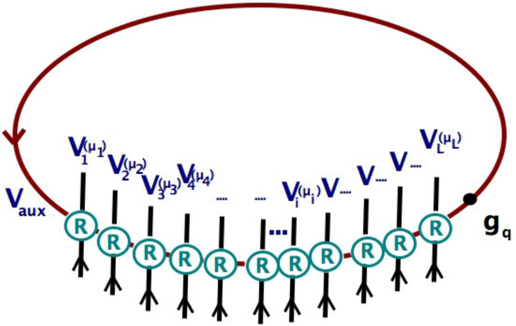
$$\widehat{T}(x) \;=\; x^L \exp \sum_{n=1}^{\infty} \frac{1}{n} x^{-n} \widehat{H}_n$$

Quantum integrability $\Leftrightarrow [\widehat{T}(x'), \widehat{T}(x'')] = 0$

Transfer matrices

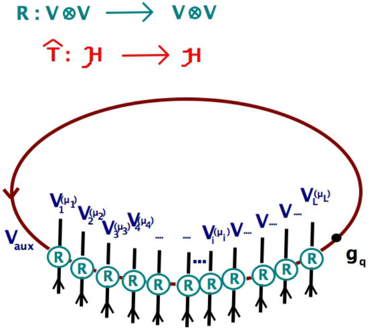
$$R : V \otimes V \longrightarrow V \otimes V$$

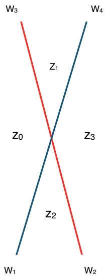
$$\hat{T} : \mathcal{H} \longrightarrow \mathcal{H}$$



Transfer matrices from the R-matrix

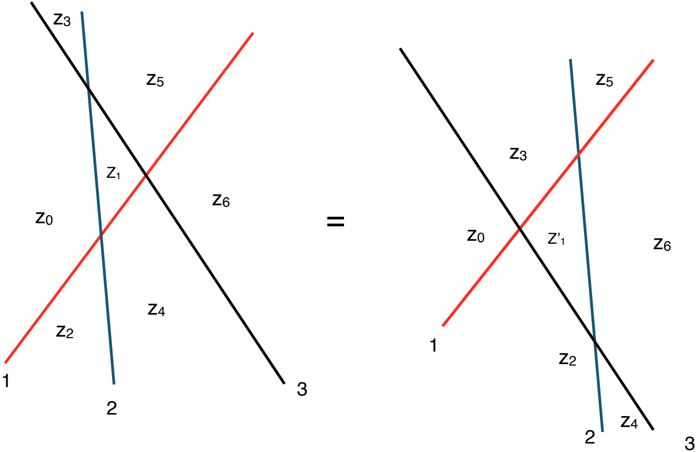
$$\widehat{T}(x) = \mathrm{Tr}_{V_{\rm aux}} \left(R(x,\mu_1) R(x,\mu_2) \ldots R(x,\mu_L) \right) \; : \; \mathcal{H} \longrightarrow \mathcal{H}$$





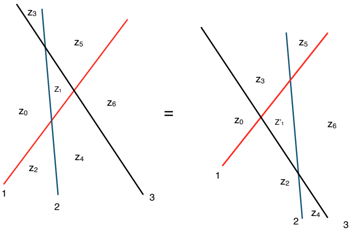
$$\mathcal{H} = V_1(\mu_1) \otimes V_2(\mu_2) \otimes \ldots \otimes V_L(\mu_L)$$

Yang-Baxter equation for the R-matrix

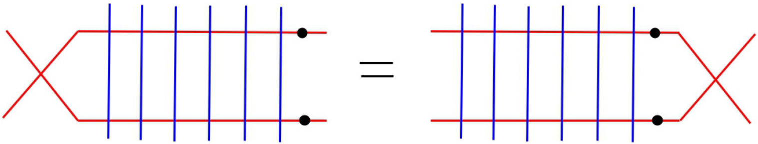


Implies $[\widehat{T}(x'), \widehat{T}(x'')] = 0$ by the train argument

Yang-Baxter equation for the R-matrix



$\Longrightarrow$



$\Longrightarrow [\widehat{T}(x'), \widehat{T}(x'')] = 0$ by the cyclicity of $\mathrm{Tr}_{V_{\mathrm{aux}}}$

Transfer matrices from the R-matrix

$$\widehat{T}(x) = \mathrm{Tr}_{V_{\mathrm{aux}}}\left(R(x,\mu_1)R(x,\mu_2)\ldots R(x,\mu_L)\right) : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\mathcal{H} = V_1(\mu_1) \otimes V_2(\mu_2) \otimes \ldots \otimes V_L(\mu_L)$$

$\mu_1, \ldots, \mu_L \in \mathbb{C}$ inhomogeneities

Heisenberg spin chain was homogeneous, i.e. $\mu_a = 0$

Twisted transfer matrices from the R-matrix

$$\widehat{T}(x;\mathfrak{q}) = \mathrm{Tr}_{V_{\mathrm{aux}}}\, g_{\mathfrak{q}}\ \left(R(x,\mu_1)R(x,\mu_2)\ldots R(x,\mu_L)\right)\ :\ \mathcal{H} \longrightarrow \mathcal{H}$$

Twisted spin chain, $\vec{\sigma}_{a+L} = Ad(g_{\mathfrak{q}})\vec{\sigma}_a$

For $SU(2)$: $\qquad g_{\mathfrak{q}} = \mathfrak{q}^{\frac{1}{2}\sigma^z}$

Anisotropic models from the trigonometric and elliptic R-matrices

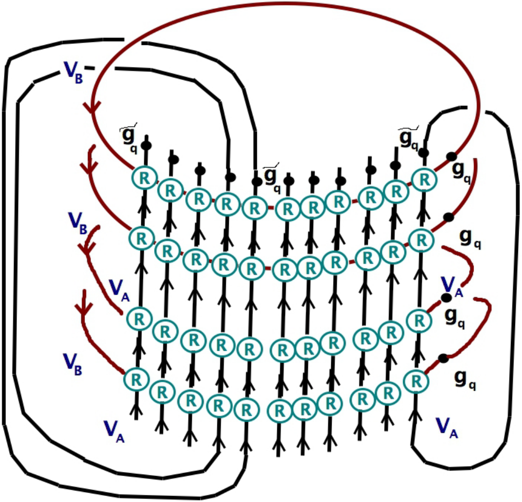
Baxter, Drinfeld, Belavin, Jimbo

$$\widehat{T}(x;\mathfrak{q}) = \mathrm{Tr}_{V_{\mathrm{aux}}}\, g_{\mathfrak{q}}\, \left(R(x,\mu_1)R(x,\mu_2)\ldots R(x,\mu_L)\right) \;:\; \mathcal{H} \longrightarrow \mathcal{H}$$

$$\widehat{H}_1 \to \sum_{a=1}^{L} \alpha\sigma_a^x \otimes \sigma_{a+1}^x + \beta\sigma_a^y \otimes \sigma_{a+1}^y + \gamma\sigma_a^z \otimes \sigma_{a+1}^z$$

$$(\alpha:\beta:\gamma) = \begin{cases} (1:1:1) & \textit{rational} & \mathbf{XXX} \\ (1:1:\Delta) & \textit{trigonometric} & \mathbf{XXZ} \\ (1:\Delta':\Delta'') & \textit{elliptic} & \mathbf{XYZ} \end{cases}$$

Lattice model



Lattice model

Partition function via transfer matrix formalism

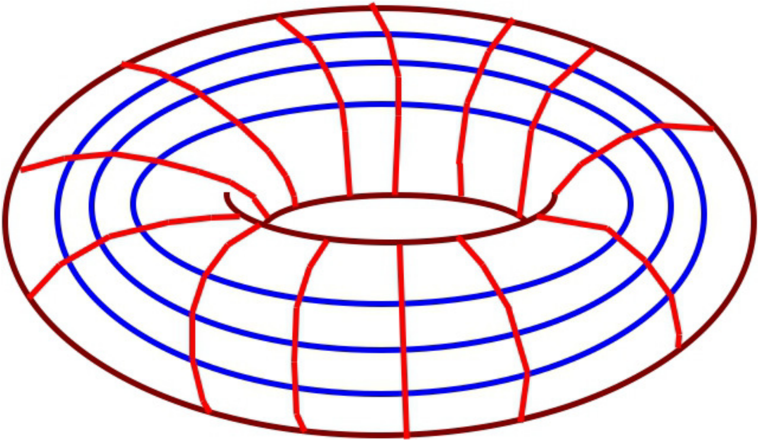
L. Onsager solution of the Ising model

$$\mathcal{Z}_{L,\tilde{L}} = \mathrm{Tr}_{\mathcal{H}_L} \left(\widehat{T}(x_1; \mathfrak{q}) \widehat{T}(x_2; \mathfrak{q}) \ldots \widehat{T}(x_{\tilde{L}}; \mathfrak{q}) \cdot g_{\tilde{\mathfrak{q}}} \right)$$

Lattice model on a torus: double trace

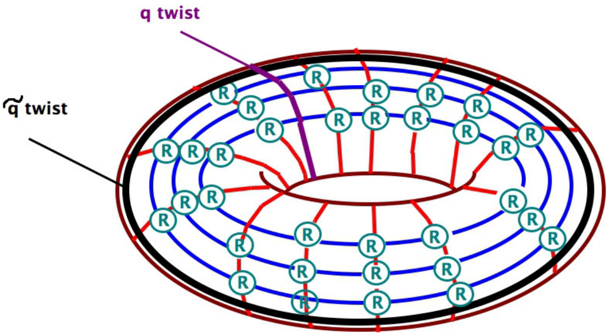
not in the sense of gauge theory

$$\mathcal{Z}_{L,\tilde{L}} = \mathrm{Tr}_{\mathcal{H}_L}\left(\widehat{T}(x_1;\mathfrak{q})\widehat{T}(x_2;\mathfrak{q})\ldots\widehat{T}(x_{\tilde{L}};\mathfrak{q})\cdot g_{\tilde{\mathfrak{q}}}\right)$$



