Periodic Monopoles and qOpers

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Geometric Langlands Correspondence

Let C be Riemann surface. Then there is a conjectured equivalence

geometric Langlands

$$\mathcal{D}\mathrm{mod}(\mathrm{Bun}_G(C)) \simeq {}'\mathrm{QCoh}'(\mathrm{Loc}_{^LG}(C))$$

Beilinson,Deligne,Drinfeld,Laumon

Arinkin,Gaitsgory,Frenkel,Lafforgue,Lurie,Mircovic,Vilonen

talk by Donagi

geometric Langlands as a mirror symmetry

$$A_{\epsilon^{-1}\Omega_I}(\mathrm{Hit}_G(C)) \simeq B_{J_\epsilon}(\mathrm{Hit}_{^LG}(C))$$

Hitchin, Hausel, Thaddeus, Donagi, Pantev, Arinkin, Bezrukavnikov, Braverman

Bershadsky,Johansen,Sadov,Vafa

Kapustin,Witten Teschner

Canonical Coisotropic Brane and Opers

geometric Langlands as a A-B mirror

$$A_{\epsilon^{-1}\Omega_I}(\mathrm{Hit}_G(C)) \simeq B_{J_\epsilon}(\mathrm{Hit}_{^LG}(C))$$

In particular

$$\mathcal{B}_{\text{cc-brane}}(A_{\epsilon^{-1}\Omega_I}) \leftrightarrow \mathcal{B}_{\text{opers}}(B_{J_\epsilon})$$

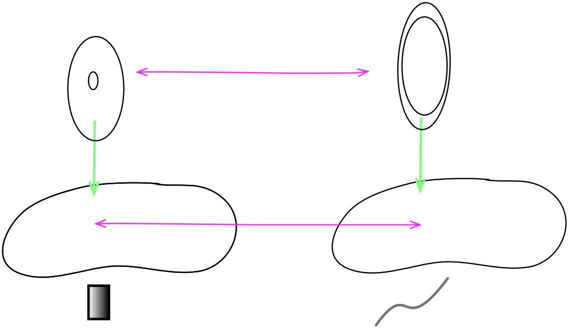
$\mathcal{B}_{\text{cc-brane}}$ is the A-model canonical space-filling brane [Kapustin-Orlov] = 'quantized algebra of functions on T^*Bun_G' = the sheaf $\mathcal{D}$ in $\mathcal{D}\mathrm{mod}(Bun_G)$

$\mathcal{B}_{\text{opers}} := \text{mirror}(\mathcal{B}_{\text{cc-brane}})$
The $\mathcal{B}_{\text{cc-brane}}$ is holomorphic Lagrangian brane in $\mathrm{Hit}_{^LG, J_\epsilon}$

In the limit $\epsilon = 0$ (see talk by Donagi) we have

'baby' geometric Langlands as B-B model mirror

$$B_I(\mathrm{Hit}_G(C)) \simeq B_I(\mathrm{Hit}_{^LG}(C))$$



$$\mathcal{B}_{\text{space-filling brane}} \longleftrightarrow \mathcal{B}_{\text{Hitchin section}} \tag{1}$$

$\mathcal{B}_{\text{opers}}$ is HyperKahler rotation of $\mathcal{B}_{\text{Hitchin section}}$ in the limit

$$\zeta_{\text{twistor}} = \epsilon R, \qquad R \to 0$$

where $R \in \mathbb{R}_+$ rescales the Higgs field in the real moment map ω_I

$$F_A - R^2 \phi \wedge \phi = 0$$

Gaoitto,Moore,Neitzke

Gaiotto

Dumitrescu, Fredrickson, Rydonakis, Mazzeo, Mulase, Neitzke

In this limit $\mathrm{Hit}_{{}^L G, J_\epsilon}$ as a complex space is the space of flat ϵ-holomorphic connections

$$\mathrm{Loc}_{{}^L G}(C)_{J_\epsilon} = \{G\text{-bundle on } C,\ \epsilon\text{-connection } \epsilon \partial_z + A_z\} \tag{2}$$

quantum geometric Langlands

Feigin, Frenkel, Gaitsgory, Kapustin, Witten

quantum geometric Langlands as a A-A mirror

$$A_{\epsilon_1^{-1}\Omega_{J_{\epsilon_2}}}(\mathrm{Hit}_G(C)) \simeq A_{\epsilon_2^{-1}\Omega_{J_{\epsilon_1}}}(\mathrm{Hit}_{^LG}(C))$$

$$W_\beta(\mathfrak{g}) \simeq W_{^L\beta}(^L\mathfrak{g})$$

where

$$^L\beta = -\frac{\epsilon_1}{\epsilon_2} = r(^Lk + {}^Lh^\vee) = \frac{1}{k + h^\vee} = \frac{r}{\beta}$$

