String Math 2017
Hamburg, Germany

Conformal Bootstrap in Two Dimensions

Xi Yin

Harvard University

based on works with Bruno Balthazar (Harvard)
Minjae Cho (Harvard)
Scott Collier (Harvard)
Petr Kravchuk (Caltech)
Ying-Hsuan Lin (Caltech)
Victor Rodriguez (Harvard)
Shu-Heng Shao (IAS)
David Simmons-Duffin (IAS & Caltech)
Yifan Wang (Princeton)

String Math 2017
Hamburg, Germany

Conformal Bootstrap in Two Dimensions

Xi Yin

Harvard University

based on works with	Bruno Balthazar (Harvard)	[1508.07305]
	Minjae Cho (Harvard)	[1511.04065]
	Scott Collier (Harvard)	[1608.06241]
	Petr Kravchuk (Caltech)	[1610.05371]
	Ying-Hsuan Lin (Caltech)	[1702.00423]
	Victor Rodriguez (Harvard)	[1703.09805]
	Shu-Heng Shao (IAS)	[1705.05865]
	David Simmons-Duffin (IAS & Caltech)	[1705.07151]
	Yifan Wang (Princeton)	

1. Motivations and questions

2. Modular constraints

3. Crossing equation and spectral function

4. Comments on superconformal theories

5. Genus two modular bootstrap

Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov ’84, Friedan-Shenker ’87, Segal ’87, Moore-Seiberg ’88]

Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov ’84, Friedan-Shenker ’87, Segal ’87, Moore-Seiberg ’88]

1. There is a local, conserved stress-energy tensor $\Rightarrow$ Virasoro symmetry with central charge c

Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov ’84, Friedan-Shenker ’87, Segal ’87, Moore-Seiberg ’88]

1. There is a local, conserved stress-energy tensor $\Rightarrow$ Virasoro symmetry with central charge c

2. Local operators organize into (unitary) representations of Virasoro algebra.

Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov '84, Friedan-Shenker '87, Segal '87, Moore-Seiberg '88]

1. There is a local, conserved stress-energy tensor $\Rightarrow$ Virasoro symmetry with central charge c

2. Local operators organize into (unitary) representations of Virasoro algebra.

3. There is an associative operator product expansion.

Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov '84, Friedan-Shenker '87, Segal '87, Moore-Seiberg '88]

1. There is a local, conserved stress-energy tensor ⇒ Virasoro symmetry with central charge c

2. Local operators organize into (unitary) representations of Virasoro algebra.

3. There is an associative operator product expansion.

4. Modular invariance (or covariance) of torus one-point function.

Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov ’84, Friedan-Shenker ’87, Segal ’87, Moore-Seiberg ’88]

1. There is a local, conserved stress-energy tensor $\Rightarrow$ Virasoro symmetry with central charge c

2. Local operators organize into (unitary) representations of Virasoro algebra.

3. There is an associative operator product expansion.

4. Modular invariance (or covariance) of torus one-point function.

Non-local operators and boundary states are beyond the scope of this talk.

2D CFTs: the known knowns

2D CFTs: the known knowns

1. The compact and rational: coset models

2D CFTs: the known knowns

1. The compact and rational: coset models

2. The compact and irrational: free orbifolds, Calabi-Yau models

2D CFTs: the known knowns

1. The compact and rational: coset models

2. The compact and irrational: free orbifolds, Calabi-Yau models

3. The noncompact and irrational: Liouville/Toda theories, noncompact cosets (e.g. cigar CFT)

2D CFTs: the known knowns

1. The compact and rational: coset models

2. The compact and irrational: free orbifolds, Calabi-Yau models

3. The noncompact and irrational: Liouville/Toda theories, noncompact cosets (e.g. cigar CFT)

4. Rational CFTs: analytic constraints on spectrum (e.g. finite dimensional modular representation) and fusion rules (Verlinde formula)

2D CFTs: the known knowns

1. The compact and rational: coset models

2. The compact and irrational: free orbifolds, Calabi-Yau models

3. The noncompact and irrational: Liouville/Toda theories, noncompact cosets (e.g. cigar CFT)

4. Rational CFTs: analytic constraints on spectrum (e.g. finite dimensional modular representation) and fusion rules (Verlinde formula)

5. Superconformal theories: much known about BPS sector and conformal manifold, not much beyond

2D CFTs: the known unknowns

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

   e.g. Landau-Ginzburg/Calabi-Yau models, deformed symmetric product orbifolds. In principle accessible using conformal perturbation theory starting from solvable points on conformal manifold, technically challenging.

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

   e.g. Landau-Ginzburg/Calabi-Yau models, deformed symmetric product orbifolds. In principle accessible using conformal perturbation theory starting from solvable points on conformal manifold, technically challenging.

   Important for AdS_3/CFT_2 e.g. D1-D5 CFT

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

e.g. Landau-Ginzburg/Calabi-Yau models, deformed symmetric product orbifolds. In principle accessible using conformal perturbation theory starting from solvable points on conformal manifold, technically challenging.

Important for AdS_3/CFT_2 e.g. D1-D5 CFT

2. Potentially irrational compact CFTs as infrared fixed points of (non-supersymmetric) RG flows

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

e.g. Landau-Ginzburg/Calabi-Yau models, deformed symmetric product orbifolds. In principle accessible using conformal perturbation theory starting from solvable points on conformal manifold, technically challenging.

Important for AdS_3/CFT_2 e.g. D1-D5 CFT

2. Potentially irrational compact CFTs as infrared fixed points of (non-supersymmetric) RG flows

e.g. coupled Potts models [Dotsenko et al '98]

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

e.g. Landau-Ginzburg/Calabi-Yau models, deformed symmetric product orbifolds. In principle accessible using conformal perturbation theory starting from solvable points on conformal manifold, technically challenging.

Important for AdS_3/CFT_2 e.g. D1-D5 CFT

2. Potentially irrational compact CFTs as infrared fixed points of (non-supersymmetric) RG flows

e.g. coupled Potts models [Dotsenko et al '98]

3. Non-unitary conformal supergroup sigma models

2D CFTs: the known unknowns

1. Strongly coupled SCFTs beyond BPS sector

e.g. Landau-Ginzburg/Calabi-Yau models, deformed symmetric product orbifolds. In principle accessible using conformal perturbation theory starting from solvable points on conformal manifold, technically challenging.

Important for AdS_3/CFT_2 e.g. D1-D5 CFT

2. Potentially irrational compact CFTs as infrared fixed points of (non-supersymmetric) RG flows

e.g. coupled Potts models [Dotsenko et al '98]

3. Non-unitary conformal supergroup sigma models

arise on the worldsheet of superstrings in AdS

[Bershadsky, Zhukov, Vaintrob '99, Berkovits, Vafa, Witten '99, Berkovits '00, '04]

2D CFTs: the unknown unknowns

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the “simplest” RCFTs - extremal meromorphic CFTs with $c=24k$, is not known for $k>1$.

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the "simplest" RCFTs - extremal meromorphic CFTs with $c=24k$, is not known for $k>1$.
(despite various efforts to construct or rule out such theories)

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the "simplest" RCFTs - extremal meromorphic CFTs with $c=24k$, is not known for $k>1$.
(despite various efforts to construct or rule out such theories)

2. What values of central charge $c>1$ can be realized by compact unitary CFTs? (Put differently, can a unitary Virasoro algebra always be embedded in a compact modular invariant theory?)

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the "simplest" RCFTs - extremal meromorphic CFTs with $c=24k$, is not known for $k>1$.
(despite various efforts to construct or rule out such theories)

2. What values of central charge $c>1$ can be realized by compact unitary CFTs? (Put differently, can a unitary Virasoro algebra always be embedded in a compact modular invariant theory?)

3. Are there compact unitary 2D CFTs with nonzero twist gap (in Virasoro primaries)? Is $(c-1)/12$ the optimal upper bound on the twist gap?

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the “simplest” RCFTs - extremal meromorphic CFTs with c=24k, is not known for k>1.
(despite various efforts to construct or rule out such theories)

2. What values of central charge c>1 can be realized by compact unitary CFTs? (Put differently, can a unitary Virasoro algebra always be embedded in a compact modular invariant theory?)

3. Are there compact unitary 2D CFTs with nonzero twist gap (in Virasoro primaries)? Is (c-1)/12 the optimal upper bound on the twist gap?

4. Are there really compact CFTs of large c and large gap in the spectrum of Virasoro primaries? (i.e. does pure quantum gravity in AdS3 exist?)

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the "simplest" RCFTs - extremal meromorphic CFTs with c=24k, is not known for k>1.
(despite various efforts to construct or rule out such theories)

2. What values of central charge c>1 can be realized by <u>compact</u> unitary CFTs? (Put differently, can a unitary Virasoro algebra always be embedded in a compact modular invariant theory?)

3. Are there <u>compact</u> unitary 2D CFTs with nonzero twist gap (in Virasoro primaries)? Is (c-1)/12 the optimal upper bound on the twist gap?

4. Are there really compact CFTs of large c and large gap in the spectrum of Virasoro primaries? (i.e. does pure quantum gravity in AdS3 exist?)

c.f. [Ooguri, Vafa '16]

2D CFTs: the unknown unknowns

1. Even for rational CFTs, far from classification: the existence of the "simplest" RCFTs - extremal meromorphic CFTs with $c=24k$, is not known for $k>1$. (despite various efforts to construct or rule out such theories)

2. What values of central charge $c>1$ can be realized by compact unitary CFTs? (Put differently, can a unitary Virasoro algebra always be embedded in a compact modular invariant theory?)

3. Are there compact unitary 2D CFTs with nonzero twist gap (in Virasoro primaries)? Is $(c-1)/12$ the optimal upper bound on the twist gap?

4. Are there really compact CFTs of large c and large gap in the spectrum of Virasoro primaries? (i.e. does pure quantum gravity in AdS3 exist?)

c.f. [Ooguri, Vafa '16]

5. To what extent does the low lying operator spectrum of a CFT pin down the entire theory? (Existence and uniqueness of UV completion of gravity+matter in AdS?)

Beyond RCFT, our analytic tools are limited.

Beyond RCFT, our analytic tools are limited.

Conformal perturbation theory, when available, is hard.

Beyond RCFT, our analytic tools are limited.

Conformal perturbation theory, when available, is hard.

Unlike the "old" bootstrap which aims to construct exact solutions, [Ferrara, Grillo, Gatto '73, Polyakov, '74, Mack, '77, Belavin, Polyakov, Zamolodchikov, '84, ...]

Beyond RCFT, our analytic tools are limited.

Conformal perturbation theory, when available, is hard.

Unlike the “old” bootstrap which aims to construct exact solutions, [Ferrara, Grillo, Gatto ’73, Polyakov, ’74, Mack, ’77, Belavin, Polyakov, Zamolodchikov, ’84, ...]

the “new” bootstrap aims to rule out theories (and constrain known theories). [Ratazzi, Rychkov, Tonni, Vichi ’08, ...many more]

Beyond RCFT, our analytic tools are limited.

Conformal perturbation theory, when available, is hard.

Unlike the “old” bootstrap which aims to construct exact solutions, [Ferrara, Grillo, Gatto ’73, Polyakov, ’74, Mack, ’77, Belavin, Polyakov, Zamolodchikov, ’84, ...]

the “new” bootstrap aims to rule out theories (and constrain known theories). [Ratazzi, Rychkov, Tonni, Vichi ’08, ...many more]

Our goal is to carve out the space of 2D (unitary) CFTs.

Beyond RCFT, our analytic tools are limited.

Conformal perturbation theory, when available, is hard.

Unlike the "old" bootstrap which aims to construct exact solutions, [Ferrara, Grillo, Gatto '73, Polyakov, '74, Mack, '77, Belavin, Polyakov, Zamolodchikov, '84, ...]

the "new" bootstrap aims to rule out theories (and constrain known theories). [Ratazzi, Rychkov, Tonni, Vichi '08, ...many more]

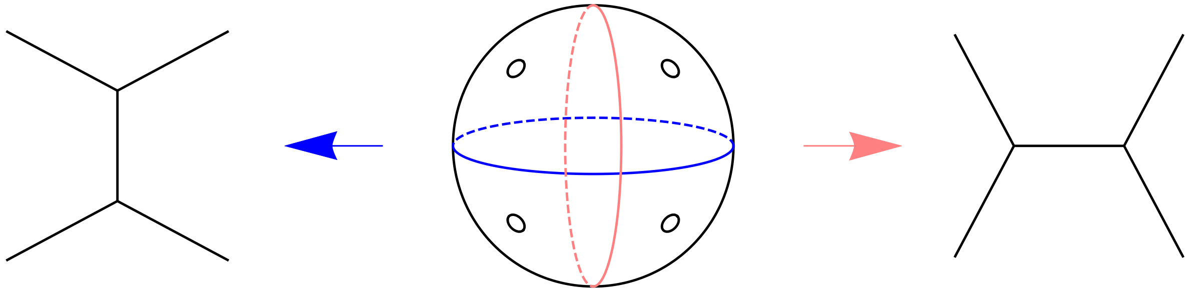
Our goal is to carve out the space of 2D (unitary) CFTs.

We would like to know: what are the possible spectra of local operators and structure constants?