Lattice model on the torus

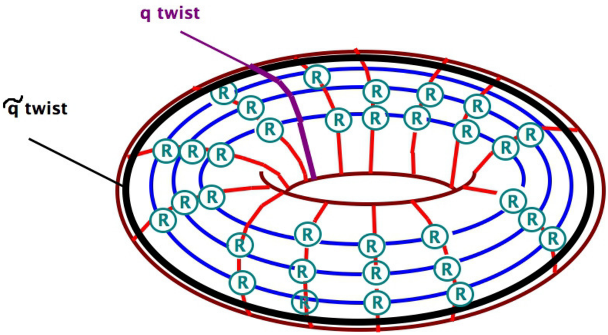
$$\mathcal{Z}_{L,\tilde{L}} = \mathrm{Tr}_{\mathcal{H}_L} \left(\widehat{T}(x_1;\mathfrak{q})\widehat{T}(x_2;\mathfrak{q}) \ldots \widehat{T}(x_{\tilde{L}};\mathfrak{q}) \cdot g_{\tilde{\mathfrak{q}}} \right)$$



Lattice model: double trace

$\mathcal{Z}_{L,\tilde{L}}(\mathfrak{q},\tilde{\mathfrak{q}}) = \sum$ over states on the edges of the lattice

Boltzmann weights = products of R-matrix elements

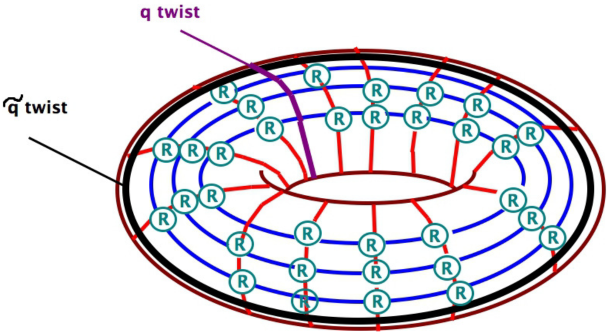


Lattice model: modularity

Exchange A and B cycles

L vs $\tilde{L}$

$\mathfrak{q}$ vs $\tilde{\mathfrak{q}}$



Lattice model: Hamiltonian viewpoint

$$\mathcal{Z}_{L,\tilde{L}} = \mathrm{Tr}_{\mathcal{H}_L} \left(\widehat{T}(x_1;\mathfrak{q}) \widehat{T}(x_2;\mathfrak{q}) \ldots \widehat{T}(x_{\tilde{L}};\mathfrak{q}) \cdot g_{\tilde{\mathfrak{q}}} \right)$$

Bethe states: $\psi_\sigma \in \mathcal{H}$

$$\widehat{T}(x,\mathfrak{q})\,\psi_\sigma = T_\sigma(x,\mathfrak{q})\,\psi_\sigma$$

$\mathcal{Z}_{L,\tilde{L}} = \sum$ over the eigenvalues of the transfer matrix

$$\mathcal{Z}_{L,\tilde{L}}(\mathfrak{q},\tilde{\mathfrak{q}}) = \sum_N \tilde{\mathfrak{q}}^N \sum_{\sigma_1,\ldots,\sigma_N} T_\sigma(x_1;\mathfrak{q}) \ldots T_\sigma(x_{\tilde{L}};\mathfrak{q})$$

Sum over the number of Bethe roots = ”magnons”

Hamiltonian viewpoint: Bethe ansatz

Bethe states: $\psi_\sigma \in \mathcal{H}$

$$\widehat{T}(x,\mathfrak{q})\,\psi_\sigma = T_\sigma(x,\mathfrak{q})\,\psi_\sigma$$

for all x

Lightnining review of Bethe ansatz

Faddeev, Sklyanin, Takhtajan

Kulish, Reshetikhin

Isergin, Korepin

Drinfeld, Jimbo, Miwa

Monodromy matrix

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = R(x, \mu_1) \dots R(x, \mu_L) : V_{\text{aux}} \otimes \mathcal{H} \to V_{\text{aux}} \otimes \mathcal{H}$$

Lightnining review of Bethe ansatz

Monodromy matrix

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} : V_{\mathsf{aux}} \otimes \mathcal{H} \to V_{\mathsf{aux}} \otimes \mathcal{H}$$

Yangian $Y(sl_2)$ generators

$$A(x), B(x), C(x), D(x) : \mathcal{H} \to \mathcal{H}$$

Lightnining review of Bethe ansatz

Monodromy matrix

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = R(x,\mu_1)\ldots R(x,\mu_L) : V_{\mathsf{aux}} \otimes \mathcal{H} \to V_{\mathsf{aux}} \otimes \mathcal{H}$$

Bethe state

$$\psi_\sigma = B(\sigma_1)B(\sigma_2)\ldots B(\sigma_N)|\downarrow\downarrow\ldots\downarrow\rangle$$

Lightnining review of Bethe ansatz

Bethe state (algebraic Bethe ansatz)

$$\psi_\sigma = B(\sigma_1)B(\sigma_2)\dots B(\sigma_N)|\downarrow\downarrow\dots\downarrow\rangle$$

Bethe roots $\sigma_1, \dots, \sigma_N$

Lightnining review of Bethe ansatz

Bethe equations

$$\mathfrak{q} \prod_{a=1}^{L} \frac{\sigma_i - \mu_a + u}{\sigma_i - \mu_a - u} = \prod_{j \neq i} \frac{\sigma_i - \sigma_j + 2u}{\sigma_i - \sigma_j - 2u}$$

Solutions = Bethe roots $\sigma_1, \ldots, \sigma_N$

Planck constant $\approx u$

Lightning review of Bethe ansatz

Functional Bethe Ansatz: $T - Q$ relation

Baxter, Sklyanin

$$P(x-u)Q_\sigma(x+2u) + \mathfrak{q}P(x+u)Q_\sigma(x-2u) = T_\sigma(x;\mathfrak{q})Q_\sigma(x)$$

$$Q_\sigma(x) = \prod_{i=1}^{N}(x-\sigma_i)\,, \qquad P(x) = \prod_{a=1}^{L}(x-\mu_a)$$

The content of this equation: $T_\sigma(x;\mathfrak{q})$ has no singularities in x

Lightnining review of Bethe ansatz

$$Q_\sigma(x) = \prod_{i=1}^{N}(x - \sigma_i) = \text{eigenvalue of Baxter operator } \widehat{Q}(x)$$

$$P(x) = \prod_{a=1}^{L}(x - \mu_a) = \text{Drinfeld polynomial}$$

Lightnining review of Bethe ansatz

q-character form of Bethe equations

E. Frenkel, Reshetikhin

$$Y_\sigma(x+2u) + \mathfrak{q}\ell(x) Y_\sigma(x)^{-1} = \frac{T_\sigma(x;\mathfrak{q})}{P(x-u)}$$

$T_\sigma(x;\mathfrak{q})$ is a polynomial in x

$$Y_\sigma(x) = \frac{Q_\sigma(x)}{Q_\sigma(x-2u)}$$

$$\ell(x) = \frac{P(x+u)}{P(x-u)}$$

Lightnining review of Bethe ansatz

q-character form of Bethe equations

$$Y_\sigma(x+2u) + \mathfrak{q}\ell(x)Y_\sigma(x)^{-1} = \frac{T_\sigma(x;\mathfrak{q})}{P(x-u)}$$

$$Y_\sigma(x) = \frac{Q_\sigma(x)}{Q_\sigma(x-2u)} =$$

eigenvalue of the operator $\widehat{Y}(x)$

q-character

$$\widehat{Y}(x+2u) + \mathfrak{q}\ell(x)\widehat{Y}_{(}x)^{-1} =$$

the fundamental q-character of $Y(sl_2)$

q-characters for general quivers

$$\widehat{Y}_i(x+2u) +$$

$$+\, \mathfrak{q}_i \ell_i(x) \widehat{Y}_i(x)^{-1} \prod_{e \in s^{-1}(i)} \widehat{Y}_{t(e)}(x+\mu_e+u) \prod_{e \in t^{-1}(i)} \widehat{Y}_{s(e)}(x-\mu_e+u) + \dots$$

= the fundamental q-character of $Y(\mathfrak{g}_\Gamma)$

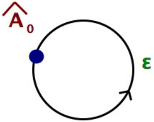
$$\ell_i(x) = \frac{P_i(x+u)}{P_i(x-u)}$$

the ℓ-weight

q-character for $\widehat{A}_0$

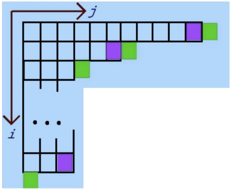
NN, Pestun, Shatashvili, 2013

E. Frenkel, D. Hernandez, 2013-2015



Additional parameter $\varepsilon = \mu_e$

$$\sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square\in\lambda} \ell(x+c_{\square}) \frac{\prod_{\blacksquare\in\partial_+\lambda} \widehat{Y}(x+2u+c_{\blacksquare})}{\prod_{\blacksquare\in\partial_-\lambda} \widehat{Y}(x+c_{\blacksquare})}$$