- $W_{\beta=\infty}(^L\mathfrak{g}) \simeq \mathcal{O}(\mathcal{B}_{^L\mathfrak{g},\mathrm{opers}}(\mathbb{C}^\times)) \simeq Z(U_{-h^\vee}(\hat{\mathfrak{g}}))$
- $W_\beta(^L\mathfrak{g}) \simeq \mathcal{O}_{\beta^{-1}}(\mathcal{B}_{^L\mathfrak{g},\mathrm{opers}}(\mathbb{C}^\times))$

$W_\mathfrak{g}$-algebra is quantization of the Poisson algebra of functions on $\mathcal{B}_{\mathfrak{g},\mathrm{opers}}(\mathbb{C}^\times)$ with quantization parameter β^{-1} Kostant, Drinfeld-Sokolov, Feigin-Frenkel

geometric Langlands

Depending on parameters ϵ_1, ϵ_2 we are dealing with different level of complexity of geometric Langlands duality:

- 'baby' geometric Langlands: $\epsilon_1 = \epsilon_2 = 0$
  classical Hitchin integrable system
- 'ordinary' geometric Langlands: $\epsilon_1 \neq 0, \epsilon_2 = 0$
  quantum Hitchin integrable system / critical level / classical monodromy / classical commutative W-algebra
- 'quantum' geometric Langlands: $\epsilon_1 \neq 0, \epsilon_2 \neq 0$
  2d CFT / chiral vertex algebra / associative W-algebra

Dealing with representation theory of $U\hat{\mathfrak{g}}$, differential equations, KZ equations, CFT conformal blocks, 4d Nekrasov partition functions

10d IIB strings on
$X_4 \times C \times \mathbb{C}^2/\Gamma_{ADE}$

↓

6d $\mathcal{N} = (0,2)$ theory
of type $\mathfrak{g}_{ADE}$ on $X_4 \times C$

↓

4d $\mathcal{N} = 2$ theory
of class $\mathcal{S}_G(C)$ on X_4

↓

$\mathrm{Hit}_G(C)$ is the Coulomb
branch of vacua for $\mathcal{S}_G(C)$
theory on $X_4 = \mathbb{R}^3 \times S^1$

The class S theory on Ω-background $X_4 = \mathbb{C}^2_{\epsilon_1,\epsilon_2}$ gives us a microscope to nail down geometric Langlands

Nekrasov,Shatashivili
Alday,Gaiotto,Tachikawa
Gaiotto,Moore,Neitzke
Teschner
Kapustin, Witten
Nekrasov,Witten

Proposal for q-geometric Langlands

(c.f. Feigin,Frenkel,Reshetikhin, more recently Aganagic, Frenkel, Okounkov and Saponov)

- $G \simeq {}^L G$ a compact Langlands self-dual Lie group
- C a flat Riemann surface, e.g. C is $\mathbb{C}$, $\mathbb{C}^*$ or $\mathcal{E}$
- $\mathrm{Mon}_G(C \times S^1)$ is the moduli space of G-monopoles on $C \times S^1$ with prescribed singularities at a coweight colored divisor on C
- flat action of abelian group $\mathbb{C}$ on C, so that $\epsilon \in \mathbb{C}$ acts by shift

$$z \mapsto z + \epsilon$$

  where C is identified with $\mathbb{C}$, $\mathbb{C}/\mathbb{Z}$, or $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$
- a constant section dz of K_C; it induces holomorphic symplectic form Ω_I on $\mathrm{Mon}_G(C \times S^1)$

Proposal for q-geometric Langlands

Conjecturally there are equivalences of the (derived) categories associated to the hyperKahler moduli space of periodic monopoles $\mathrm{Mon}_G(C \times S^1)$

'baby' q-geometric Langlands is B-B mirror

$$B_I(\mathrm{Mon}_G(C \times S^1)) \simeq B_I(\mathrm{Mon}_{^LG}(C \times_{\epsilon_1} S^1))$$