Conformal Bootstrap

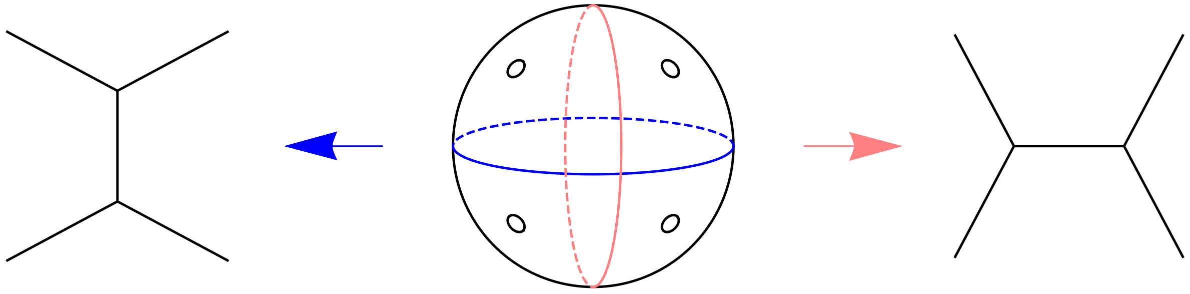
Conformal Bootstrap

crossing invariance
(associativity of OPE)

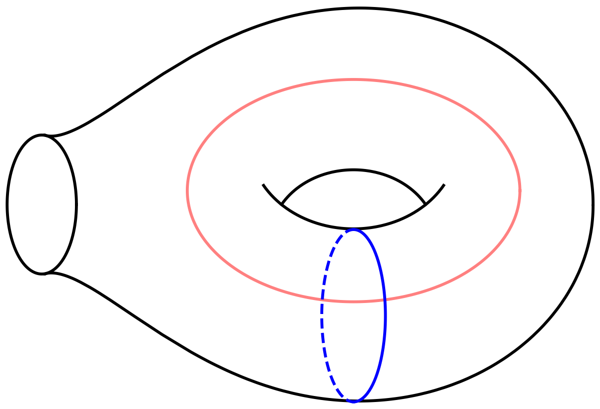


Conformal Bootstrap

crossing invariance
(associativity of OPE)

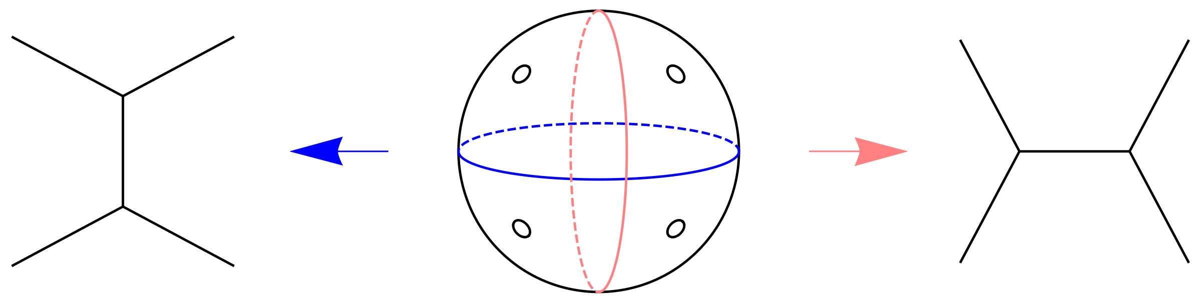


modular invariance



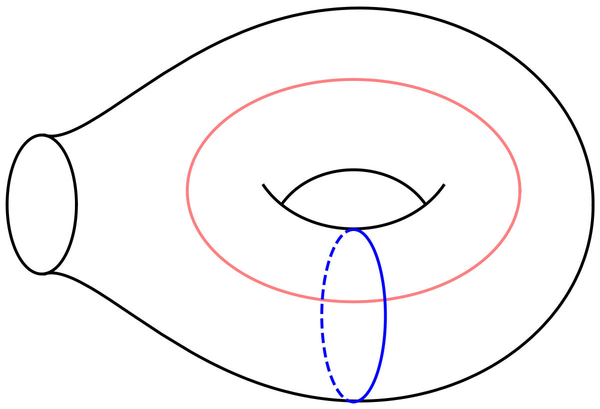
Conformal Bootstrap

crossing invariance
(associativity of OPE)



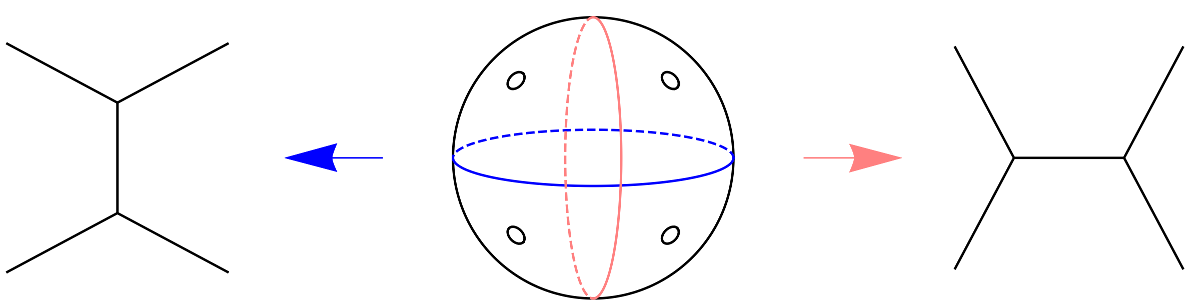
(any spacetime dimension)

modular invariance



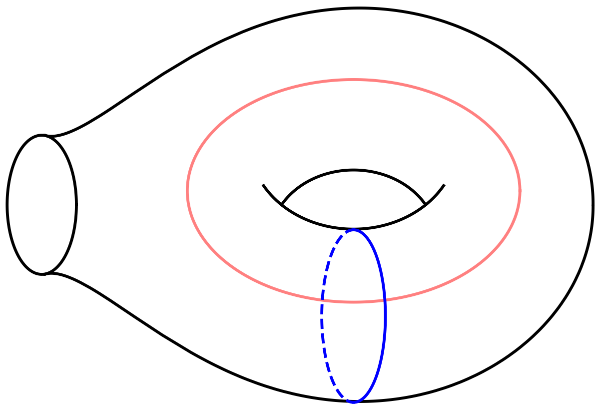
Conformal Bootstrap

crossing invariance
(associativity of OPE)



(any spacetime dimension)

modular invariance



(only properly understood in 2D)

Let us begin with a simple example

Let us begin with a simple example

Modular invariance of the torus partition function

$$Z(\tau,\bar{\tau}) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i \tau}$$

Let us begin with a simple example

Modular invariance of the torus partition function

$$Z(\tau,\bar{\tau}) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i \tau}$$

Invariance under $\tau \to \tau + 1$ is equivalent to the requirement that all local operator have integer spins.

Let us begin with a simple example

Modular invariance of the torus partition function

$$Z(\tau,\bar{\tau}) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i \tau}$$

Invariance under $\tau \to \tau + 1$ is equivalent to the requirement that all local operator have integer spins.

(Fermionic theories are subject to GSO projection.)

Let us begin with a simple example

Modular invariance of the torus partition function

$$Z(\tau,\bar{\tau}) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i \tau}$$

Invariance under $\tau \to \tau + 1$ is equivalent to the requirement that all local operator have integer spins.

(Fermionic theories are subject to GSO projection.)

The invariance under $\tau \to -1/\tau$ imposes much more nontrivial constraints on the operator spectrum.

Let us begin with a simple example

Modular invariance of the torus partition function

$$Z(\tau, \bar{\tau}) = \mathrm{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\tilde{c}}{24}}, \quad q = e^{2\pi i \tau}$$

Invariance under $\tau \to \tau + 1$ is equivalent to the requirement that all local operator have integer spins.

(Fermionic theories are subject to GSO projection.)

The invariance under $\tau \to -1/\tau$ imposes much more nontrivial constraints on the operator spectrum.

For instance, it relates the growth of the number of operators at large scaling dimensions to the operator content of the lowest scaling dimensions. [Cardy '86]

Let us begin with a simple example

Modular invariance of the torus partition function

$$Z(\tau,\bar{\tau}) = \mathrm{Tr} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}, \quad q = e^{2\pi i\tau}$$

Invariance under $\tau \to \tau+1$ is equivalent to the requirement that all local operator have integer spins.

(Fermionic theories are subject to GSO projection.)

The invariance under $\tau \to -1/\tau$ imposes much more nontrivial constraints on the operator spectrum.

For instance, it relates the growth of the number of operators at large scaling dimensions to the operator content of the lowest scaling dimensions. [Cardy '86]

The modular constraint on the operator spectrum goes much further!

[Hellerman '09, Friedan-Keller '13, Qualls-Shapere '13, Collier-Lin-XY '16]

Modular Bootstrap

Modular Bootstrap

The torus partition function admits character decomposition:

$$Z(\tau, \bar{\tau}) = \sum_{h, \tilde{h}} d_{h, \tilde{h}} \chi_h(\tau) \bar{\chi}_{\tilde{h}}(\bar{\tau})$$

Modular Bootstrap

The torus partition function admits character decomposition:

$$Z(\tau,\bar{\tau}) = \sum_{h,\tilde{h}} d_{h,\tilde{h}} \chi_h(\tau) \bar{\chi}_{\tilde{h}}(\bar{\tau})$$

(Assuming c>1 here; h=0 character is degenerate.)

Modular Bootstrap

The torus partition function admits character decomposition:

$$Z(\tau,\bar{\tau}) = \sum_{h,\tilde{h}} d_{h,\tilde{h}} \chi_h(\tau) \bar{\chi}_{\tilde{h}}(\bar{\tau})$$

(Assuming c>1 here; h=0 character is degenerate.)

We impose the positivity of $d(h,\bar{h})$ and modular invariance of $Z(\tau,\bar{\tau})$, namely,

$$Z(-1/\tau, -1/\bar{\tau}) = Z(\tau,\bar{\tau})$$

Modular Bootstrap

The torus partition function admits character decomposition:

$$Z(\tau,\bar{\tau}) = \sum_{h,\tilde{h}} d_{h,\tilde{h}} \chi_h(\tau) \bar{\chi}_{\tilde{h}}(\bar{\tau})$$

(Assuming c>1 here; h=0 character is degenerate.)

We impose the positivity of $d(h,\bar{h})$ and modular invariance of $Z(\tau,\bar{\tau})$, namely,

$$Z(-1/\tau,-1/\bar{\tau}) = Z(\tau,\bar{\tau})$$

and that the spin $s = h - \bar{h}$ takes integer values.

Modular Bootstrap

The torus partition function admits character decomposition:

$$Z(\tau,\bar{\tau}) = \sum_{h,\tilde{h}} d_{h,\tilde{h}}\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau})$$

(Assuming c>1 here; h=0 character is degenerate.)

We impose the positivity of $d(h,\bar{h})$ and modular invariance of $Z(\tau,\bar{\tau})$, namely,

$$Z(-1/\tau,-1/\bar{\tau}) = Z(\tau,\bar{\tau})$$

and that the spin $s = h - \bar{h}$ takes integer values.

We aim to rule out all spectra $\mathcal{I}$ with some hypothetical properties.

Modular crossing equation

$$\sum_{h,\tilde{h}} d_{h,\tilde{h}} \left[\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\bar{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] = 0$$

Modular crossing equation

$$\sum_{h,\tilde{h}} d_{h,\tilde{h}} \left[\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\bar{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] = 0$$

Strategy: seek a linear functional

$$\alpha = \sum_{m+n=\mathrm{odd}} a_{m,n}\, \partial_z^m \partial_{\bar{z}}^n \big|_{z=\bar{z}=0}\,, \quad \tau \equiv i e^z$$

Modular crossing equation

$$\sum_{h,\tilde{h}} d_{h,\tilde{h}} \left[\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\bar{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] = 0$$

Strategy: seek a linear functional

$$\alpha = \sum_{m+n=\mathrm{odd}} a_{m,n}\, \partial_z^m \partial_{\bar{z}}^n \big|_{z=\bar{z}=0}\,, \quad \tau \equiv i e^z$$

such that

$$\alpha \left[\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\bar{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] > 0, \quad \forall (h, \tilde{h}) \in \mathcal{I}.$$

Modular crossing equation

$$\sum_{h,\tilde{h}} d_{h,\tilde{h}} \left[\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\bar{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] = 0$$

Strategy: seek a linear functional

$$\alpha = \sum_{m+n=\mathrm{odd}} a_{m,n}\, \partial_z^m \partial_{\bar{z}}^n|_{z=\bar{z}=0}\,, \quad \tau \equiv i e^z$$