= the fundamental q-character of $Y(\widehat{u(1)})$

$$c_{\square} = \varepsilon(i-j) - u(i+j-2)\,, \qquad \square = (i,j)$$

$$\ell(x) = \frac{P(x+u)}{P(x-u)}$$

Bethe/gauge correspondence

Bethe/gauge correspondence

NN, Shatashvili 2007

Bethe/gauge correspondence

Prior work: Moore, NN, Shatashvili, 1997

Givental, 1993

Gorsky, NN, 1992-1994

Gerasimov, Shatashvili, 2006

$\mathcal{N} = (2, 2), d = 2$ super-Poincare invariant gauge theory

Bethe/gauge correspondence

Quantum integrable system

Supersymmetric vacua (in finite volume)

$$\uparrow$$

Bethe/gauge correspondence

$$\downarrow$$

Stationary states = joint eigenvectors of quantum integrals of motion

Twisted chiral ring, e.g. $\mathcal{O}_n = \frac{1}{(2\pi \mathrm{i})^n n!} \mathrm{Tr}\sigma^n$

$\uparrow$

Bethe/gauge correspondence

$\downarrow$

Quantum integrals of motion $\widehat{H}_n$, e.g. $\widehat{\mathrm{Tr} L^n}$ for Lax operator L

Effective twisted superpotential $\mathcal{W}(\sigma_1, \ldots, \sigma_N)$

$$\uparrow$$

Bethe/gauge correspondence

$$\downarrow$$

The Yang-Yang functional $\mathcal{Y}(\sigma_1, \ldots, \sigma_N)$

$\mathcal{N} = (2,2), d = 2$ super-Poincare invariant gauge theory

$$\Longleftarrow \qquad ? \qquad \Longrightarrow$$

Quantum integrable system

$\mathcal{N} = 4, d = 2\ U(N)$ gauge theory
with L hypermultiplets in the fundamental representation

Example of Bethe/gauge correspondence

Inhomogeneous twisted length $L\ SU(2)$ spin $\frac{1}{2}$ chain
in the sector with N spins up

Softly broken $\mathcal{N} = 4 \to \mathcal{N} = 2, d = 2\ U(N)$ gauge theory
by the twisted mass u, corresponding to the $U(1)$ symmetry

$$Q, \tilde{Q} \mapsto e^{\mathrm{i}u} Q, e^{\mathrm{i}u} \tilde{Q}$$

$$\Phi \mapsto e^{-2\mathrm{i}u} \Phi$$

Inhomogeneities μ_a = twisted masses
$\leftrightarrow U(L)$ flavor symmetry of $\mathcal{N} = 4$ theory

the twist parameter $\mathfrak{q}$ = Kähler modulus

$$\mathfrak{q} = e^{2\pi \mathrm{i} t} = e^{\mathrm{i}\vartheta - 2\pi r}$$

Bethe equations

= quantum cohomology (twisted chiral ring) relations

$$\mathfrak{q} \prod_{a=1}^{L} \frac{\sigma_i - \mu_a + u}{\sigma_i - \mu_a - u} = \prod_{j \neq i} \frac{\sigma_i - \sigma_j + 2u}{\sigma_i - \sigma_j - 2u}$$

Solutions =
eigenvalues of the complex scalar in the $U(N)$ vector multiplet:

$$\sigma \sim \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$$

up to permutations of σ_i's –
the remainder of the $U(N)$ gauge symmetry

Bethe equations

= quantum cohomology (twisted chiral ring) relations

$$1 = \mathfrak{q} \prod_{a=1}^{L} \frac{\sigma_i - \mu_a + u}{\sigma_i - \mu_a - u} \prod_{j \neq i} \frac{\sigma_i - \sigma_j - 2u}{\sigma_i - \sigma_j + 2u} = \exp\left(\frac{\partial \widetilde{W}}{\partial \sigma_i} \right)$$

$\widetilde{W}(\sigma_1, \ldots, \sigma_N) =$ effective twisted superpotential

one-loop exact computation!

Baxter Q-operator

= characteristic polynomial of the adjoint Higgs

$$Q(x) = \mathrm{Det}(x - \sigma)$$

Gauged linear sigma model on $T^*\mathrm{Gr}(N, L)$

low energy description of our gauge theory for $r \gg 0$

$$Q(x)[p] = c_x(\mathcal{E}_p) = \text{Chern polynomial of the tautological bundle}$$

$$Q(x)[p] = x^N - c_1(\mathcal{E}_p)x^{N-1} + c_2(\mathcal{E}_p)x^{N-2} - \dots$$

local operator $Q(x)[p]$, $p \in \Sigma$
in the sigma model with worldsheet Σ, roughly:

$$\mathcal{E}_p \to \mathcal{M}\,, \qquad \mathcal{E}_p = ev_p^* \mathbf{E}$$

$\mathbf{E} =$ rk N tautological bundle over $T^*\mathrm{Gr}(N, L)$

$$ev : \Sigma \times \mathcal{M} \longrightarrow T^*\mathrm{Gr}(N, L) \qquad \text{evaluation map}$$

Lift to three dimensions

$$\Sigma \longrightarrow \mathbf{S}^1 \times \Sigma$$

Twisted masses $\rightarrow$ Wilson loops + real masses

XXX $\rightarrow$ **XXZ** = trigonometric case

Lift to four dimensions

$$\Sigma \longrightarrow E \times \Sigma$$

Elliptic curve E

Twisted masses $\to$ Holomorphic $GL(L) \times \mathbb{C}^{\times}$ bundle on E

XXX $\to$ **XYZ** = elliptic case

Lift to four dimensions

$$\Sigma \longrightarrow E \times \Sigma$$

Elliptic curve E

Twisted masses $\to$ Holomorphic $GL(L) \times \mathbb{C}^\times$ bundle on E

XYZ = elliptic case — anomalous when $L \neq 2N$

What is the meaning of $T_\sigma(x)$?

What is the meaning of $T - Q$ relations?

Quiver gauge theory

$\mathcal{N} = (4,4)$ quiver gauge theory

$\mathcal{N} = 4$ softly broken down to $\mathcal{N} = 2$

Quiver γ with the set Vert_γ of vertices
and the set Edge_γ of edges

$\mathcal{N} = (4,4)$ quiver gauge theory

$\mathcal{N} = 4$ softly broken down to $\mathcal{N} = 2$

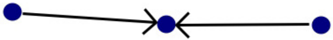
Quiver γ with the set Vert_γ of vertices
and the set Edge_γ of edges

$$e \in \mathrm{Edge}_\gamma, \qquad s(e), t(e) \in \mathrm{Vert}_\gamma$$

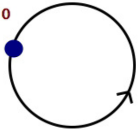
source and target

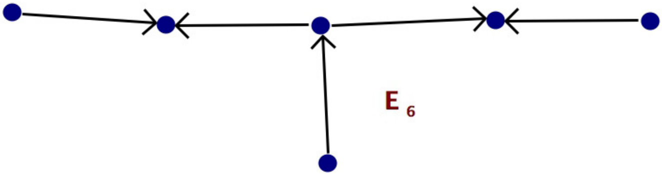
Examples of quivers

A_3



$\hat{A}_0$





E_6

Apologies for notations

$$N_i, L_i$$

Stand both for vector spaces (colors $\mathbb{C}^{N_i}$ and flavors $\mathbb{C}^{L_i}$), their dimensions, sometimes characters

$$N_i \sim \sum_{\alpha \in [N_i]} e^{\sigma_{i,\alpha}}$$

$$L_i \sim \sum_{\mathfrak{f} \in [L_i]} e^{\mu_{i,\mathfrak{f}}}$$

$$[p] := \{1, 2, \ldots, p\}$$

Gauge group

$$G = \times_{i \in \mathrm{Vert}_\gamma} U(N_i)$$

Vector multiplet scalars

$$\Phi_i, \sigma_i \in \mathrm{Lie}GL(N_i)$$

Matter hypermultiplets

Fundamentals $Q_i \in \mathrm{Hom}(L_i, N_i), \tilde{Q}_i \in \mathrm{Hom}(N_i, L_i)$

Matter hypermultiplets

Fundamentals $Q_i \in \mathrm{Hom}(L_i, N_i), \tilde{Q}_i \in \mathrm{Hom}(N_i, L_i)$

Bi-fundamentals $Q_e \in \mathrm{Hom}(N_{s(e)}, N_{t(e)}), \tilde{Q}_e \in \mathrm{Hom}(N_{t(e)}, N_{s(e)})$

Matter superpotential

$$W = \sum_{i \in \mathrm{Vert}_\gamma} \mathrm{Tr}_{L_i} \left(\tilde{Q}_i \Phi_i Q_i \right) +$$

$$+ \sum_{e \in \mathrm{Edge}_\gamma} \mathrm{Tr}_{N_{s(e)}} \left(\tilde{Q}_e \Phi_{t(e)} Q_e \right) - \mathrm{Tr}_{N_{t(e)}} \left(Q_e \Phi_{s(e)} \tilde{Q}_e \right)$$

Matter masses, compatible with $\mathcal{N} = 4$

$$\mathfrak{M}_i \in \mathrm{End}(L_i)\,, \qquad \mu_e \in \mathbb{C}$$

Twisted masses of the fundamental and the bi-fundamental hypermultiplets, respectively

$$\left(Q_i, \tilde{Q}_i, Q_e, \tilde{Q}_e\right) \longrightarrow \left(Q_i e^{-\mathrm{i}\mathfrak{M}_i}, e^{\mathrm{i}\mathfrak{M}_i}\tilde{Q}_i, e^{\mathrm{i}\mu_e} Q_e, e^{-\mathrm{i}\mu_e}\tilde{Q}_e\right)$$