'ordinary' q-geometric Langlands is A-B mirror

$$A_{\epsilon_1^{-1}\Omega_I}(\mathrm{Mon}_G(C \times S^1)) \simeq B(\mathrm{Mon}_{^LG}(C \times_{\epsilon_1} S^1))$$

'quantum' q-geometric Langlands is A-A mirror

$$A_{\epsilon_1^{-1}\Omega_{J_{\epsilon_2}}}(\mathrm{Mon}_G(C \times_{\epsilon_2} S^1)) \simeq A_{\epsilon_2^{-1}\Omega_{J_{\epsilon_1}}}(\mathrm{Mon}_{^LG}(C \times_{\epsilon_1} S^1))$$

Here $C \times_{\epsilon_1} S^1$ is C fibered over S^1 with a twist ϵ.

rational/trigonometric/elliptic

Type of $C = Jac(\check{C})$

$\check{C}$	$\check{\mathcal{E}}_{\text{cusp}}$	$\check{\mathcal{E}}_{\text{nod}}$	$\check{\mathcal{E}}$
C	$\mathbb{C}$	$\mathbb{C}^*$	$\mathcal{E}$
	4d	5d	6d
	rational	trigonometric	elliptic
	cohomology	K-theory	elliptic cohomology
	Yangian algebra	quantum affine algebra	elliptic quantum group
	difference	q-difference	elliptic q-difference

q-lift / categorification / K-theory version of various objects of (quantum) geometric Langlands and CFT.

Integrable system of periodic monopoles

Phase space

The space $\mathrm{Mon}_G(C \times S^1)_I$ is the holomorphic phase space of integrable system of group valued Higgs bundles on C

Hurtubuise,Markman   Kapustin, Cherkis   Charbonneau, Hurtubise
Nekrasov, VP

Group Higgs bundles

$$\mathrm{Mon}_G(C \times S^1)_I = \{G\text{-bundle on } C, \text{ meromorphic section } g \text{ of } Ad_G \text{ bundle}\}$$

Let G be simply connected simple Lie group.

Hamiltonians

The ring of commuting Hamiltonians is generated by fundamental characters $\chi_{R_i}(g(z))$ where R_i denotes the irreducible representation with fundamental highest weight λ_i.

$\mathrm{Mon}_G(C \times S^1_R)$ is hyperKahler.

There is a family of holomorphic structures on $\mathrm{Mon}_G(C \times S^1)$ fibered over $\mathbb{CP}^1_{\text{twistor}}$.

There is a convenient description of $\mathrm{Mon}_G(C \times S^1_R)_\zeta$ in the limit $\zeta \to 0, R \to \infty$ with fixed

$$\epsilon = \zeta R$$

aka 'conformal limit' Gaiotto, and talks by Holland and Bridgeland

The resulting complex space $\mathrm{Mon}(C \times S^1)_{J_\epsilon}$ is equivalent to the complex space of ϵ-difference connections

Finite difference ϵ-connection

$$\mathrm{Mon}_{^LG}(C \times S^1)_{J_\epsilon} = \{{}^LG\text{-bundle } \mathcal{P} \text{ on } C, \text{ mero morphism } \epsilon^*\mathcal{P} \to \mathcal{P}\}$$

The space $\mathrm{Mon}_{^LG}(C \times S^1)_{J_\epsilon}$ is q-geometric Langlands version of the space of flat ϵ-connections $\mathrm{Loc}_{^LG}(C)_{J_\epsilon}$ of the B-side of the ordinary geometric Langlands.

Main question

In the ordinary geometric Langlands the key role plays the question of finding the image $\mathcal{B}_{\text{opers}}$ of the quantization brane $\mathcal{B}_{\text{cc-brane}}$ under the mirror map. The answer leads to effective solution of quantum integrable system.

Beilinson,Drinfeld, Kapustin,Witten, Gukov, Witten, Nekrasov,Rosly,Shatashvili Teschner, Gaiotto, Neitzke-Holland

q-geometric Langlands A-B mirror

Can we compute the **q-oper brane**, that is can we find the image of the canonical coisotropic quantization brane

$$\mathcal{B}_{\text{cc-brane}} \to \mathcal{B}_{q\text{-oper}}$$

under the q-geometric Langlands equivalence

$$A_{\epsilon_1^{-1}\Omega_I}(\mathrm{Mon}_G(C \times S^1)) \to B(\mathrm{Mon}_{^L G}(C \times_{\epsilon_1} S^1)) \qquad ?$$

What does it mean to compute?