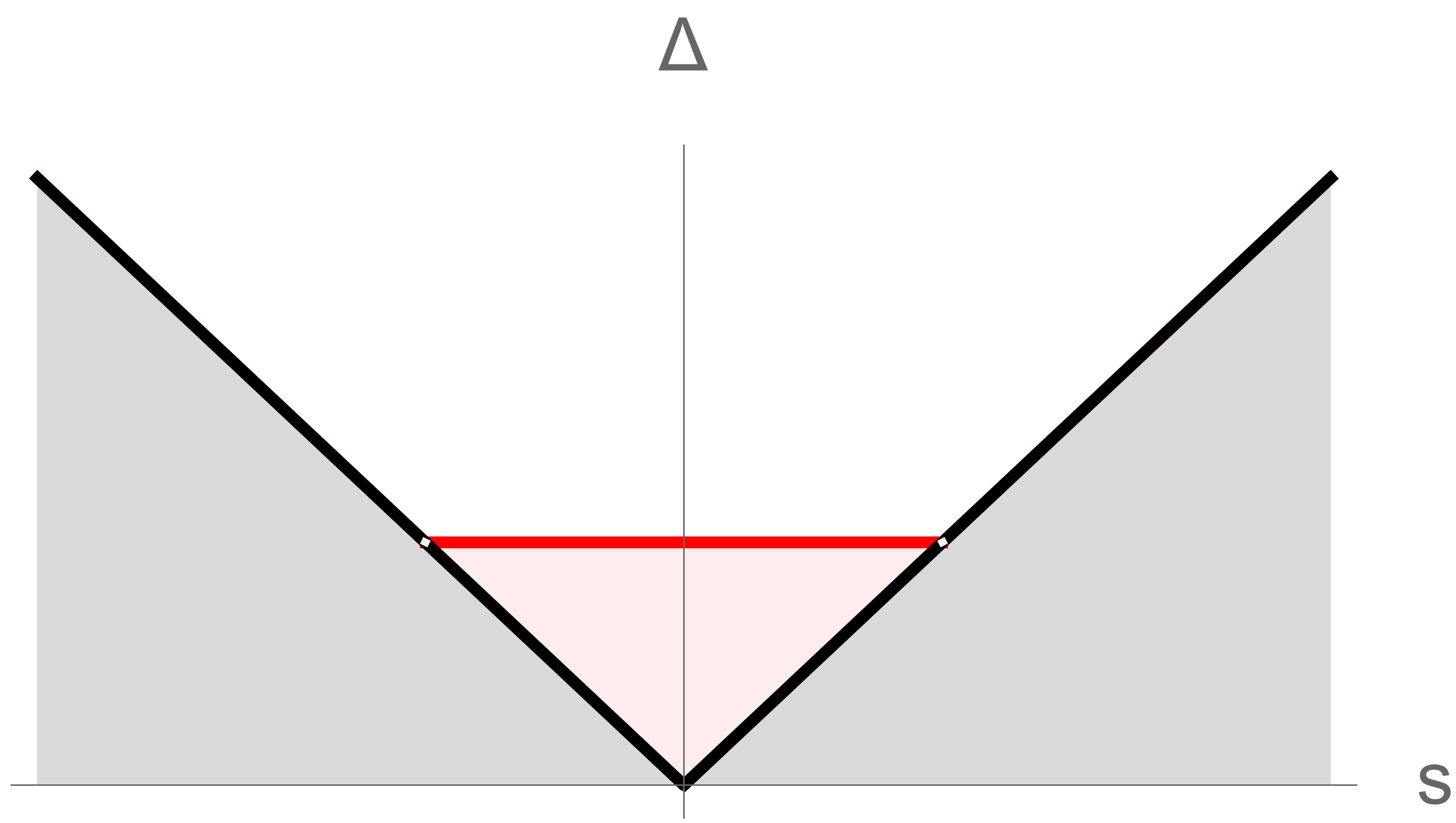
such that

$$\alpha \left[\chi_h(\tau)\bar{\chi}_{\tilde{h}}(\bar{\tau}) - \chi_h(-1/\tau)\bar{\chi}_{\tilde{h}}(-1/\bar{\tau})\right] > 0, \quad \forall (h,\tilde{h}) \in \mathcal{I}.$$

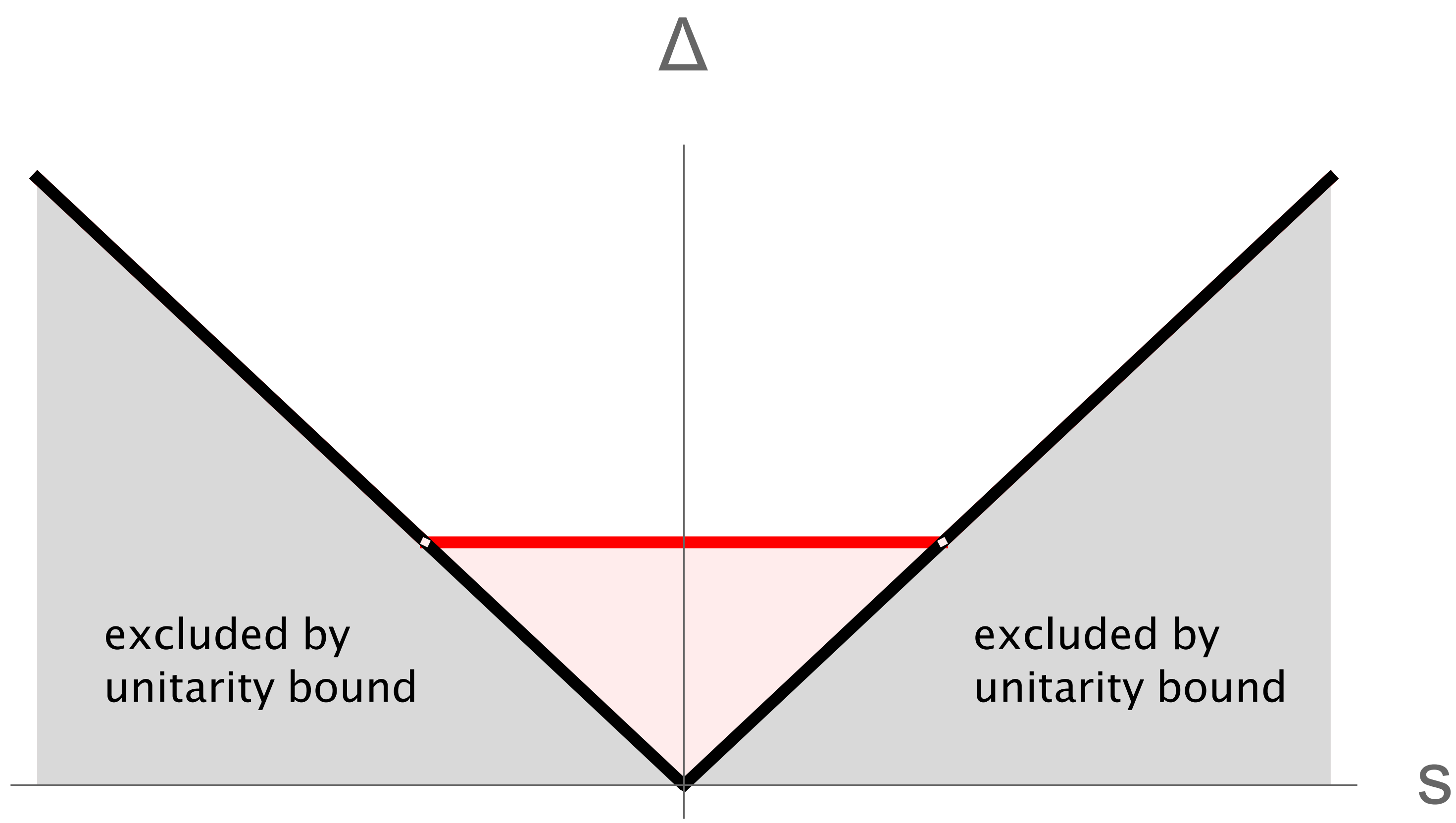
Depending on the hypothesis we choose to make on the spectrum, such a linear functional α may or may not exist. If α is found, then the modular crossing equation cannot be satisfied, thereby ruling out the hypothetical spectrum.

For instance, one may hypothesize a gap Δ_0 in the spectrum of scaling dimensions.

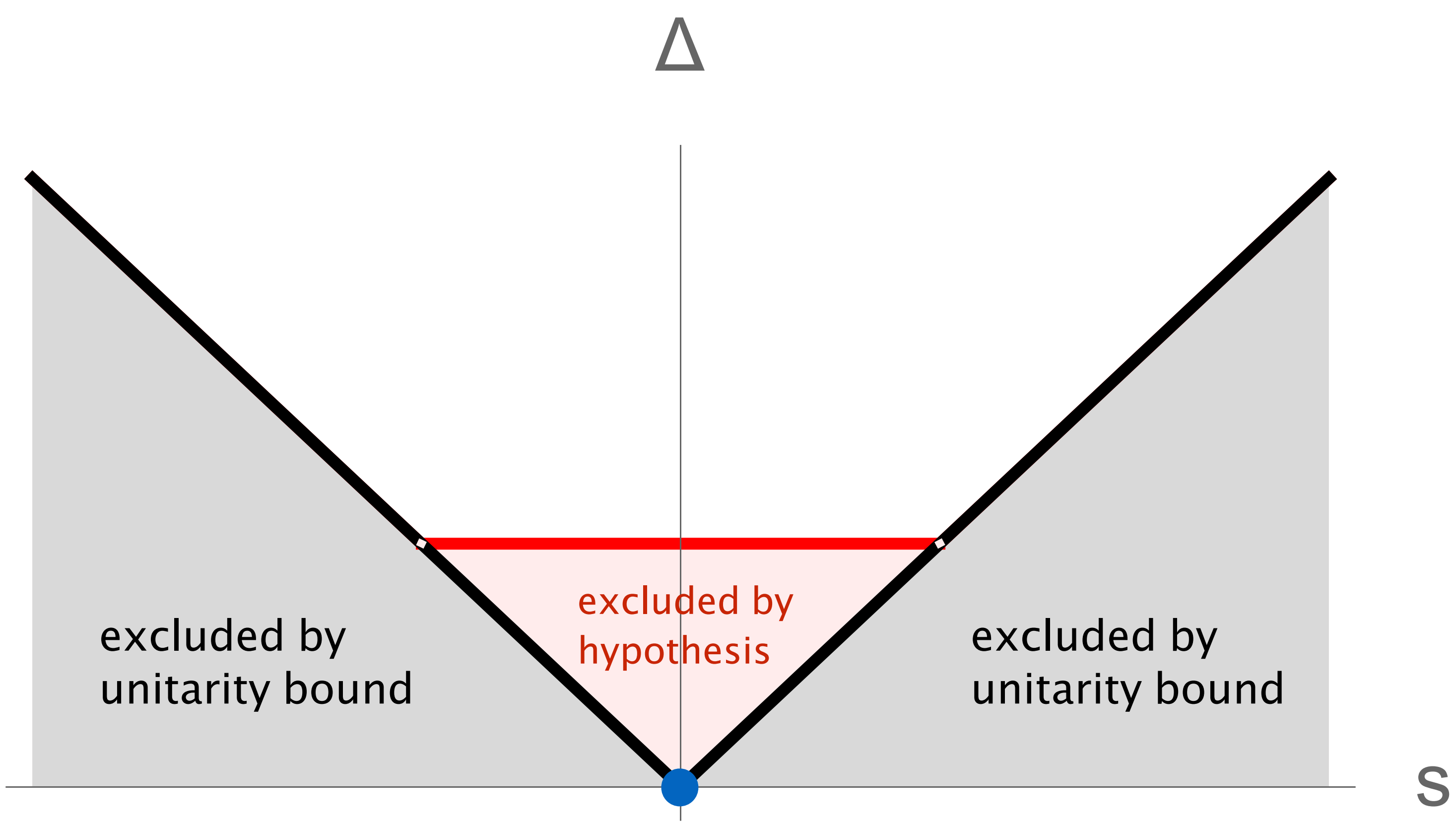
For instance, one may hypothesize a gap Δ_0 in the spectrum of scaling dimensions.



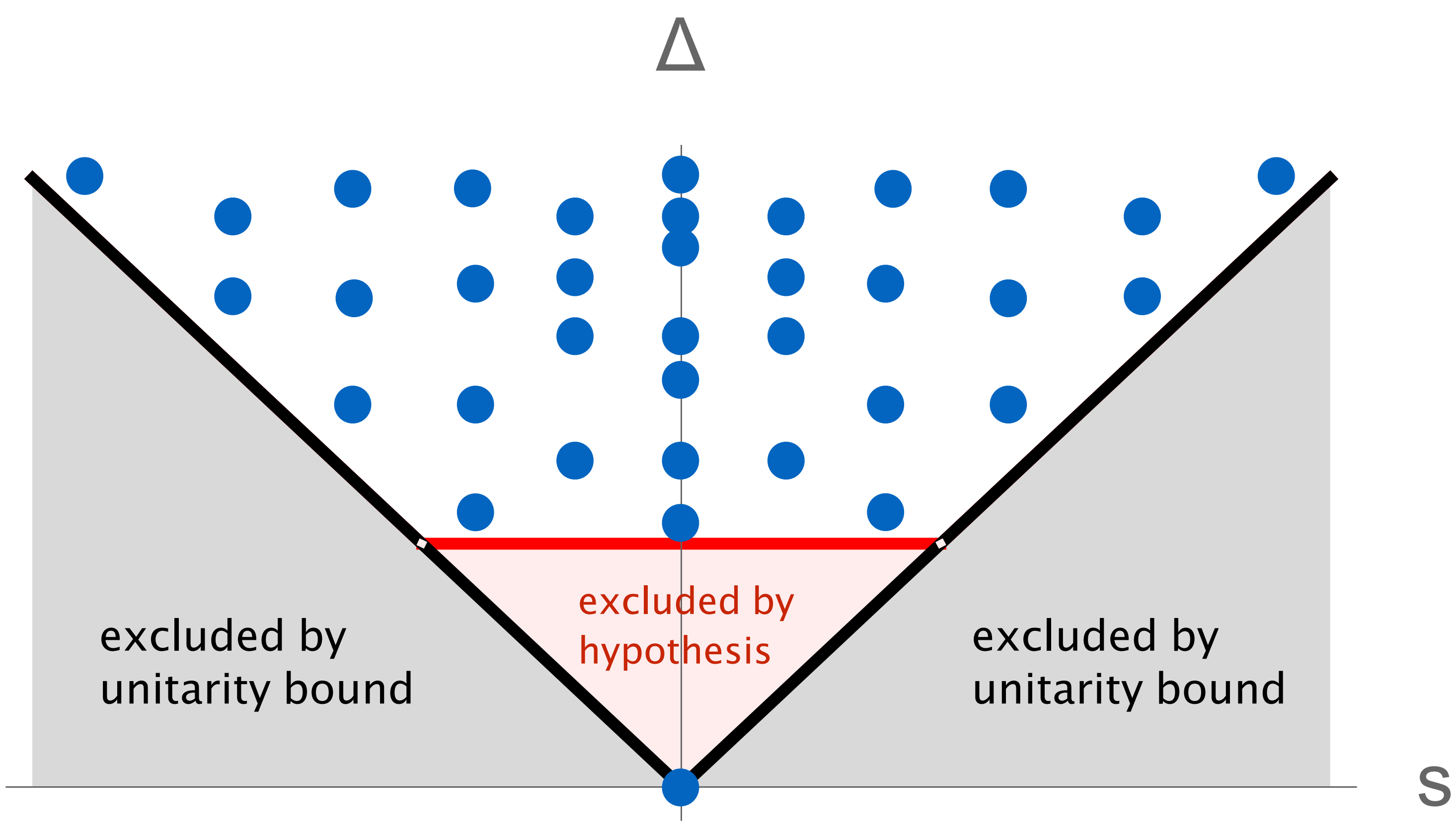
For instance, one may hypothesize a gap Δ_0 in the spectrum of scaling dimensions.