Susy breaking by the twisted mass u

$$W = \sum_{i \in \mathrm{Vert}_\gamma} \mathrm{Tr}_{M_i}\left(\tilde{Q}_i \Phi_i Q_i\right) +$$

$$+ \sum_{e \in \mathrm{Edge}_\gamma} \mathrm{Tr}_{N_{s(e)}}\left(\tilde{Q}_e \Phi_{t(e)} Q_e\right) - \mathrm{Tr}_{N_{t(e)}}\left(Q_e \Phi_{s(e)} \tilde{Q}_e\right)$$

The most important $U(1)$ symmetry

$$\left(Q_i, \tilde{Q}_i, Q_e, \tilde{Q}_e, \Phi_i\right) \longrightarrow \left(e^{\mathrm{i}u} Q_i, e^{\mathrm{i}u} \tilde{Q}_i, e^{\mathrm{i}u} Q_e, e^{\mathrm{i}u} \tilde{Q}_e, e^{-2\mathrm{i}u} \Phi_i\right)$$

Integrate out massive matter

$$\widetilde{W}(\sigma_{i,\alpha}) =$$

$$\sum_{i \in \mathrm{Vert}_\gamma} \sum_{\alpha \in [N_i]} \left(\log(\mathfrak{q}_i)\, \sigma_{i,\alpha} + \sum_{\beta \in [N_i]} \varpi\left(-2u + \sigma_{i,\alpha} - \sigma_{i,\beta}\right) + \right.$$

$$\left. + \sum_{\mathsf{f} \in [L_i]} \left(\varpi\left(u + \sigma_{i,\alpha} - \mu_{i,\mathsf{f}}\right) + \varpi\left(u - \sigma_{i,\alpha} + \mu_{i,\mathsf{f}}\right)\right) \right)$$

$$+ \sum_{e \in \mathrm{Edge}_\gamma} \sum_{\alpha \in [N_{t(e)}]} \sum_{\beta \in [N_{s(e)}]} \left(\varpi\left(u + \mu_e + \sigma_{t(e),\alpha} - \sigma_{s(e),\beta}\right)\right.$$

$$\left. + \varpi\left(u - \mu_e + \sigma_{s(e),\beta} - \sigma_{t(e),\alpha}\right)\right)$$

Rational case

$$\varpi(z) = z\ (\log(z) - 1)\ , \qquad \exp \varpi'(z) = z$$

Trigonometric case

$$\varpi_R(z) =$$

$$= R\frac{z^2}{2} - \log(2R)z - \frac{1}{2R}\operatorname{Li}_2\left(e^{-2Rz}\right) - \frac{\pi^2}{12R} \ ,$$

$$\exp\varpi'_R(z) = \frac{\sinh(Rz)}{R}$$

Elliptic case

$$\varpi_{R,\rho}(z) = R\frac{z^2}{2} - \log(2R)z - \frac{\pi^2}{12R} +$$

$$+\sum_{n=0}^{\infty} \frac{1}{2R} \left(\mathrm{Li}_2\left(e^{2\pi \mathrm{i} n\rho} e^{-2Rz}\right) - \mathrm{Li}_2\left(e^{2\pi \mathrm{i}(n+1)\rho} e^{2Rz}\right)\right),$$

$$\exp \varpi'_{R,\rho}(z) = \frac{1}{2\mathrm{i}R} \frac{\theta_{11}(2\mathrm{i}Rz;\rho)}{\theta'_{11}(0;\rho)}$$

Supersymmetric vacua of the quiver gauge theory

$$\exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1\,, \qquad i \in \mathrm{Vert}_{\gamma}, \alpha \in [N_i]$$

Supersymmetric vacua of the quiver gauge theory

$$\exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1\,, \qquad i \in \mathrm{Vert}_{\gamma}, \alpha \in [N_i]$$

Correspond to Bethe equations of a spin chain with $Y(\mathfrak{g}_{\gamma})$ symmetry

Supersymmetric vacua of the quiver gauge theory

$$\exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1 \,, \qquad i \in \mathrm{Vert}_{\gamma}, \alpha \in [N_i]$$

q-character formulation

$$\exp \frac{\partial \tilde{W}}{\partial \sigma_{i,\alpha}} = 1\,, \qquad i \in \mathrm{Vert}_{\gamma}, \alpha \in [N_i]$$

Can be reformulated as the system of conditions for the q-characters

$$\mathcal{T}_i(x) := \mathcal{Y}_i(x + 2u) +$$

$$+\mathfrak{q}_i \ell_i(x) \frac{\prod_{e \in s^{-1}(i)} \mathcal{Y}_{t(e)}(x + \mu_e + u) \prod_{e \in t^{-1}(i)} \mathcal{Y}_{s(e)}(x - \mu_e + u)}{\mathcal{Y}_i(x)} + \ldots$$

to have no singularities in x

except for the poles coming from $\ell_i(x)$'s

Partition function on T^2

$$\mathrm{Tr}_{\mathcal{H}_{\mathsf{susy}}[(N_i)]}(-1)^F \exp - \sum_k t_k \mathcal{O}_k^{(0)}$$

$\uparrow$

Bethe/gauge correspondence

$\downarrow$

Gibbs ensemble partition function in the weight $\vec{N}$ subspace

$$\mathrm{Tr}_{\mathcal{H}_{\mathsf{QIS}}[(N_i)]} \exp - \sum_k t_k \widehat{H}_k$$

Partition function on T^2

$$\mathrm{Tr}_{\mathcal{H}_{\mathsf{susy}}[(N_i)]}(-1)^F \exp - \sum_k t_k \mathcal{O}_k^{(0)} \sim \sum_{\mathrm{vac}} e^{-\sum_k t_k \langle \mathcal{O}_k \rangle_{\mathrm{vac}}}$$

assuming all vacua are bosonic

$$\uparrow$$

Bethe/gauge correspondence

$$\downarrow$$

Gibbs ensemble partition function in the weight $\vec{N}$ subspace

$$\mathrm{Tr}_{\mathcal{H}_{\mathsf{QIS}}[(N_i)]} \exp - \sum_k t_k \widehat{H}_k$$

Partition function on T^2 of the ensemble of gauge theories

$$\mathcal{Z} = \sum_{(N_i)} \prod_i \tilde{\mathfrak{q}}_i^{N_i} \, \mathrm{Tr}_{\mathcal{H}_{\mathsf{susy}}[(N_i)]} (-1)^F \exp - \sum_k t_k \mathcal{O}_k^{(0)}$$

$$\uparrow$$

Bethe/gauge correspondence

$$\downarrow$$

Toroidal Lattice model Partition function

$$\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}_{\mathsf{QIS}}} \prod_i \tilde{\mathfrak{q}}_i^{\widehat{N}_i} \exp - \sum_k t_k \widehat{H}_k$$

Questions

$\downarrow$

- Why sum over $\vec{N}$?

- Why choose $t_k \mathcal{O}_k$ in such a way, that

$$\exp - \sum_k t_k \widehat{H}_k = \widehat{T}(x_1; \mathfrak{q}) \ldots \widehat{T}(x_{\tilde{L}}; \mathfrak{q})\,?$$

Questions

- Why choose $t_k \mathcal{O}_k$ in such a way, that

$$\exp - \sum_k t_k \widehat{H}_k = \widehat{T}(x_1; \mathfrak{q}) \ldots \widehat{T}(x_{\tilde{L}}; \mathfrak{q})\,?$$

$\widehat{T}(x; \mathfrak{q})$ turns out to be a natural observable
within the twisted chiral ring

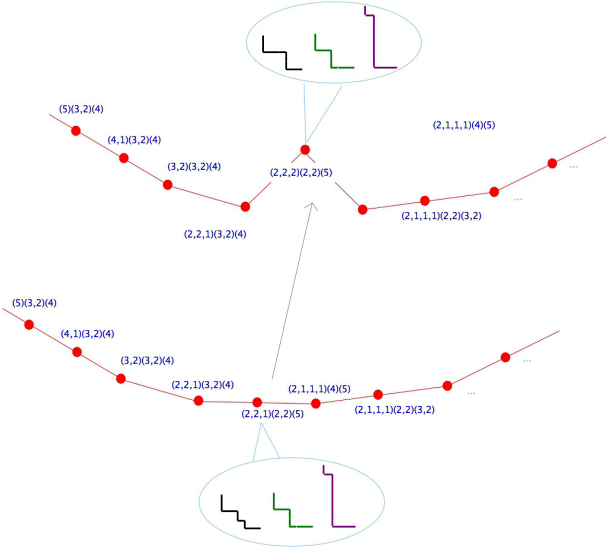
q-characters

$\widehat{T}(x;\mathfrak{q})$ turns out to be a natural observable
within the twisted chiral ring

Well-behaved with respect to the

non-perturbative Dyson-Schwinger relations

Non-perturbative Dyson-Schwinger relations



Non-perturbative Dyson-Schwinger relations

Contributions of topologically distinct sectors to the path integral are related to each other

$$\Leftrightarrow$$

Analytic properties of $\langle \widehat{T}(x;\mathfrak{q})\rangle$, e.g. no poles in x

Remarks

$\downarrow$

- When the quiver γ is one of the affine Dynkin diagrams

- Bethe equations correspond to the spin chains
  with Kac-Moody spin groups

- There are gauge theories corresponding to the super-Lie algebras

- For general γ — a wild Lie algebra $\mathfrak{g}_\gamma$

Remarks

What has changed compared to the old results of Nakajima et al.