(α, β) coordinates and the generating function $W(\alpha)$ of **q-opers**

To 'compute' Lagrangian brane of **q-opers** means for us to find a suitable system of Darboux coordinates (α^i, β_i) in $\mathrm{Mon}_{^LG}(C \times_{\epsilon_1} S^1)$ and the generating function $W(\alpha)$ such that

$$\beta_i = \frac{\partial W}{\partial \alpha^i}$$

In the case of ordinary $\mathfrak{sl}_2$ **opers** on $\mathbb{P}^1$ with 4 regular singularities the problem has been solved by Nekrasov, Rosly, Shatashvili.
The generalization to $\mathfrak{sl}_n$ **opers** and irregular punctures has been addressed by Neitzke-Hollands, presented in the talk by Hollands, also see talk by Bridgeland.

q-oper solution

In this talk we will find the solution of **q-oper** problem.

1. Give geometric definition of the space of q-difference connections that are **q-opers**
$$\mathcal{B}_{\text{q-opers}} \subset \mathrm{Mon}_{{}^L G}(C \times_\epsilon S^1)$$
2. Define Darboux coordinates (α^i, β_i) in $\mathrm{Mon}_{{}^L G}(C \times_\epsilon S^1)$
3. Present the generating function $W(\alpha)$ such that
$$\beta_i = \frac{\partial W}{\partial \alpha^i}$$
is the graph of $\mathcal{B}_{\text{q-opers}}$.

The presented technic applies to any compact Lie group G, and any flat curve C, in the talk we consider concrete examples.

q-oper definition

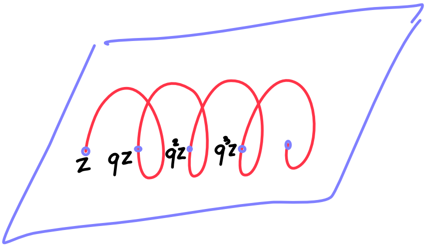
From now on fix the q-difference case $C = \mathbb{C}^\times$, denote by $z \in \mathbb{C}^\times$ the multiplicative coordinate, and set

$$q = e^\epsilon$$

to be multiplicative shift

$$z \mapsto qz$$

A q-connection $A(z)$ defines the parallel transport, i.e. q-difference equation on a trivializing section



$$s(qz) = A(z)s(z)$$

where $A \in G(\mathbb{C}^*)$ is a G-valued function of z

c.f. talk by Okounkov

gauge transformation:
$A(z) \to g(qz)A(z)g(z)^{-1}$

What does it mean for G-group valued q-connection $A(z)$ to be a q-oper?

Recall that

A $\mathfrak{g}$-oper for $\mathfrak{g}$ valued connection $\partial_z + a_z$ comes from Kostant section (Kostant'59-'63)

$$s_K : \mathfrak{g}/G \to \mathfrak{g}$$

of the adjoint Lie algebra quotient $\pi : \mathfrak{g} \to g/G$, see talk by Ben-Zvi.

After fixing a Borel subalgebra, Kostant section yields an element $a \in \mathfrak{g}$ by its conjugacy class.

Example of Kostant section for $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}_2$ and fix a conjugacy class $u = \frac{1}{2}\mathrm{tr}\, x^2$ of a regular $a \in \mathfrak{g}$. Then Kostant section is

$$a = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}$$

The $\mathfrak{sl}_2$-oper is a connection of the form $\partial_z + \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}$, talk Hollands, Ben-Zvi

Parallel to the work of Kostant'59-'63 for the Lie algebra, there is construction of Steinberg'65 section for the Lie group.

Example of Steinberg section for SL_2

Let $G = SL_2$ and fix a conjugacy class $t_1 = \mathrm{tr} g$ of a regular $g \in G$. Then Steinberg section is

$$g = \begin{pmatrix} t_1 & 1 \\ -1 & 0 \end{pmatrix}$$

Example of Steinberg section for SL_n

Fix a conjugacy class of a regular $g \in G$ by the fundamental characters $t_k = \chi_{R_k}(g)$ where $\chi_{R_k} = \mathrm{tr}_{\Lambda^k \mathbb{C}^n} g$. Then Steinberg section is

$$g = \begin{pmatrix} t_1 & t_2 & t_3 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