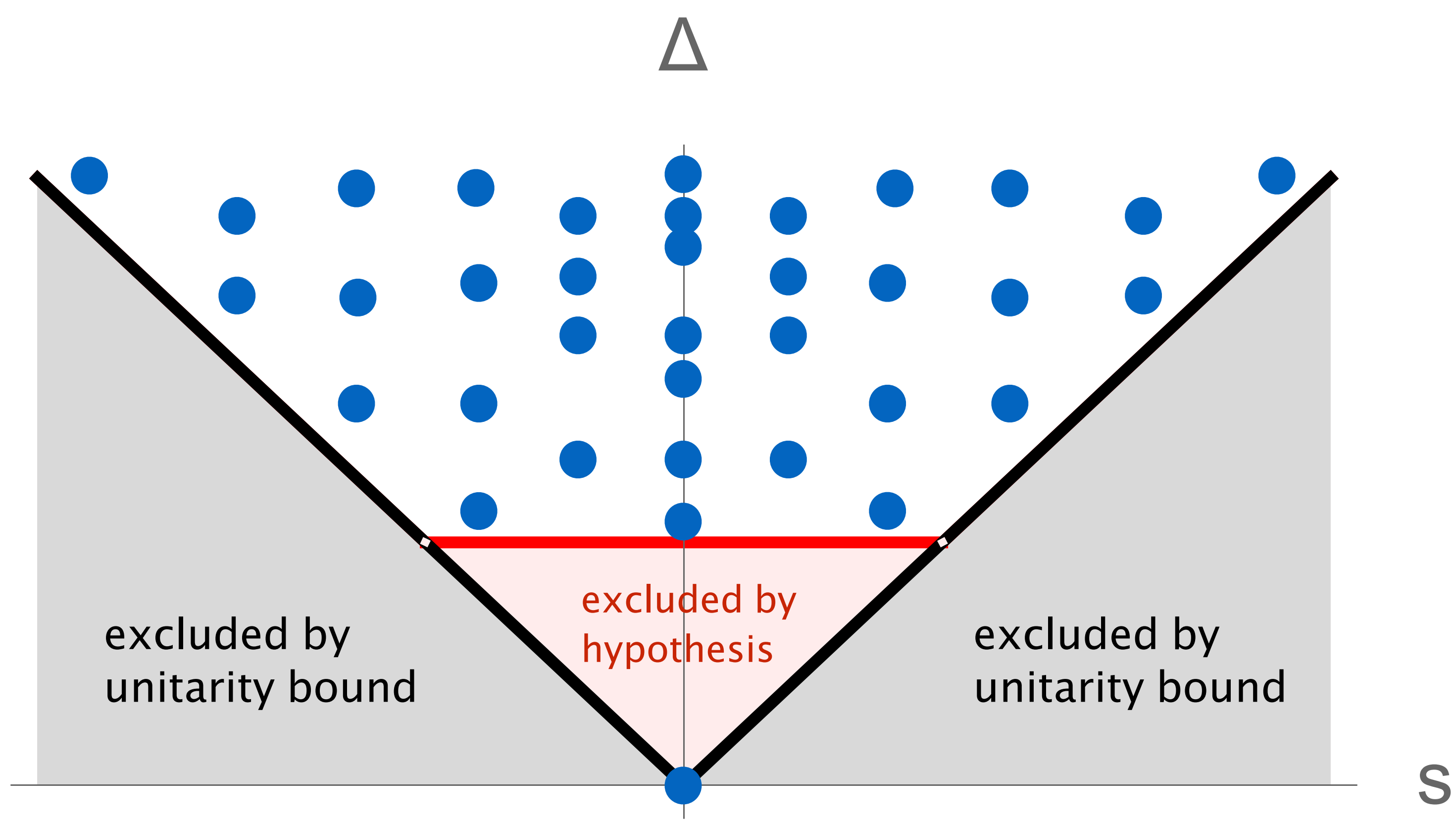
For instance, one may hypothesize a gap Δ_0 in the spectrum of scaling dimensions.



For instance, one may hypothesize a gap Δ_0 in the spectrum of scaling dimensions.



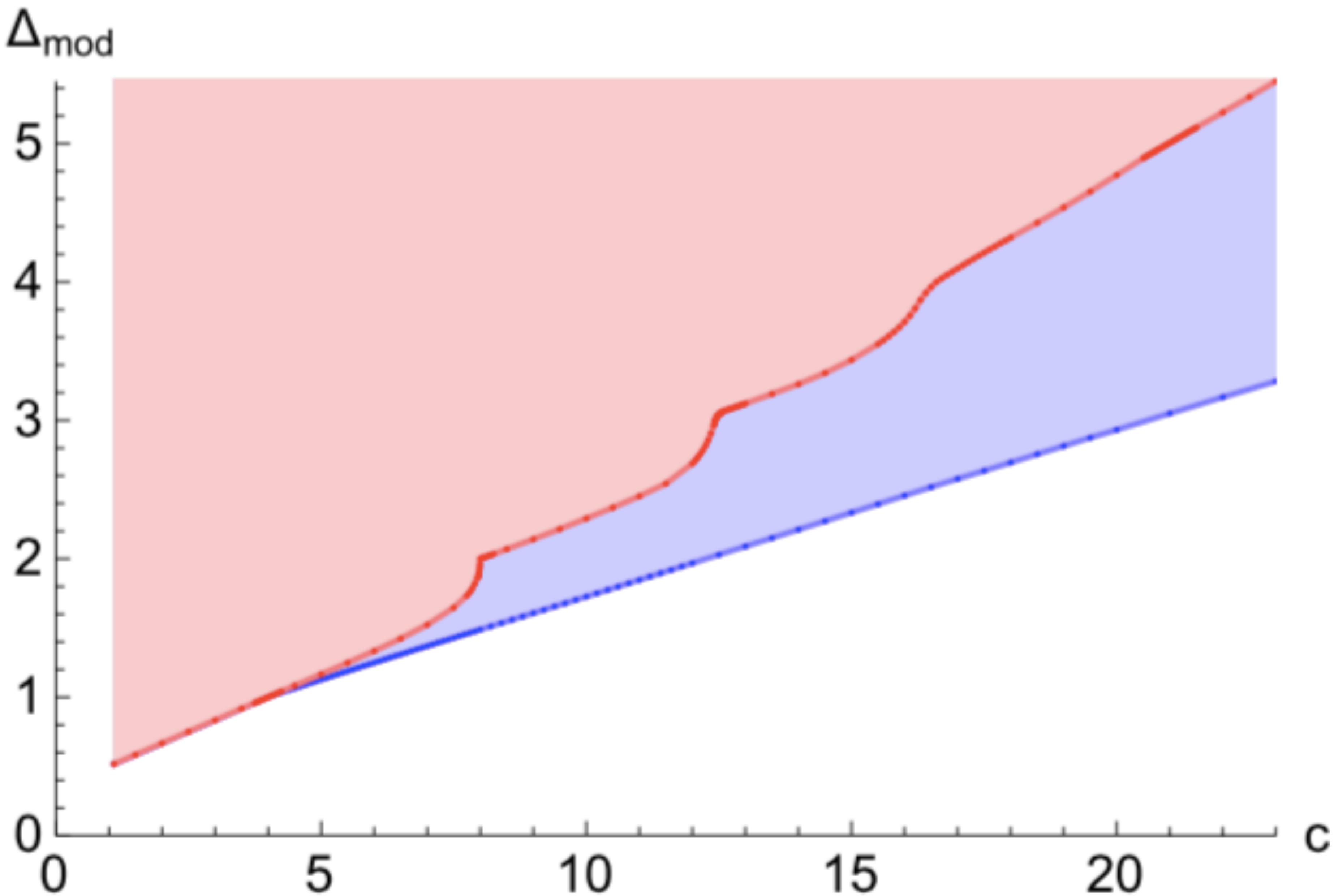
For instance, one may hypothesize a gap Δ_0 in the spectrum of scaling dimensions.



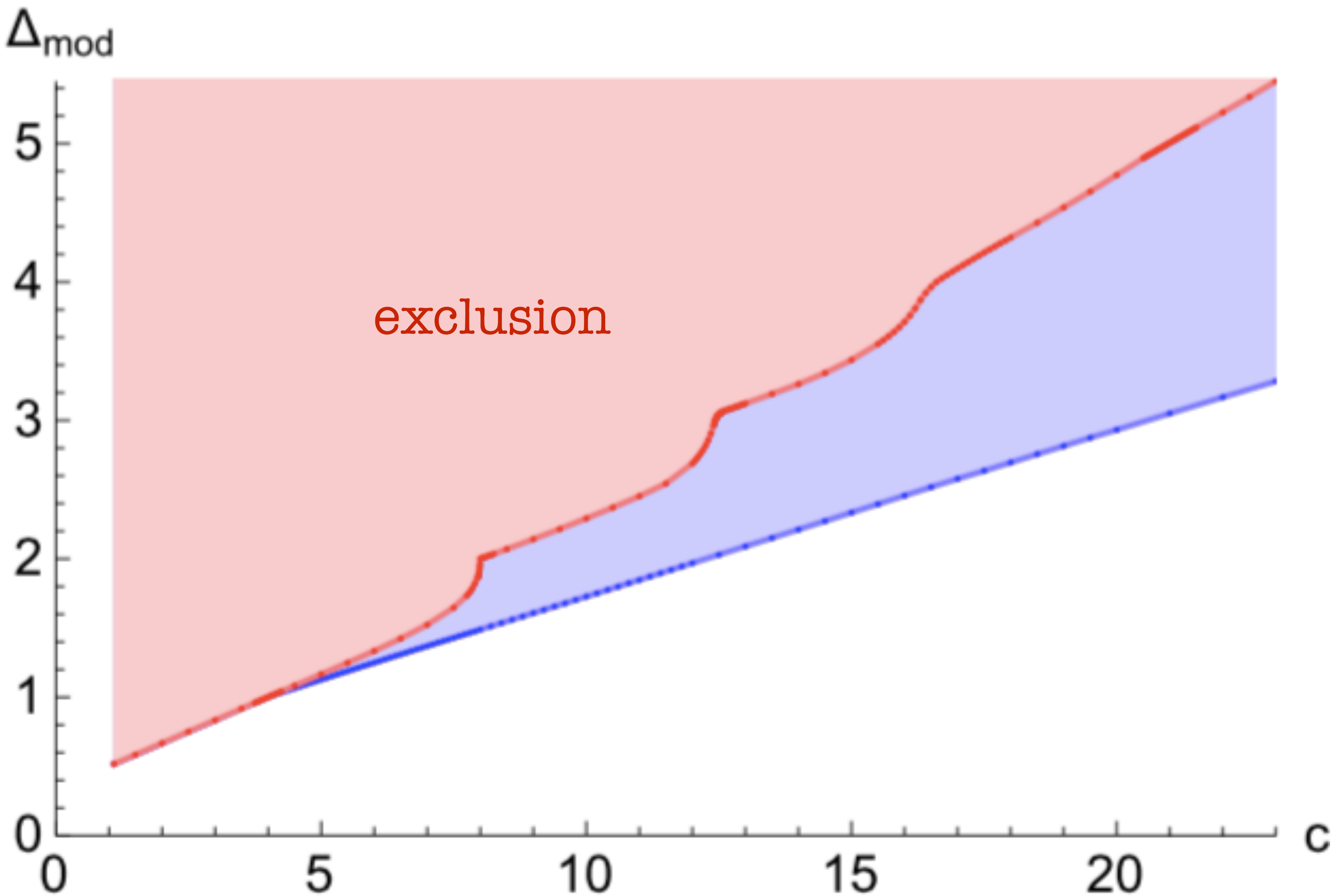
In practice, can work with a basis of linear functionals up to a finite derivative order, truncate on the range of spin (must check stability wrt increasing spin truncation), and use semidefinite programming [SDPB by D. Simmons-Duffin] to optimize the bound numerically.

Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]

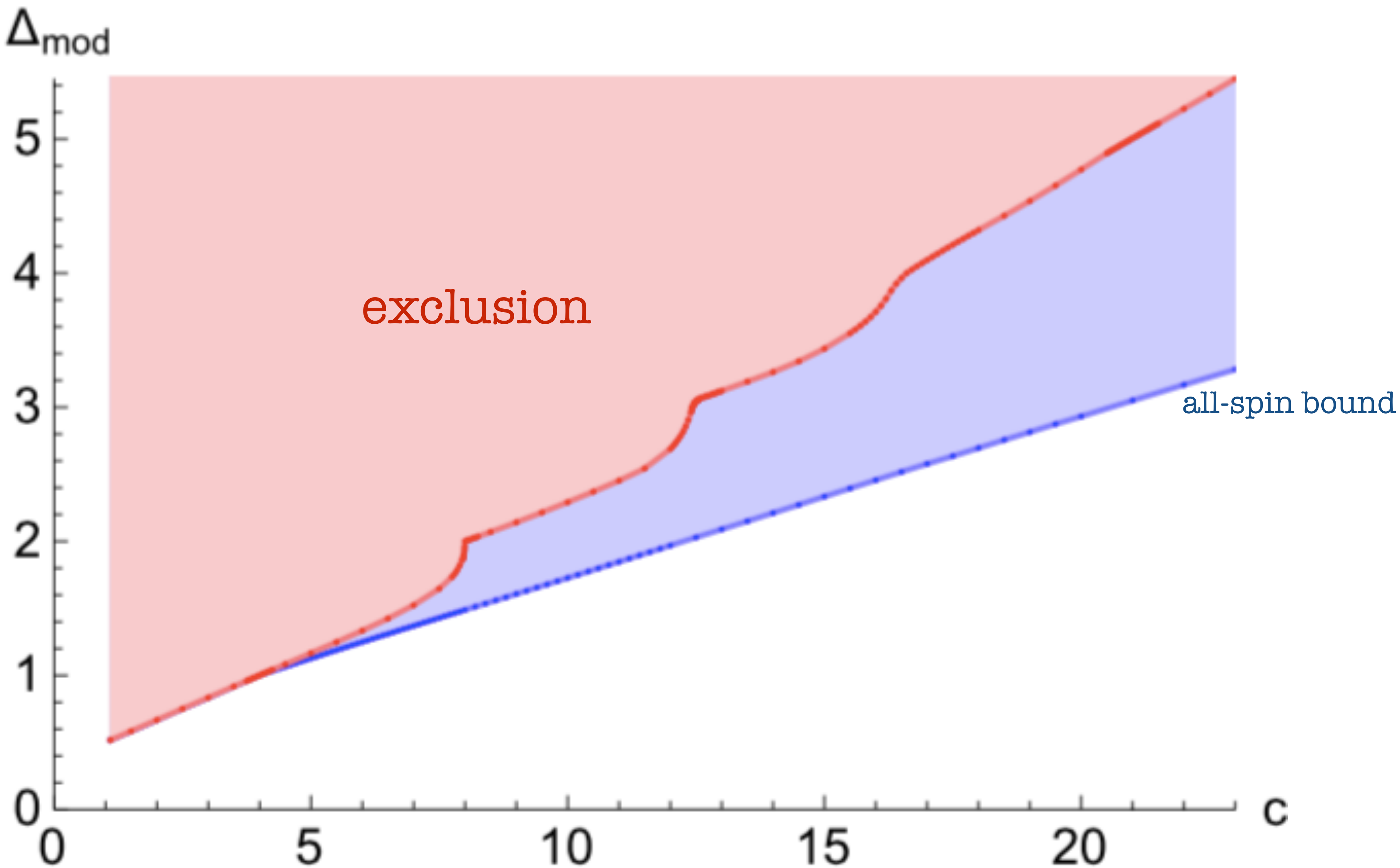
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



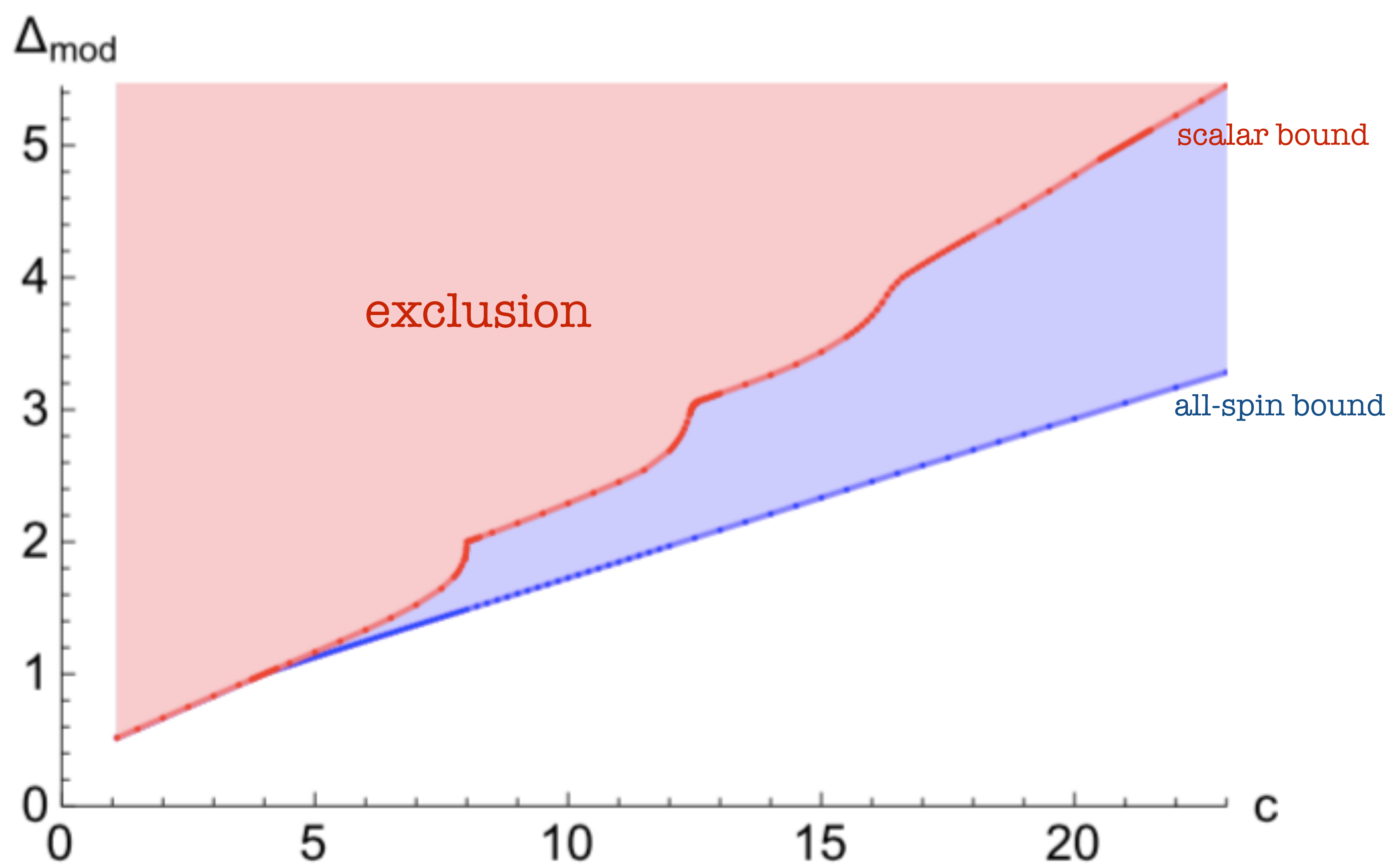
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



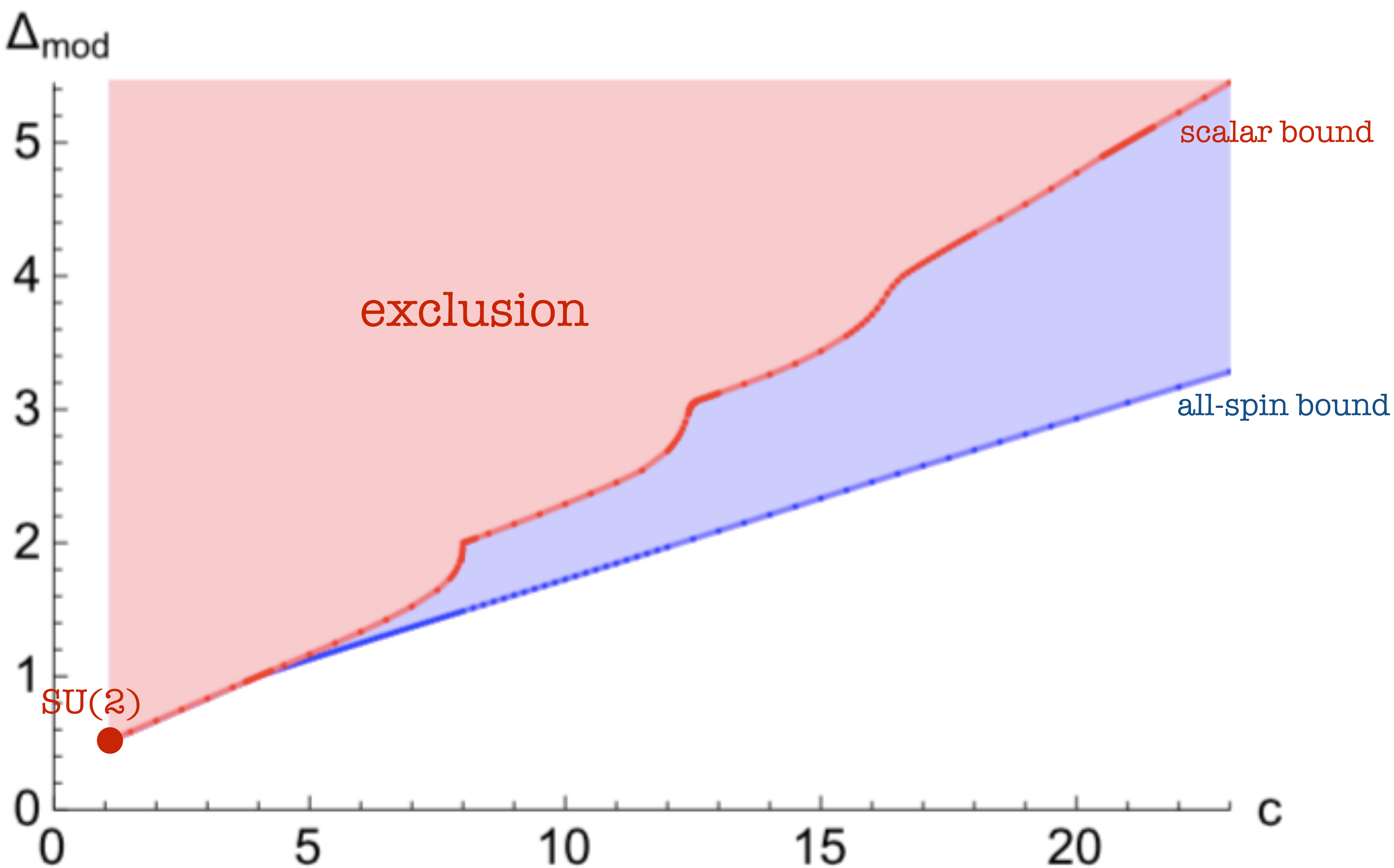
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



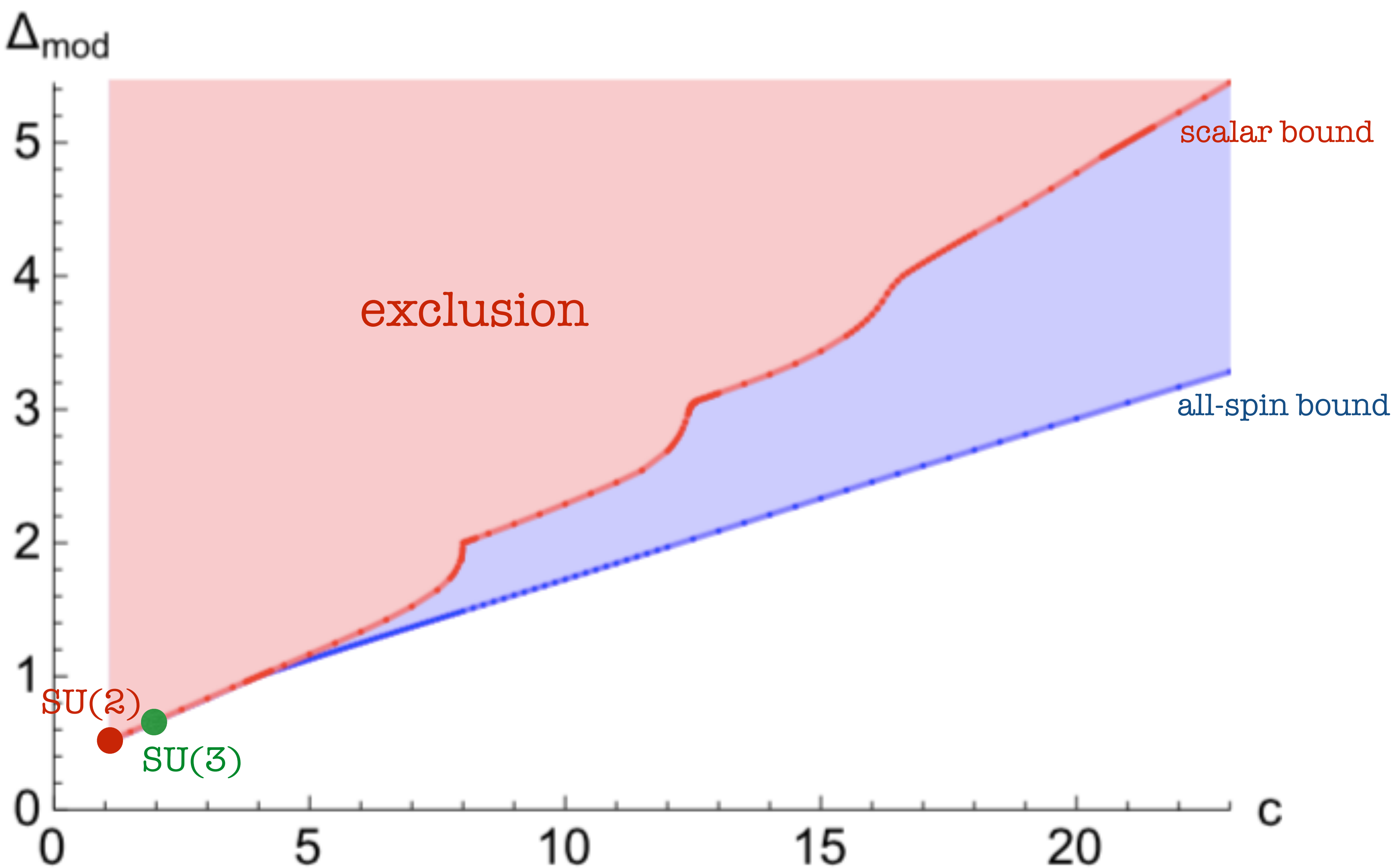
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