Unlike simple Lie groups, Yangians, quantum affine algebras, etc. have inequivalent maximal commutative subalgebras

To see them all, we need

the $\mathfrak{q}$-parameters: Kähler moduli

of the two dimensional theory

not visible at the level of supersymmetric quantum mechanics!

Remarks

Max commutative = Bethe subalgebras

at the level of supersymmetric quantum mechanics, $\mathfrak{q} \to 0$

become Gelfand-Zetlin subalgebras

Nazarov, 1995

Remarks

The original formulation of Bethe/gauge correspondence

mostly concerned with the commutative (quantum integrals) subalgebra

The non-abelian structure provides rigidity
and offers an exciting perspective

on the string landscape of vacua

Remarks

The original formulation of Bethe/gauge correspondence:

The non-abelian structure comes from domain walls

viewed as operators in the spirit of S-branes

Gutperle, Strominger, 2002

Remarks

Recent progress:

The non-abelian structure, i.e. R-matrices

can be understood mathematically using the stable envelope basis

Maulik, Okounkov 2012; Aganagic, Okounkov 2016

Questions

$\downarrow$

If one replaces the R-matrices with spectral parameters by the R-matrices without (finite quantum group $U_q(\mathfrak{g})$), one can describe the lattice model using Chern-Simons theory with gauge group G in three dimensions

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{Tr} \left(A dA + \frac{2}{3} A^3 \right)$$

Witten 1989

- How to introduce the spectral parameter into Chern-Simons theory?

Cohomological field theory perspective

Start with CohFT with the moduli space $\mathcal{M}$ of solutions

Fields/Equations/Symmetries paradigm:
d-dimensional fields

$$\mathcal{Q}^2 = 0$$

Correlations functions: integrals of products of cohomology classes of $\mathcal{M}$

$$\langle \mathcal{O}_1 \dots \mathcal{O}_p \rangle^d \sim \int_{\mathcal{M}} \Omega_1 \wedge \dots \wedge \Omega_p$$

$$\{\mathcal{Q}, \mathcal{O}_i\} = 0, \qquad \mathcal{O}_i \leftrightarrow \Omega_i, \qquad d\Omega_i = 0$$

Loop upgrade

NN, PhD. thesis 1996

Baulieu, NN, Losev, 1997

Oxidation of cohomological field theory:
make fields t-dependent
K-theory of $\mathcal{M}$
Fields/Equations/Symmetries paradigm:
loop space, i.e. $d+1$-dimensional fields

Correlations functions: pushforwards of K-theory classes of $\mathcal{M}$

$$\mathcal{Q}^2 = \partial_t$$

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle^{d+1} \sim \int_{\mathcal{M}} \widehat{A}(\mathcal{M}) \wedge \Omega_1 \wedge \ldots \wedge \Omega_p$$

Double Loop upgrade

NN, PhD. thesis, 1996

Baulieu, NN, Losev, 1997

Costello, 2013

Oxidation of cohomological field theory:

make fields $z, \bar{z}$-dependent

Ell-cohomology of $\mathcal{M}$

Fields/Equations/Symmetries paradigm:

double loop space, i.e. $d+2$-dimensional fields

Correlations functions: pushforwards in elliptic cohomology of $\mathcal{M}$

$$\mathcal{Q}^2 = \partial_{\bar{z}}$$

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle^{d+2} \sim \int_{\mathcal{M}} \widehat{Ell}(\mathcal{M}) \wedge \Omega_1 \wedge \ldots \wedge \Omega_p$$

3d CS = loop upgrade of 2d YM

$\mathcal{M}$ = moduli space of $G-$ flat connections on Σ

$\mathcal{N} = 2$ $d = 2$ super-Yang-Mills theory, twisted version

$$\mathcal{Q}A = \Psi \, , \mathcal{Q}\Psi = D_A \sigma \, , \mathcal{Q}\sigma = 0$$

$$\mathcal{Q}\chi = H \, , \mathcal{Q}H = [\sigma, \chi] \, , \mathcal{Q}\bar{\sigma} = \eta \, , \mathcal{Q}\eta = [\sigma, \bar{\sigma}]$$

Review of the cohomological field theory on

$\mathcal{M} =$ moduli space of $G-$ flat connections on Σ

$\mathcal{N} = 2$ $d = 2$ super-Yang-Mills theory, twisted version

$$\mathcal{Q}A = \psi \, , \mathcal{Q}\psi = D_A \sigma \, , \mathcal{Q}\sigma = 0$$

$$\mathcal{Q}\chi = H \, , \mathcal{Q}H = [\sigma, \chi] \, , \mathcal{Q}\bar{\sigma} = \eta \, , \mathcal{Q}\eta = [\sigma, \bar{\sigma}]$$

$$S_0 = \mathcal{Q} \int_\Sigma \mathrm{Tr} \left(\chi \left(\mathrm{i} F_A - g^2_{\mathsf{YM}} \star H \right) + \psi \wedge \star D_A \bar{\sigma} + \eta [\sigma, \bar{\sigma}] \right)$$

$\uparrow$

Bare action

Review of 2d YM as deformation of SYM

$\mathcal{M} =$ moduli space of $G -$ flat connections on Σ

$\mathcal{N} = 2$ $d = 2$ super-Yang-Mills theory, twisted version

$$S_0 + \mathrm{i}\kappa \int_\Sigma \mathrm{Tr}\left(\sigma F_A + \frac{1}{2}\psi \wedge \psi\right)$$

$$\uparrow$$

2-observable, viewed as deformation of the action

Twisted F-term in the physical theory, $\widetilde{W} = \frac{\kappa}{2}\mathrm{Tr}\sigma^2$

Review of 2d YM

$\mathcal{N} = 2$ $d = 2$ super-Yang-Mills theory, twisted version

$$S_0 + \mathrm{i}\kappa \int_\Sigma \mathrm{Tr}\left(\sigma F_A + \frac{1}{2}\psi \wedge \psi\right)$$

$$\uparrow$$

2-observable, viewed as deformation of the action
Twisted F-term in the physical theory, $\widetilde{W} = \frac{\kappa}{2}\mathrm{Tr}\sigma^2$

Twisted $\bar{F}$-term, $\widetilde{W}^* = \frac{\bar{\kappa}}{2}\mathrm{Tr}\bar{\sigma}^2$
shifts the action by the $\mathcal{Q}$-exact term

$$+\mathrm{i}\bar{\kappa}\int_\Sigma \mathrm{Tr}\left(\bar{\sigma}H + \eta\chi\right)$$

Review of 2d YM

Take the limit $\bar{\kappa} \to \infty$

$$\mathrm{i}\bar{\kappa} \int_\Sigma \mathrm{Tr}\left(\bar{\sigma} H + \eta\chi\right)$$

The quartet $\bar{\sigma}, H, \eta, \chi$ decouples: and we are left with A, ψ, σ

Witten 1992

$$\mathcal{Q}A = \psi\,, \mathcal{Q}\psi = D_A\sigma\,, \mathcal{Q}\sigma = 0$$

$$S = \mathrm{i}\kappa \int_\Sigma \mathrm{Tr}\left(\sigma F_A + \frac{1}{2}\psi \wedge \psi\right)$$

$\uparrow$

2-observable, becomes the action

Add 0-observable $t\,\mathrm{Tr}\sigma^2 \Longrightarrow$ 2d Yang-Mills theory

Loop upgrade

$$S = \mathrm{i}\kappa \int_\Sigma \mathrm{Tr}\left(\sigma F_A + \frac{1}{2}\psi \wedge \psi\right)$$

$$\downarrow$$

$$S_{CS} = \frac{k}{4\pi} \int_{\Sigma \times \mathbb{S}^1} \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 + \psi\psi\right)$$

Double Loop upgrade

NN, PhD. thesis 1996, proposed to explain the representation theory of quantum affine algebras

$$S = \mathrm{i}\kappa \int_{\Sigma} \mathrm{Tr}\left(\sigma F_A + \frac{1}{2}\psi \wedge \psi\right)$$

$$\downarrow$$

$$S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 + \psi\psi\right)$$

where dz is a holomorphic one-differential on E:

an elliptic curve, a cylinder, or a plane

Recent revival, Costello 2013

Susy of the Double Loop upgrade

$$S_{4dCS} = \kappa \int_{\Sigma\times E} dz \wedge \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 + \psi\psi \right)$$

is $\mathcal{Q}$-invariant, with

$$\mathcal{Q}A_m = \psi_m\,,\ \mathcal{Q}\psi_m = F_{m\bar{z}}\,,\ \mathcal{Q}A_{\bar{z}} = 0$$

$$\mathcal{Q}A_z = \eta\,,\ \mathcal{Q}\eta = F_{z\bar{z}}\,,$$

$$\mathcal{Q}\chi = H\,,\ \mathcal{Q}H = D_{\bar{z}}\chi$$

$m = 1, 2 \longrightarrow$ coordinates on Σ

Anomaly of the Double Loop upgrade

$$S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 + \psi\psi \right)$$

when E is an elliptic curve, is not gauge invariant

$$S_{4dCS} \longrightarrow S_{4dCS} + \kappa \int_{\Sigma \times E} dz \wedge \text{integral } 3 - \text{form}$$

under large gauge transformations

Anomaly of the Double Loop upgrade

$$S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 + \psi\psi \right)$$

when E is an elliptic curve, is not gauge invariant

$$S_{4dCS} \longrightarrow S_{4dCS} + \kappa \int_{\Sigma \times E} dz \wedge \text{integral } 3 - \text{form}$$

under large gauge transformations, incommensurate periods...