Definition of Steinberg section for simple G

Fix $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ and let $e_i \in \mathfrak{g}_+$ be the standard generators of weight α_i where α_i are simple roots with $i = 1 \dots r$. Let $s_i \in N(T)$ be the Weyl reflections in simple roots α_i, in particular $s_1 \dots s_r$ is Coxeter element. Then an element

$$g(t) = \prod_{i=1}^{r} s_i \exp(-e_i t_i), \qquad t_i \in \mathbb{C}$$

is Steinberg section: there is an isomorphism (i.e. polynomial map in both directions) between affine spaces of the parameters $(t_1, \dots, t_r)$ and the affine space of the fundamental characters $(\chi_1, \dots, \chi_r)$

Caution

In the SL_{r+1} example we find

$$t_i = \chi_i, \qquad i = 1 \dots r$$

but in general the map between χ_i and t_i is not identity.

For example for $SO(8)$ with Dynkin graph **Y** we find

$$\begin{aligned} t_1 &= \chi_1 \\ t_2 &= \chi_2 + 1 \\ t_3 &= -\chi_3 \\ t_4 &= -\chi_4 \end{aligned}$$

where χ_1 is vector, χ_2 is adjoint, χ_3, χ_4 are spinors.

Definition of q-oper

A q-oper in $\mathrm{Mon}_G(\mathbb{C}^* \times_q S^1)$ on C is the following data

- a reduction of the structure group of the G-bundle to a Borel subgroup B
- a q-connection $A(z)$ in the form of Steinberg section

$$A(z) = \prod_{i=1}^{r} s_i \exp(-e_i t_i(z))$$

Frenkel, Semenov-Tian-Shansky, Sevostyanov

Example of SL_2 q-oper

$$A(z) = \begin{pmatrix} t_1(z) & 1 \\ -1 & 0 \end{pmatrix}$$

Example of SL_n q-oper

$$A(z) = \begin{pmatrix} t_1(z) & t_2(z) & t_3(z) & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

Example of SO_8 q-oper

Pick a basis in the fundamental representation of SO_8 such that the metric has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and choose the conventional basis of simple roots. Then SO_8 q-oper is

$$A(z) = \begin{pmatrix} t_1(z) & t_2(z) & t_3(z)t_4(z) & t_4(z) & 0 & 0 & -1 & t_3(z) \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t_4(z) & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t_1(z) & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -t_2(z) & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -t_3(z) & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Isomorphism to the base

The Steinberg affine map from the parameters $(t_1(z), \ldots, t_r(z))$ of Steinberg section to the space of adjoint invariants defined by the fundamental characters $(\chi_1(z), \ldots, \chi_r(z))$ provides canonical holomorphic isomorphism between the brane of q-opers $\mathcal{B}_{\text{q-opers}}$ and the base of $\mathrm{Mon}_G(C \times S^1)$ integrable system.

Darboux coordinates in the q-Char variety

The power of Nekrasov, Rosly, Shatashvili construction comes from certain system of distinguished coordinates (α, β) in the character variety $\mathrm{Char}_G(C)$ (representation of the fundamental group $\pi_1(C)$ in G).

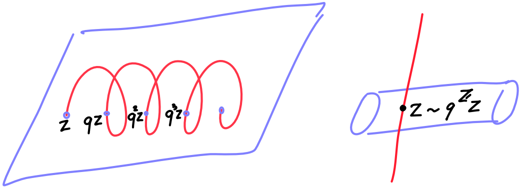
Riemann-Hilbert

The character variety $\mathrm{Char}_G(C)$ is isomorphic to the space $\mathrm{Loc}_G(C)$ of the pairs (holomorphic G-bundle, holomorphic flat connection $\partial_z + a_z$) but the isomorphism is complex analytic, rather than algebraic.

This isomorphism requires to compute the monodromies of the flat connection $(\partial_{\bar{z}}, \partial_z + a_z)$ and is called Riemann-Hilbert correspondence.

Unwrap the spirals

Similarly, for the $\mathrm{Mon}_G(\mathbb{C}^* \times_q S^1)$ we need to introduce the coordinates (α, β) in the space $\mathrm{qChar}_G(\mathbb{C}^*)$ – the analogue of the character variety.



To construct qChar variety we look at space $\mathbb{C}^* \times_q S^1$ as the family of spirals$\simeq \mathbb{R}$ fibered over the elliptic curve $\tilde{C} = \mathbb{C}^*/q^{\mathbb{Z}}$.