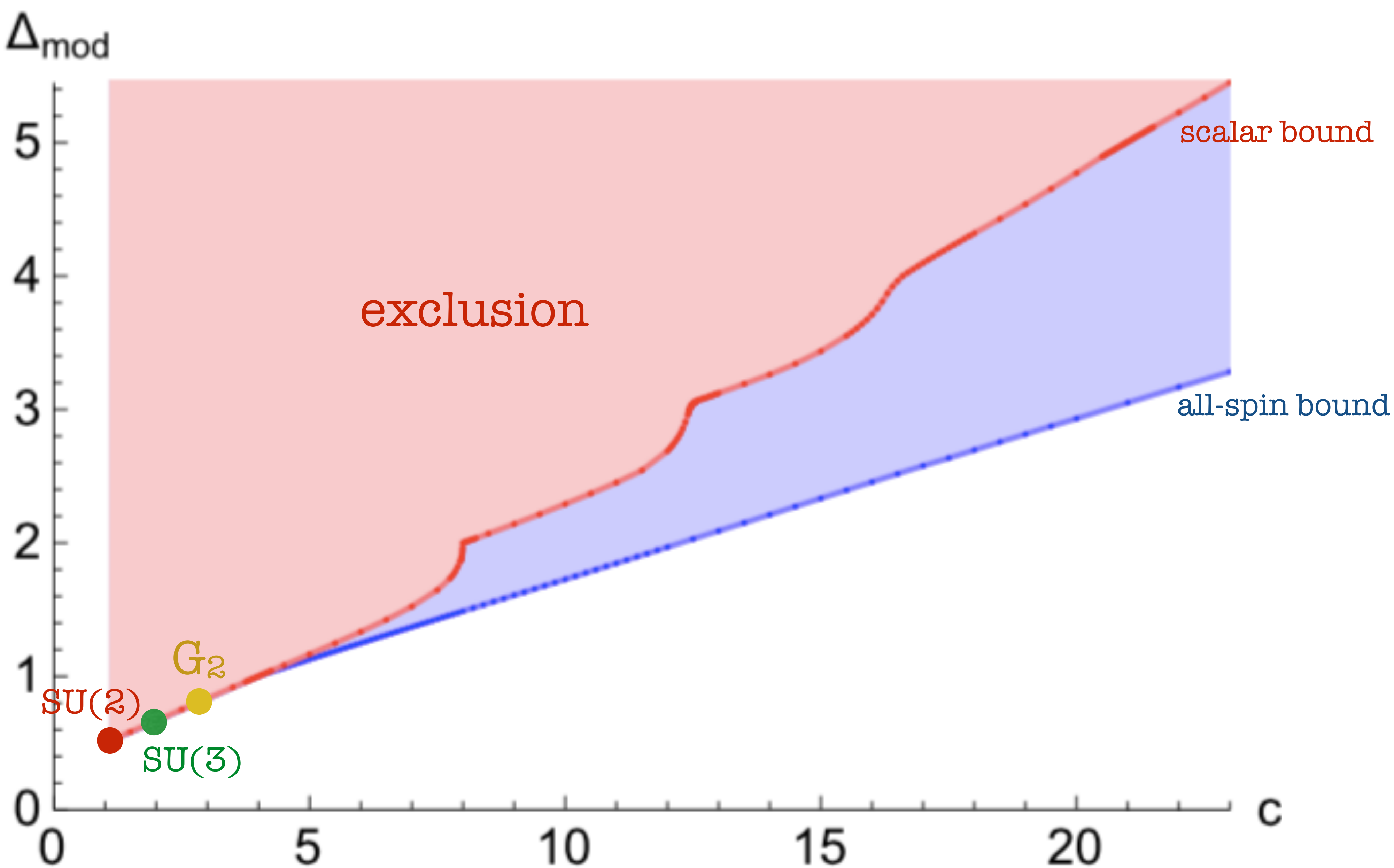
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY ’16]



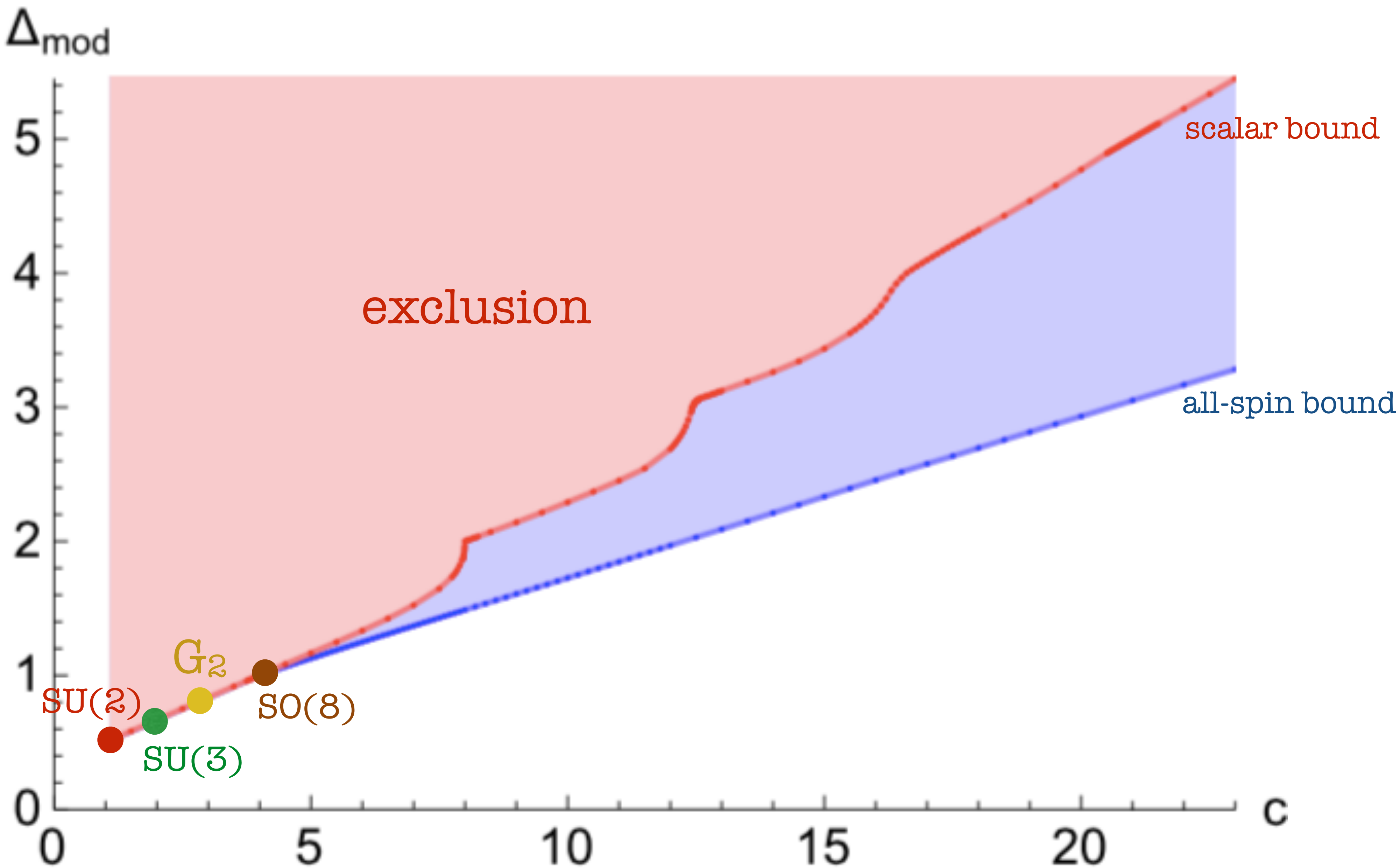
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



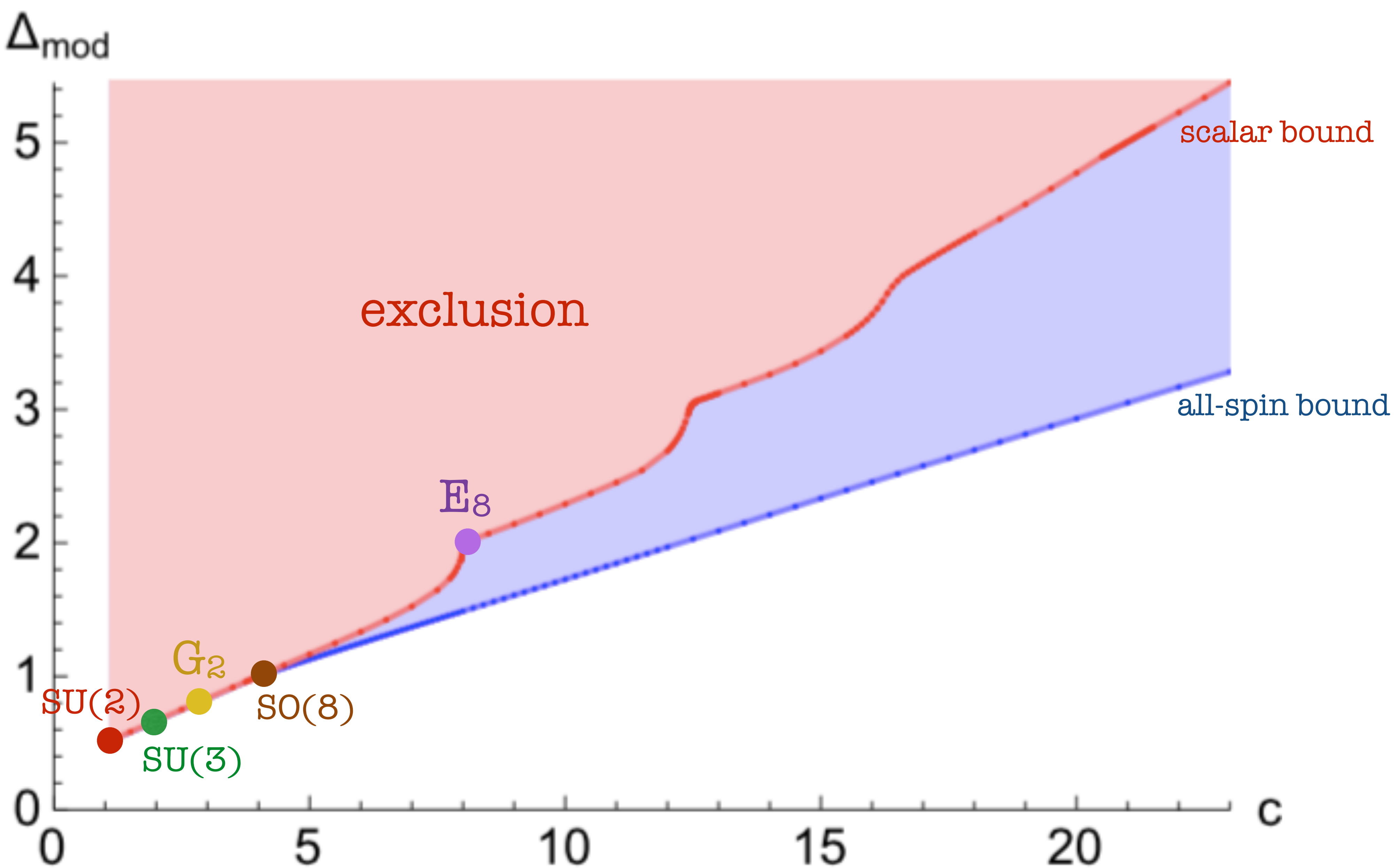
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