Double Loop upgrade of $\mathcal{N} = 4$ $d = 4$ theory

$$S_{4dCS} = \kappa \int_{\Sigma \times E} dz \wedge \mathrm{Tr}\left(AdA + \frac{2}{3}A^3 + \psi\psi \right)$$

when E is an elliptic curve, is not gauge invariant

$$S_{4dCS} \longrightarrow S_{4dCS} + \kappa \int_{\Sigma \times E} dz \wedge \text{integral } 3 - \text{form}$$

under large gauge transformations: incommensurate periods...

String theory realization

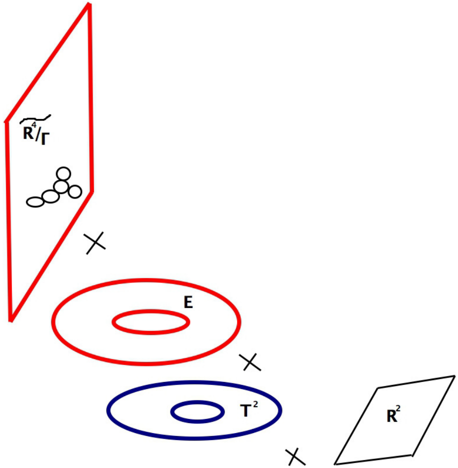
String theory realization

Various puzzles will be resolved

Different approaches will be connected

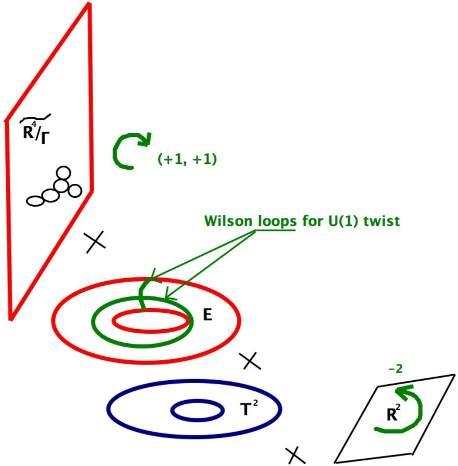
String theory realization

IIB string on $ALE \times \mathbb{R}^2 \times E \times T^2$



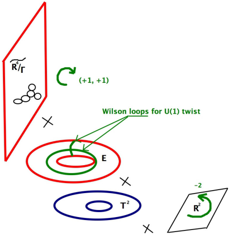
String theory realization

IIB string on $\left(\textcolor{teal}{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u \textcolor{red}{E} \times T^2$



String theory realization

IIB string on $\left(\mathrm{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u E \times T^2$

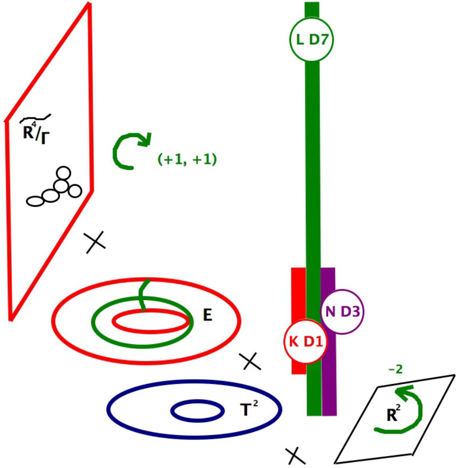


ALE with $\zeta_i^{\mathbb{C}} = 0 \Longrightarrow U(1)$-isometry
ALE is twisted with a line bundle $\mathcal{L}_u$ over E
$\mathbb{R}^2$ is twisted with a line bundle $\mathcal{L}_u^{-2}$ over E

String theory realization of our gauge theory

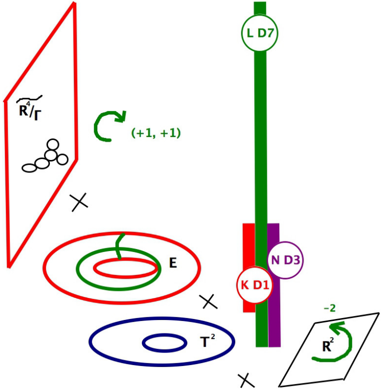
IIB string on $\left(\mathrm{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u \, E \times T^2$

with fractional D-branes



String theory realization of our gauge theory

IIB string on $\left(\mathrm{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u E \times T^2$
with $LD7$-branes on $\mathrm{ALE} \times E \times T^2$
with $ND3$-branes on $E \times T^2$
with $KD1$-branes on E



String theory realization of our gauge theory

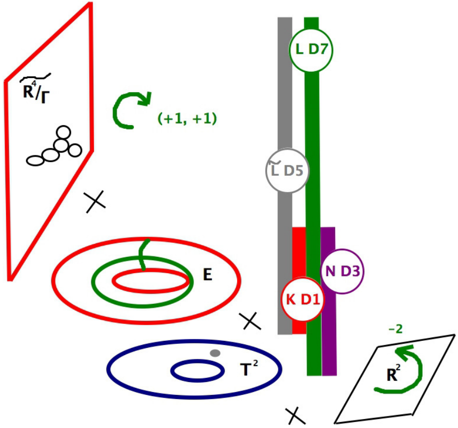
IIB string on $\left(\mathrm{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u E \times T^2$
with $LD7$-branes on $\mathrm{ALE} \times E \times T^2$
with $ND3$-branes on $E \times T^2$
with $KD1$-branes on E
Fractionalization: $(L, N, K) \longrightarrow (L_i, N_i, K_i)$
Compact branes to be summed over

String theory realization of the q-character

NN, 2015

IIB string on $\left(\mathrm{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u E \times T^2$

with fractional $D7, D3, D1$-branes and $D5$-branes in addition



String theory realization of the q-character

IIB string on $\left(\mathrm{ALE} \times \mathbb{R}^2\right) \tilde{\times}_u E \times T^2$
with $LD7$-branes on $\mathrm{ALE} \times E \times T^2$
with $ND3$-branes on $E \times T^2$
with $KD1$-branes on E
with $\tilde{L}D5$-branes on $\mathrm{ALE} \times E$

String theory realization of the q-character

with $ND3$-branes on $E \times T^2$

with $\tilde{L}D5$-branes on $\mathrm{ALE} \times E$

This is similar to the construction of crossed instantons
qq-character and q-character observables
in 4d and 5d supersymmetric gauge theories

NN, Pestun 2012; NN, Pestun Shatashvili 2013; NN 2015-

String theory realization

Now we shall get a 6d-ish version

of Chern-Simons theory, dual

to the collection of quiver gauge theories

String theory realization

Now we shall get a 6d-ish version of CS theory

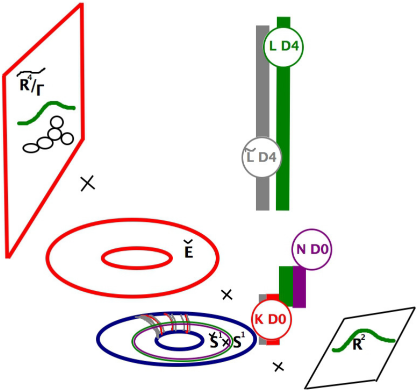
using T-dual string background(s)

T-dual description: electric frame

T-duality along E and one of the circles in T^2

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1$

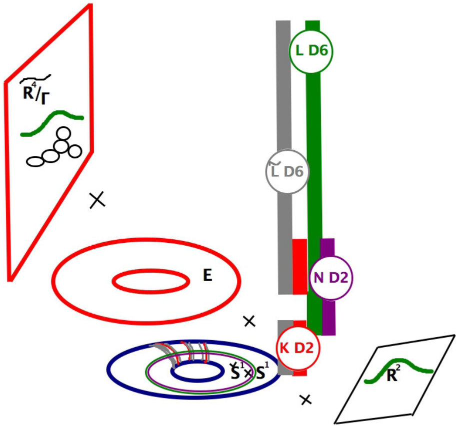
with fractional D-branes



T-dual description: magnetic frame

T-dualize one of the circles in T^2

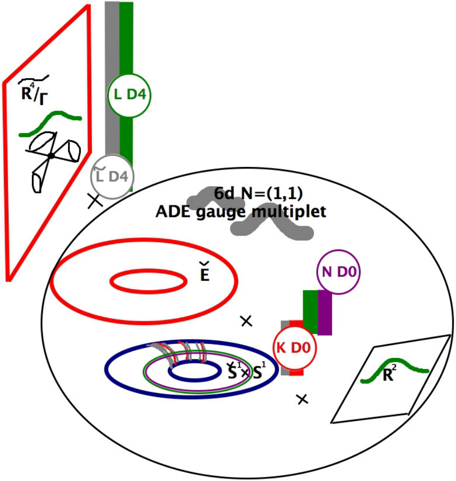
IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times E \times \check{S}^1 \times S^1$
with fractional D-branes