The qChar variety

$$\mathrm{qChar}_G(\mathbb{C}^*) = \mathrm{Mon}_G(\mathcal{E}_q \times \mathbb{R}_t)$$

The holomorphic description is given along the rays $\mathbb{R}_t$ from $t = -\infty$ to $t = +\infty$.

Monopole scattering data

So we shall look for canonical coordinates (α, β) in the space $\mathrm{Mon}_G(\tilde{C} \times \mathbb{R})$.

This space is well-understood after the work of Hitchin on monoles in $\mathbb{R}^3$. In fact, if $\tilde{C}$ were $\mathbb{C}$

$$\mathrm{Mon}_G(\mathbb{C} \times \mathbb{R})_n \simeq \mathrm{Maps}_n(\mathbb{P}^1, G/B)$$

where the monopole charge n takes values in the coroot lattice of G. The key idea is that we can filter the solutions to the parallel transport equation along the rays $\mathbb{R}$

$$D_t s = 0$$

and construct two flags according to the asymptotics of growth as $t \to +\infty$ or as $t \to -\infty$. Birkhoff,Stokes,Hitchin,Hurtubise,Jarvis, c.f. talk by Hollands. To specify a flag is equivalent to specify a reduction of G-bundle structure to B-bundle.

Example of qChar for SL_2

For SL_2 charge k monopoles we expect $Maps(\tilde{C}, \mathbb{P}^1)_k$, i.e. degree k rational rational functions. Suppose that scattering monodromy from $t = -\infty$ to $t = \infty$ modulo B transformations is

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

The invariant data is the ratio

$$\frac{b(z)}{a(z)} = \sum_{i=1}^{n} \frac{\beta_i}{z - \alpha_i}$$

where $\alpha_i \in \tilde{C}$ are locally flat relative to dz, and β_i are the residues.

The system (α, β) provides canonical coordinates for SL_2 qChar

$$\{\alpha_i, \beta_j\} = \delta_{ij}\beta_j$$

Hitchin, Donaldson, Hurtubise, Jarvis, Gerasimov, Harchev, Lebedev, Oblezin, Finkelberg, Kuznetsov, Markarian, Mirkovic, Braverman

Hitchin scattering

For SL_2 monopoles on $\tilde{C} \times \mathbb{R}$ there is two-dimensional space of solutions of parallel transport along $\mathbb{R}$ parametrize by the points $z \in \tilde{C}$.

$$D_t s = 0$$

Let $s_\pm(z,t)$ be the two solutions of minimal growth as $t \to \pm\infty$, they specify two lines $L_+ \subset \mathbb{C}^2$ and $L_- \subset \mathbb{C}^2$.

$$0 \subset L_+(z) \subset \mathbb{C}^2$$
$$0 \subset L_-(z) \subset \mathbb{C}^2$$

For generic z the two lines $L_+(z)$ and $L_-(z)$ are in generic position with $L_+ \cup L_- = 0$.
Still it could happen that at some point $z_* \in \tilde{C}$ the lines $L_+(z_*)$ and $L_-(z_*)$ coincide. The set of such points z_* are α_i coordinates.

From this special solution $s(\alpha_i, t)$ of minimal growth at $t \to \pm\infty$ we find the conjugated coordinate β_i as the abelian monodromy, that is ratio

$$\beta_i = \frac{\lim_{t\to\infty} s(\alpha_i, t) e^{-\lambda_+ t}}{\lim_{t\to-\infty} s(\alpha_i, t) e^{-\lambda_- t}}$$

in situation when the minimal growth solution has regular asymptotics with fixed values of $\lambda_\pm$ coming from the boundary data of monopoles Mon at infinity.

For non-regular growth we use more general suitable basis of normalizing coefficents.

The GL_n-difference equations and difference Hilbert-Riemann correspondence have been adressed since the ancient times Birkhoff'1913, and more recent work by Baranovsky-Ginzburg, Jimbo-Sakai, Borodin, Krichever,Ramis, Sauloy, Zhang, Etingof, Singer, Vizio, Kontsevich, Soibelman , and c.f. talk by Okounkov
The twistor geometry of the periodic monopoles provides a new perspective on this ancient story.

The construction (α, β) coordinates for monopole scattering problem of $Maps(\tilde{C}, G/B)$ has natural generalization for arbitrary simple Lie algebra G, and introduction of singularities.