Six dimensional super-Yang-Mills

Witten; Strominger; Greene, Morrison, Strominger; Bershadsky, Sadov, Vafa, 1995

IIA string on ALE

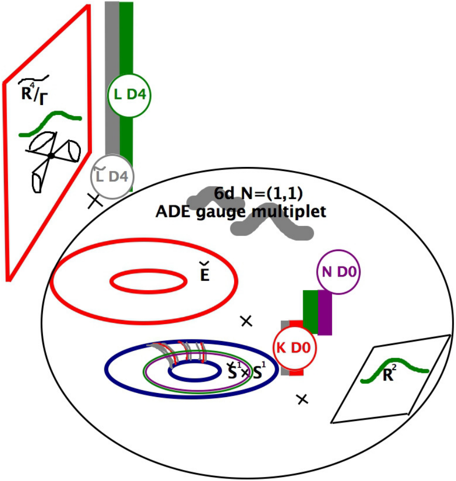


Six dimensional super-Yang-Mills: electric frame

IIA string on $\text{ALE} \times \mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1$

with fractional D-branes $\Longrightarrow$ electric sources

more details below



Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times E \times \check{S}^1 \times S^1$

with fractional D-branes $\Longrightarrow$ magnetic sources

Twist by $\mathcal{L}_u \Longrightarrow$ 6d Ω-deformation of SYM

Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times E \times \check{S}^1 \times S^1$

with fractional D-branes $\Longrightarrow$ magnetic sources

Twist by $\mathcal{L}_u \Longrightarrow$ 6d Ω-deformation of SYM
preserving $\mathcal{N} = (2,2)$ $d = 2$ super-Poincare invariance
with $\mathbf{2}$ out of 4 scalars remaining massless: root of a Higgs branch

Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times E \times \check{S}^1 \times S^1$

with fractional D-branes $\Longrightarrow$ magnetic sources

Twist by $\mathcal{L}_u \Longrightarrow$ 6d Ω-deformation of SYM

In the limit of vanishing size $E \Longrightarrow \mathcal{N} = 2^*$ theory in 4d
with special Ω-deformation, $m = -\varepsilon \Longrightarrow$ massless chiral in 2d

Six dimensional super-Yang-Mills: magnetic frame

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times E \times \check{S}^1 \times S^1$

with fractional D-branes $\Longrightarrow$ magnetic sources

Twist by $\mathcal{L}_u \Longrightarrow$ 6d Ω-deformation of SYM

In the limit of vanishing size $E \Longrightarrow \mathcal{N} = 2^*$ theory in 4d
with special Ω-deformation, $m = -\varepsilon \Longrightarrow$ massless chiral in 2d
magnetic membranes reduce to
susy 't Hooft operators wrapped on A and B cycles on $\check{S}^1 \times S^1$

Six dimensional super-Yang-Mills: electric frame

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1$

with fractional D-branes $\Longrightarrow$ electric sources

Twist by $\mathcal{L}_u$ upon T-duality on E

produces the Neveu-Schwarz B-field, with $H = dB \neq 0$

Six dimensional Chern-Simons theory

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1$

with electric sources

with the Neveu-Schwarz B-field, with $H = dB \neq 0 \Longrightarrow$

$$\int_{\mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1} H \wedge CS(A)$$

from the $\int C \wedge G \wedge G$ Chern-Simons term in 11d

Four dimensional Chern-Simons theory

IIA string on $\mathrm{ALE} \times \mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1$

Neveu-Schwarz B-field, so that $H = dB \neq 0$

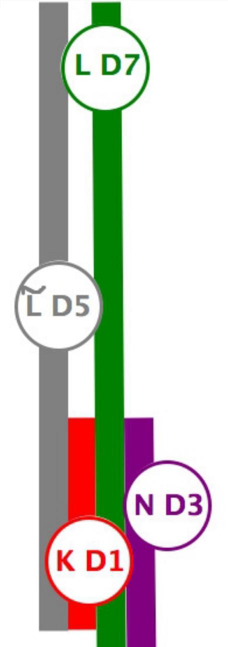
$$\int_{\mathbb{R}^2} H \sim \mathrm{Re}\frac{dz}{u}$$

Supersymmetric localization $\Longrightarrow$

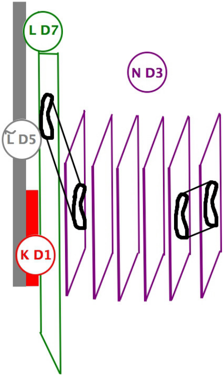
$$\int_{\mathbb{R}^2 \times \check{E} \times \check{S}^1 \times S^1} H \wedge CS(\mathcal{A}) = \frac{1}{u} \int_{\check{E} \times \check{S}^1 \times S^1} dz \wedge CS(\mathcal{A})$$