In generic case we have $\sum_{i=1}^{r} n_i$ pairs of coordinates $(\alpha_{i,j}, \beta_{i,j})$ where $\sum n_i \alpha_i^{\vee}$ is monopole charge, with $j = 1 \dots n_i$.
The coordinates $\alpha_{i,j}$ are the points on $\tilde{C}$ in which the map to G/B lands in the divisor colored by the simple root α_i.

Some versions of *qChar*-varieties for $\mathrm{Mon}_G(\tilde{C} \times \mathbb{R})$, have appeared under different names such as rational/trigonometric/elliptic Zastava Finkelberg et.al, Braverman et.al, Beilinson-Drinfeld Grassmanian Gerasimov et.al, or the fiber of Hecke correspondence Kapustin-Witten.

Separation of Variables / Abelianization / q-Miura transformation

By gauge transformation $\tilde{A}(z) = g(qz)A(z)g^{-1}(z)$ of the q-connection $A(z)$ in the equation $s(qz) = A(z)s(z)$
the q-oper

$$A(z) = \begin{pmatrix} t_1(z) & 1 \\ -1 & 0 \end{pmatrix}$$

can be converted into the lower triangular form with

$$\tilde{A}(z) = \begin{pmatrix} Y^{-1}(z) & 0 \\ -1 & Y(z) \end{pmatrix}$$

and

$$t_1(z) = Y(qz) + Y^{-1}(z)$$

The variables $Y_i(z)$ can be thought as generalized eigenvalues of Kac-Moody group element represented by the Steinberg section $t_1(z), \ldots, t_r(z)$.

Now we can integrate abelianized equation. Define $Q_i(z)$ such that

$$Q_i(qz) = Y_i(qz)Q_i(z)$$

and take the solution $Q_i(z) \to 1$ at $z \to 0$ (assuming that 0 is regular singuarity with generalized eigenvalues $Y_i(z) \to 1$).

The $Q_i(z)$ generically blows up along the ray z/q^n as $n \to +\infty$.

But for certain rays $\alpha_{i,j}q^{\mathbb{Z}}$ the function $Q_i(\alpha_{i,j}q^k)$ has the asymptotics of minimal growth, say

$$Q_i(zq^k) \sim \beta_{i,j}\mathfrak{q}_{\mathfrak{i}}{}^k$$

where $\mathfrak{q}_{\mathfrak{i}}$ is the minimal eigenvalue of generalized root type eigenvalue $Y_i(z)$ at $z \to \infty$.
This gives canonical coordinates $(\alpha_{i,j}, \beta_{i,j})$

Generating function

Proposition

The 5d K-theoretic ADE quiver gauge theory partition function Z on $\mathbb{C}^2_{q_1,q_2} \times S^1$ is the generating function of the $\mathcal{B}_{q-oper}$ in the qChar_G in coordinates (α, β) in a sense that $\mathcal{B}_{q-oper}$ is Lagrangian defined by the graph

$$\beta_{i,j} = \lim_{q_2 \to 1} Z(q_2 a_{i,j}, \dots,)/Z(a_{i,j}, \dots)$$

The expresion of $t_i(z)$ in terms of the generalized eigenvalues $Y_i(z)$ is called q-character Frenkel, Reshetikhin, Semenov-Tian-Shansky, Sevostyanov.

It coincides with the q-character coming from the quiver gauge theory construction Nekrasov, VP, Shatashvili'13.

Quiver gauge theory in Ω-background

Nakajima Douglas, Moore Kapustin, Cherkis VP, Nekrasov '12

6d $\mathcal{N} = (0,1)$ Γ_{ADE}-quiver gauge theory on $X_4 \times \check{C}$

↓

4d $\mathcal{N} = 2$ $\Gamma_{ADE}(C)$ quiver gauge theory on X_4

↓

$\mathrm{Mon}_G(C \times S^1)$ is the Coulomb branch of vacua for $\Gamma_{ADE}(C)$ theory on $X_4 = \mathbb{R}^3 \times S^1$

The Γ-quiver gauge theory Ω-background $X_4 = \mathbb{C}^2_{\epsilon_1,\epsilon_2}$ for $C = \mathbb{C}^*$ opens the window into the q-geometric quantum Langlands

Nekrasov,VP, Shatashivili '13

Nekrasov'15-'16

VP, Kimura'15-'16

Dankeschön!