Up to $\mathcal{Q}$-exact terms, $\mathcal{A} = A + \mathrm{i}(\dots)$

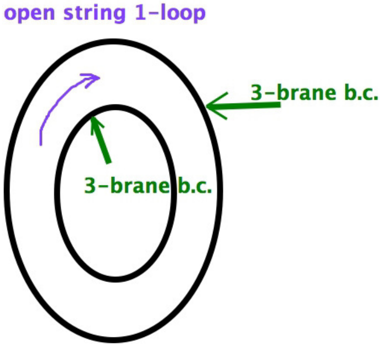
Open-closed string duality



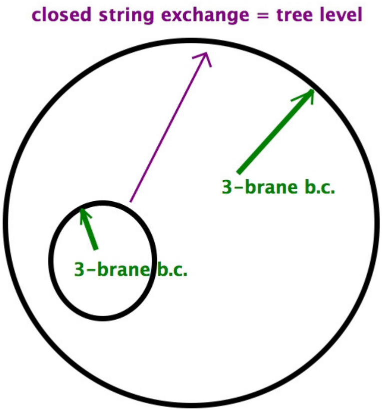
Open-closed string duality



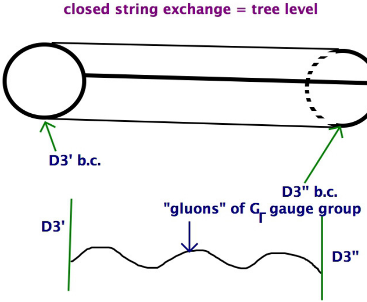
Open-closed string duality



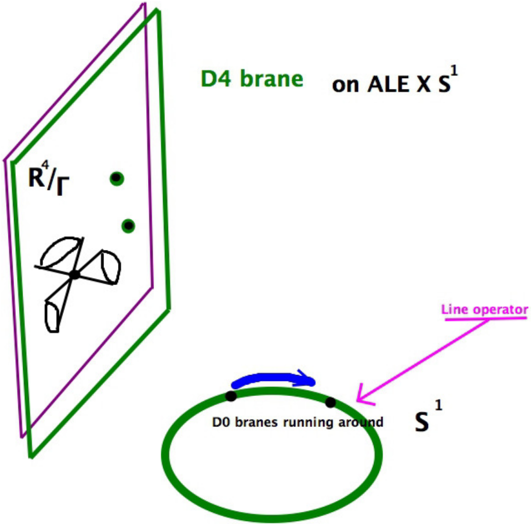
Open-closed string duality



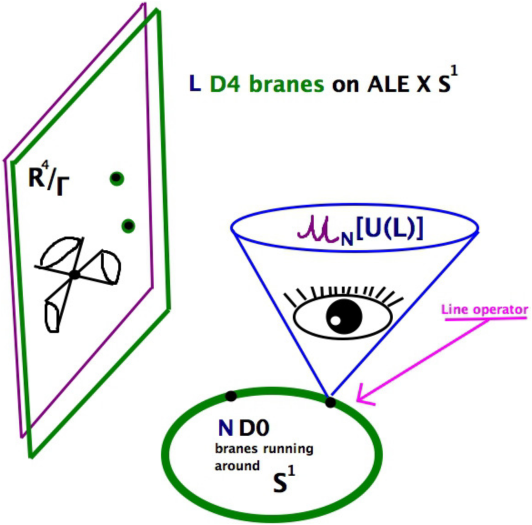
Open-closed string duality



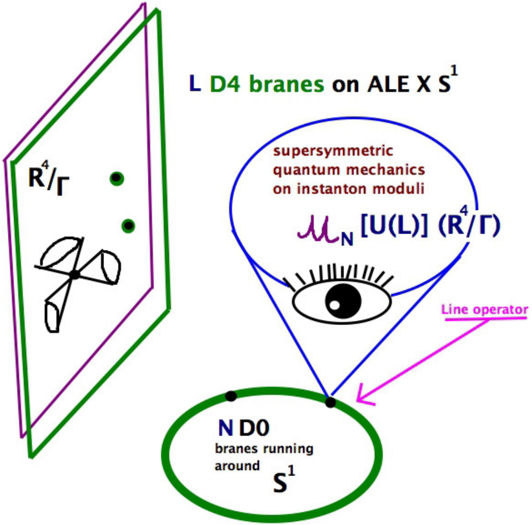
Line operators



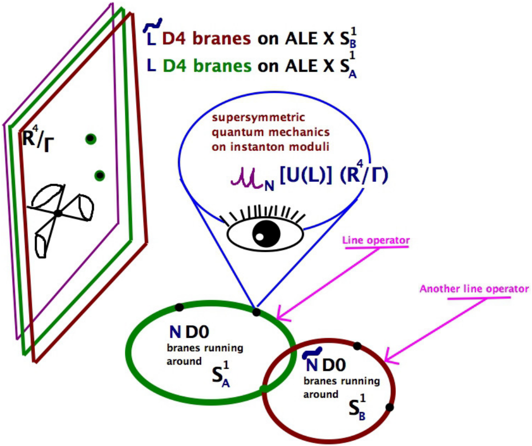
Line operators and instanton moduli



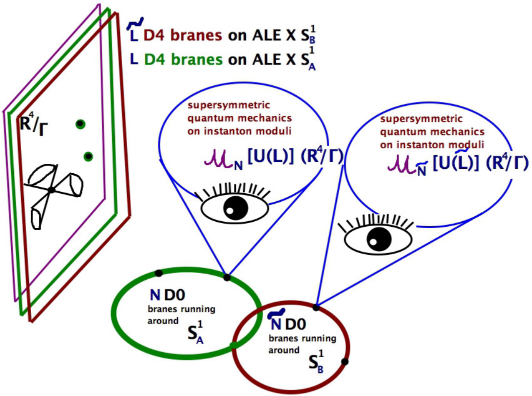
Line operators and instanton moduli



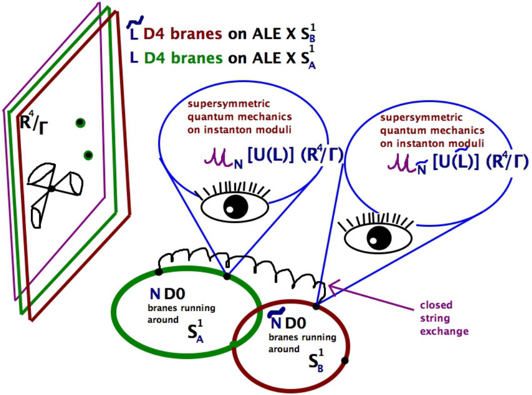
Two kinds of line operators



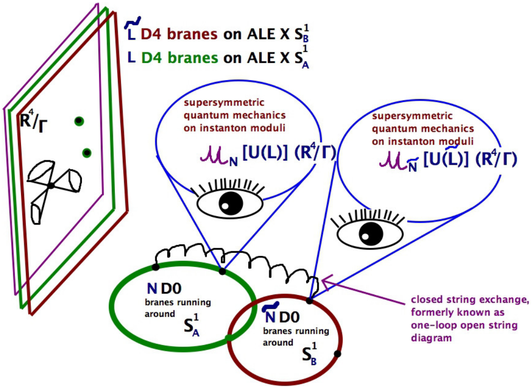
Two kinds of line operators and instanton moduli



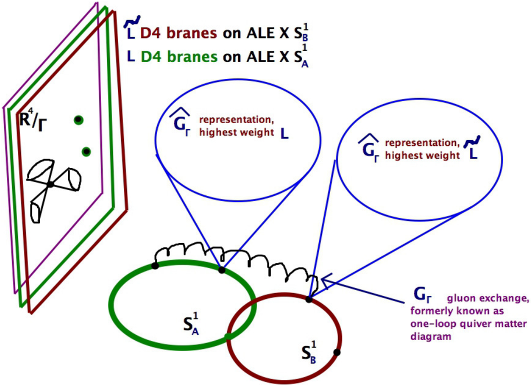
String exchanges between line operators



String exchanges between line operators



Gluon exchanges between $\widehat{\text{Wilson}}$ loops



$\widehat{\text{Wilson}}$ **loop** = **Wilson loop with a hat**

$$\widehat{W}_{R_{\widehat{\lambda}}}[C,q] = \mathrm{Tr}_{R_{\widehat{\lambda}}}\left(q^{L_0}\, P\exp\oint_C A \right) ,$$

for the highest weight representation $R_{\widehat{\lambda}}$ of $\widehat{G_\Gamma}$ Kac-Moody group

$\widehat{\text{Wilson}}$ **loop** = **Wilson loop with a hat**

$$\widehat{W}_{R_{\widehat{\lambda}}}[C,q] = \mathrm{Tr}_{R_{\widehat{\lambda}}}\left(q^{L_0}\, P\exp\oint_C A \right) ,$$

Can be expanded in ordinary G_Γ - Wilson loops,

with higher spin representations suppressed by powers of q

Dictionary

$$\widehat{W}_{R_{\widehat{\lambda}}}[C,q] = \mathrm{Tr}_{R_{\widehat{\lambda}}} \left(q^{L_0}\, P \exp \oint_C A \right) ,$$

In our story, the two kinds of line operators we encounter correspond to

$$\widehat{\lambda} = \sum_{i=0}^{r} L_i \varpi_i \,, \qquad \text{and} \qquad \widehat{\lambda} = \sum_{i=0}^{r} \tilde{L}_i \varpi_i$$

respectively, with ϖ_i being the fundamental weights of $\widehat{\mathfrak{g}_\Gamma}$

Dictionary: weight subspaces

$$R_{\widehat{\lambda}} = \bigoplus_{\widehat{w}} R_{\widehat{\lambda}}^{\widehat{w}},$$

In our story, the two weight subspaces we encounter correspond to

$$\widehat{w} = \sum_{i=0}^{r} L_i \varpi_i - N_i \alpha_i \,, \qquad \text{and} \qquad \widehat{w} = \sum_{i=0}^{r} \tilde{L}_i \varpi_i - \tilde{N}_i \alpha_i$$

respectively, with $\tilde{N}_i = K_i$, and α_i being the simple roots of $\widehat{\mathfrak{g}_\Gamma}$

Twist parameters

The parameters $\mathfrak{q}_i$ and $\tilde{\mathfrak{q}}_i$

$$\leftrightarrow$$

background $G_\Gamma^{\mathbb{C}}$-flat connection on $\check{S}^1 \times S^1$

Twist parameters

The parameters $\mathfrak{q}_i$ and $\tilde{\mathfrak{q}}_i$

$$\leftrightarrow$$

background $G_\Gamma^{\mathbb{C}}$-flat connection on $\check{S}^1 \times S^1$

Can be fixed in the six-dimensional setup (in 4d problematic)

Virasoro

$\mathfrak{q}_i$ and $\tilde{\mathfrak{q}}_i$ $\leftrightarrow$ flat $G_{\Gamma}^{\mathbb{C}}$-connection on $\check{S}^1 \times S^1$

and the parameters q and $\tilde{q}$ of the $\widehat{\text{Wilson}}$ loop operators

$$q = \prod_{i \in \mathrm{Vert}_{\gamma}} \mathfrak{q}_i^{a_i}\,, \qquad \tilde{q} = \prod_{i \in \mathrm{Vert}_{\gamma}} \tilde{\mathfrak{q}}_i^{a_i}$$

In string theory:

$$q = \exp - \frac{L_{\check{S}^1} M_s}{g_s} + \mathrm{i} \int_{\check{S}^1} C_{(1)}\,, \qquad \tilde{q} = \exp - \frac{L_{S^1} M_s}{g_s} + \mathrm{i} \int_{S^1} C_{(1)}$$

$C_{(1)}$ = Background IIA Ramond-Ramond $U(1)$ flat gauge field on $\check{S}^1 \times S^1$

Virasoro and M-theory

$\mathfrak{q}_i$ and $\tilde{\mathfrak{q}}_i$ $\leftrightarrow$ flat $G_\Gamma^{\mathbb{C}}$-connection on $\check{S}^1 \times S^1$

and the parameters q and $\tilde{q}$ of the $\widehat{\text{Wilson loop}}$ operators

$$q = \prod_{i \in \mathrm{Vert}_\gamma} \mathfrak{q}_i^{a_i}\,, \qquad \tilde{q} = \prod_{i \in \mathrm{Vert}_\gamma} \tilde{\mathfrak{q}}_i^{a_i}$$

Lift to M-theory: line operators become $M5$ branes

wrapped on $\mathrm{ALE} \times \left(\check{S}^1 \,\mathrm{or}\, S^1\right) \times S^1_{10}$, respectively

q, $\tilde{q}$ – elliptic curve nodes for $\left(\check{S}^1 \,\mathrm{or}\, S^1\right) \times S^1_{10}$

Little strings

Berkooz, Rosali, Seiberg; Seiberg 1997

Losev, Moore, Shatashvili 1997

Reviews Aharony 1999; Kutasov 2001

in BPS/CFT context, (2,0) version, Aganagic, Haouzi, 2016

Take the limit $g_s \to 0$, keeping M_s finite: $q, \tilde{q} \to 0$

The $\widehat{\text{Wilson}}$ loop operator becomes the ordinary one

Decouple one of the nodes, e.g. the affine one

$$\widehat{\mathfrak{g}_\Gamma} \to \mathfrak{g}_\Gamma\,,\ \mathfrak{q}_0 \to 0\,,\ \tilde{\mathfrak{q}}_0 \to 0$$

$$L_0 = K_0 = 0 = \tilde{L}_0 = N_0$$

In lieu of conclusions

The Chern-Simons $\int dz \wedge CS(A)$ approach to quantum groups has the advantage of making the group whose quantum deformation one is seeking, visible in the structure of sources

Seems problematic for groups outside the ADE (BCFG) classification

Bethe/gauge correspondence does not have the explicit G_{Γ} symmetry but is more general (covers all quivers and also super-algebras)

Lots of things to learn and understand better...

One wild speculation

Naively, to describe affine quiver theories

One would attempt to study the LG_Γ, or $\widehat{G_\Gamma}$ gauge theory

One additional dimension: 7d theory? natural in M-theory on ALE

However, we learned: $U(1)_{L_0} \subset \widehat{G_\Gamma}$ is the 10d RR $U(1)$ gauge field

Perhaps we'll learn about the origin of the (12d?) E_8 gauge field

Hořava, Witten 1996; Witten 1997; Diaconescu, Moore, Witten 2000

Lots of things to learn and understand better...

THANK YOU