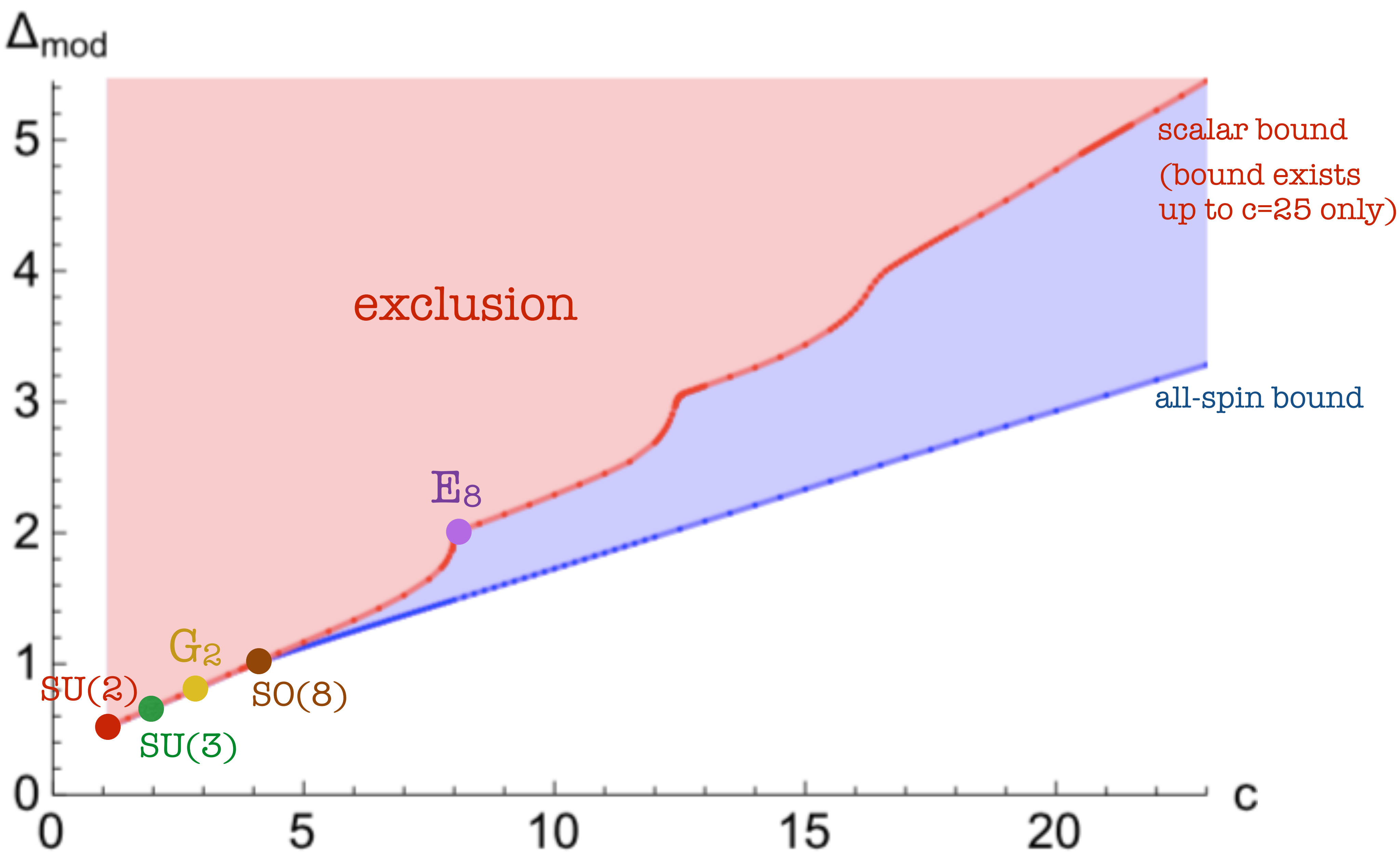
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



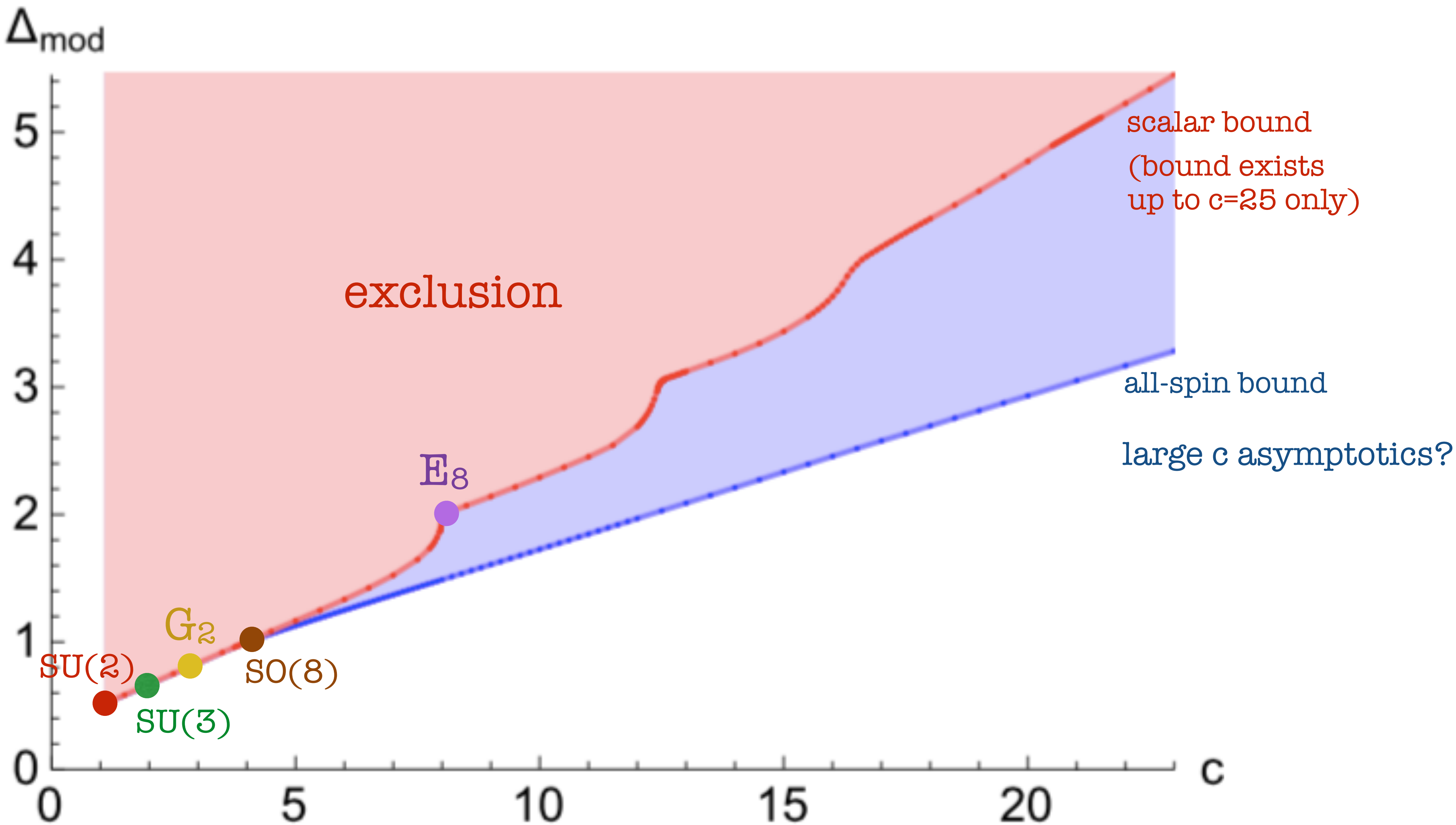
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



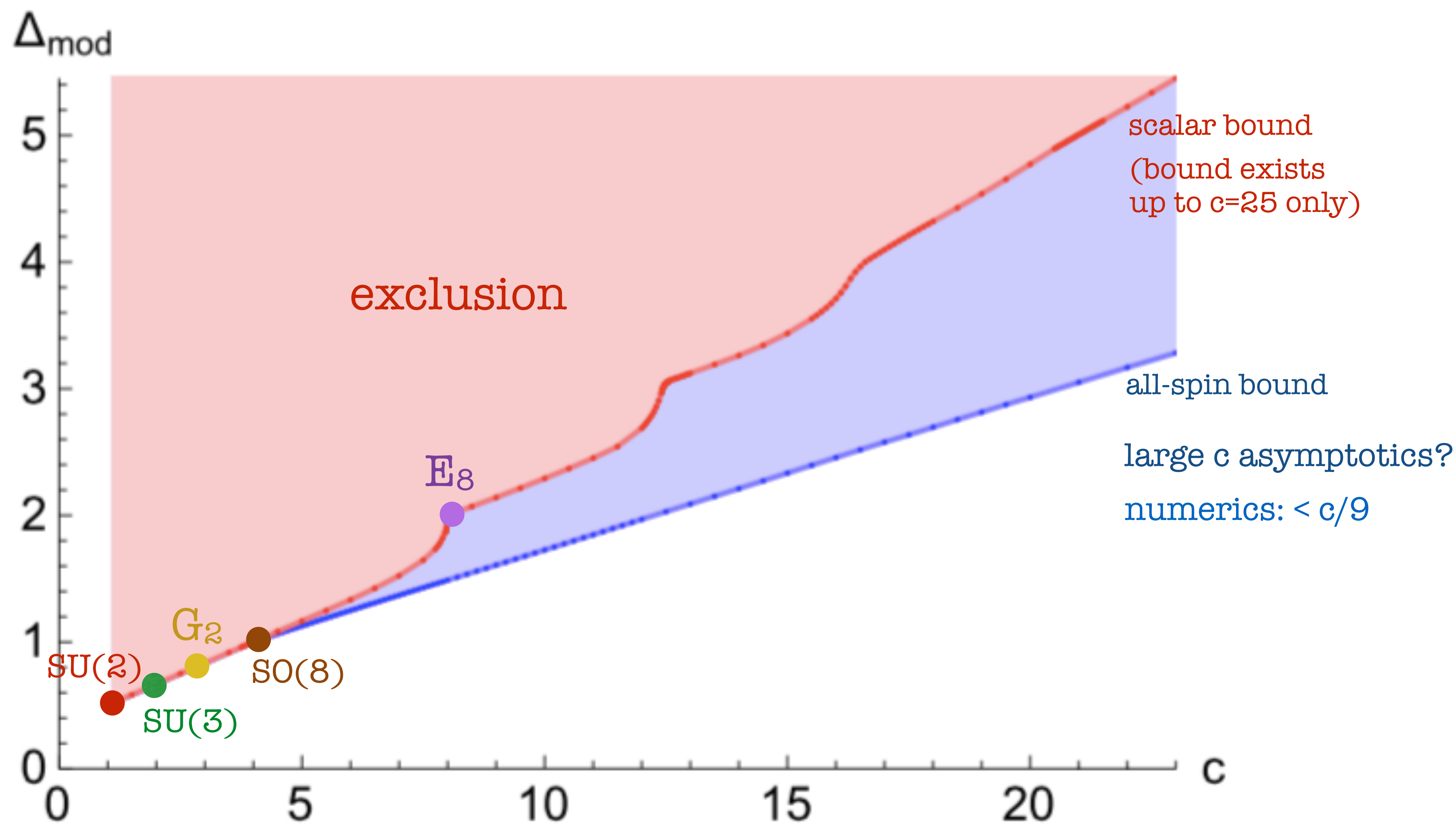
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



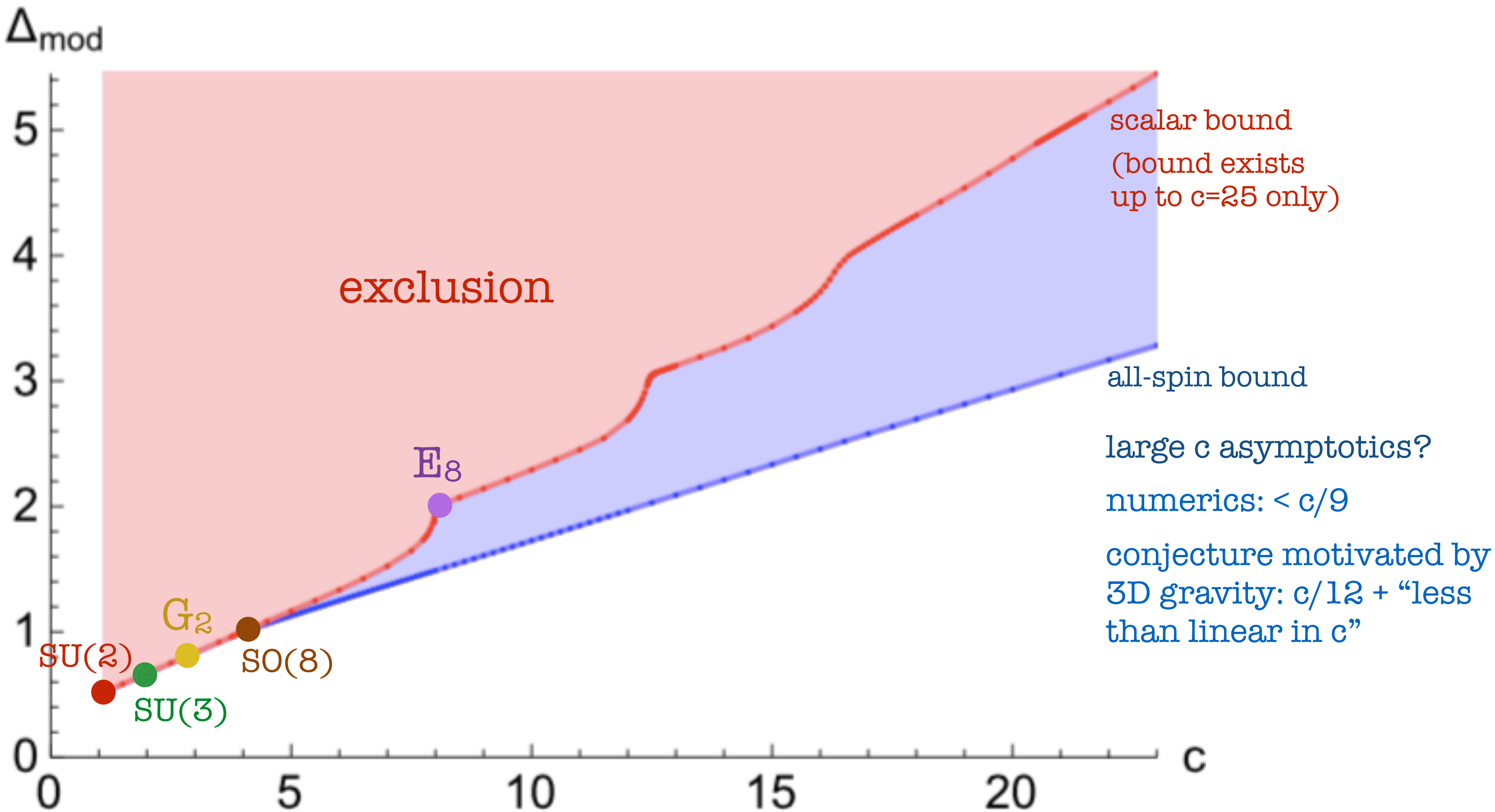
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]



Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY '16]

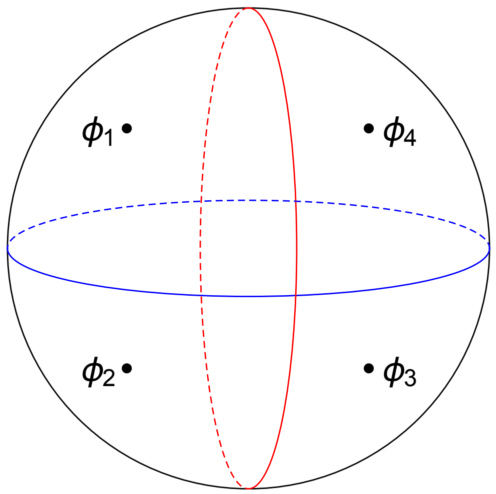


Now we turn to the analysis of OPE

(via crossing invariance of sphere 4-point function)

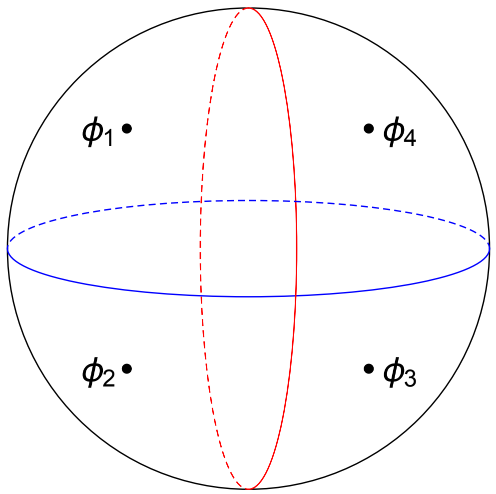
Now we turn to the analysis of OPE

(via crossing invariance of sphere 4-point function)



Now we turn to the analysis of OPE

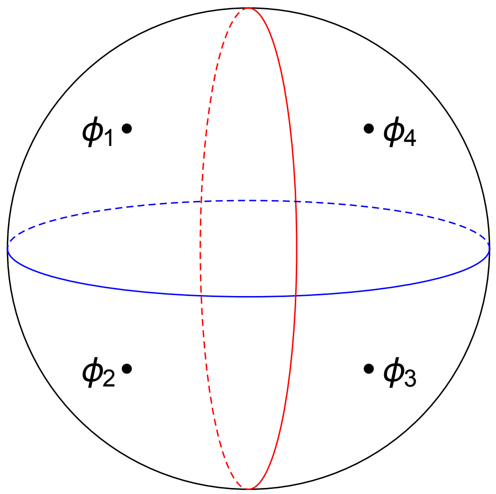
(via crossing invariance of sphere 4-point function)



$$\langle \phi_1(0)\phi_2(z)\phi_3(1)\phi_4(\infty)\rangle = \sum_i C_{12i}C_{34i}F_{12|i|34}(z,\bar{z})$$

Now we turn to the analysis of OPE

(via crossing invariance of sphere 4-point function)

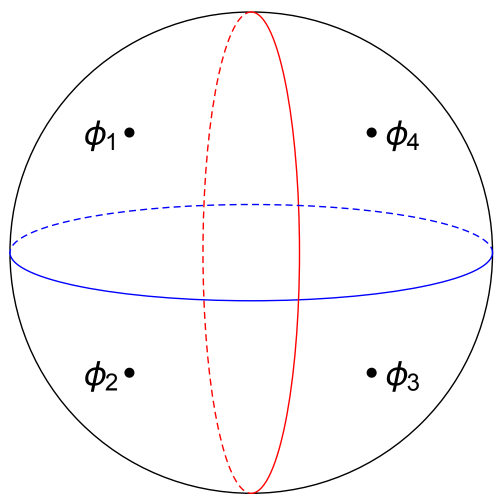


Virasoro conformal block

$$\langle \phi_1(0)\phi_2(z)\phi_3(1)\phi_4(\infty)\rangle = \sum_i C_{12i}C_{34i}F_{12|i|34}(z,\bar{z})$$

Now we turn to the analysis of OPE

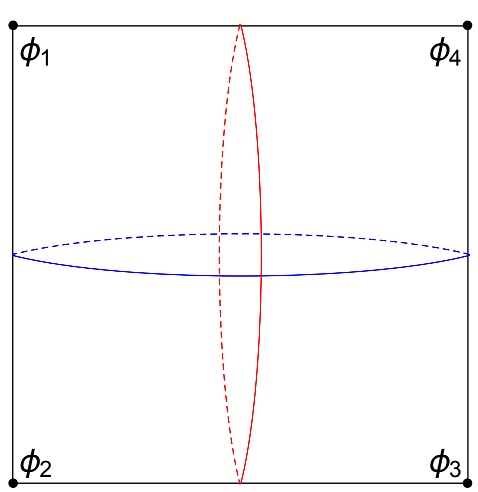
(via crossing invariance of sphere 4-point function)



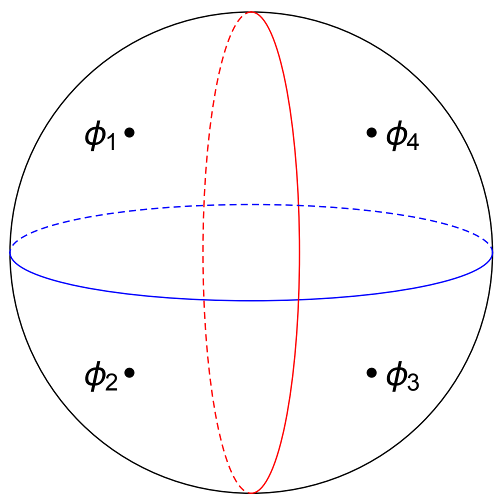
Virasoro conformal block

$$\langle \phi_1(0)\phi_2(z)\phi_3(1)\phi_4(\infty)\rangle = \sum_i C_{12i}C_{34i}F_{12|i|34}(z,\bar{z})$$

useful to map to
pillow geometry



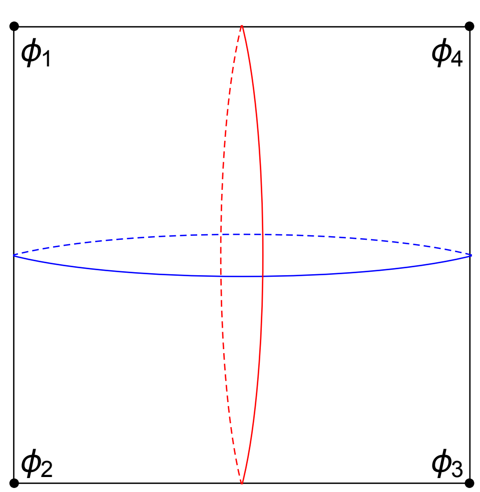
Now we turn to the analysis of OPE
(via crossing invariance of sphere 4-point function)



Virasoro conformal block

$$\langle \phi_1(0)\phi_2(z)\phi_3(1)\phi_4(\infty)\rangle = \sum_i C_{12i}C_{34i}F_{12|i|34}(z,\bar{z})$$

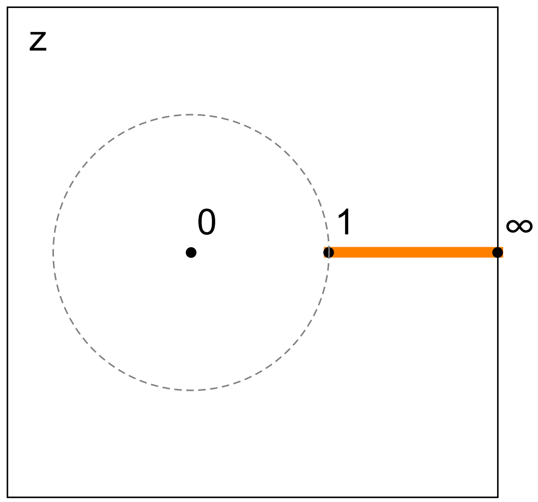
useful to map to pillow geometry

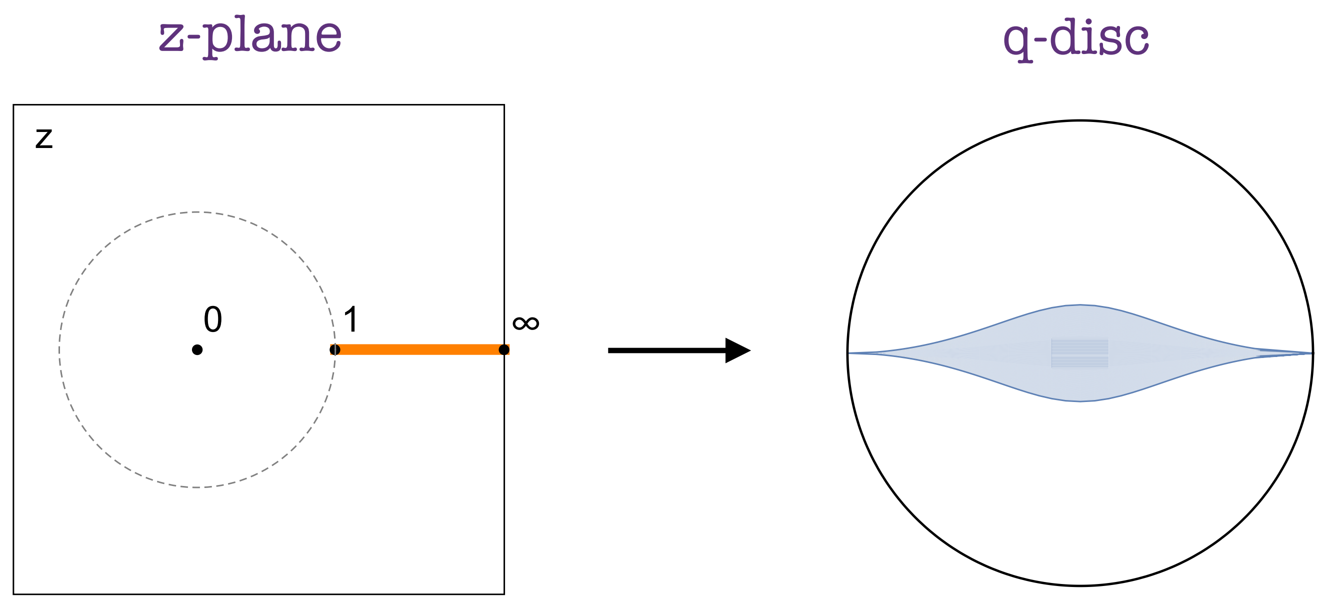


The 4-punctured sphere is conformally mapped to the pillow geometry (T^2/Z_2), with the identification of moduli

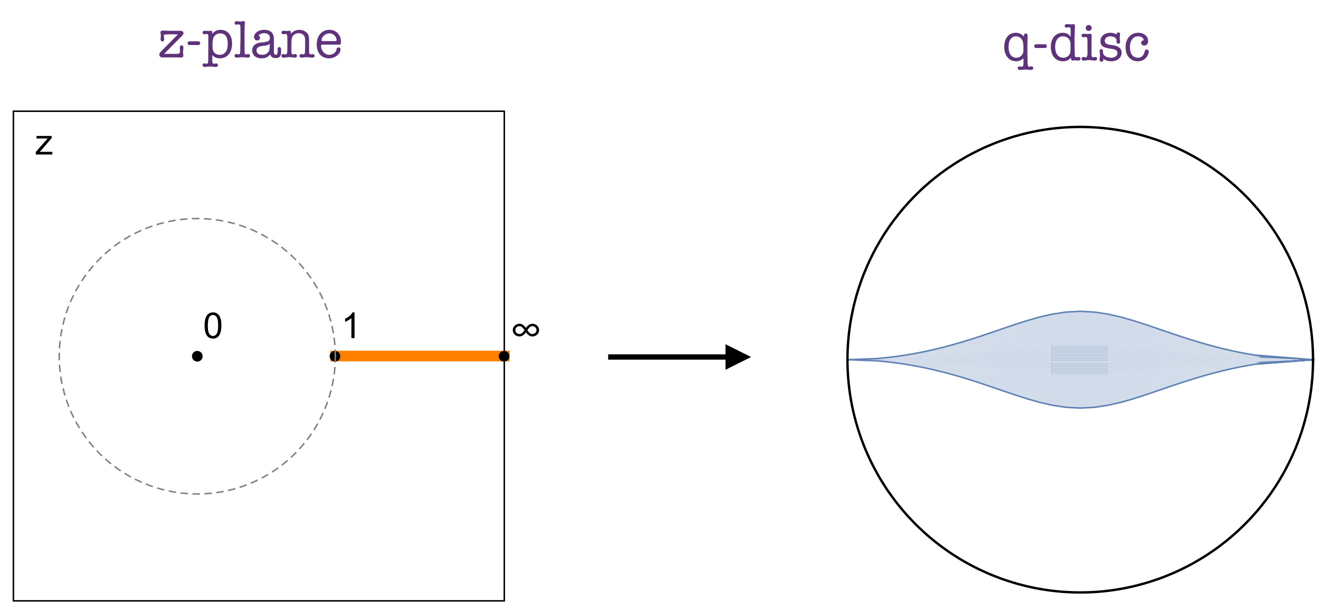
$$\tau = i\frac{K(1-z)}{K(z)}, \quad K(z) = {}_2F_1(\frac{1}{2},\frac{1}{2};1;z) \qquad q = e^{\pi i\tau}$$

z-plane

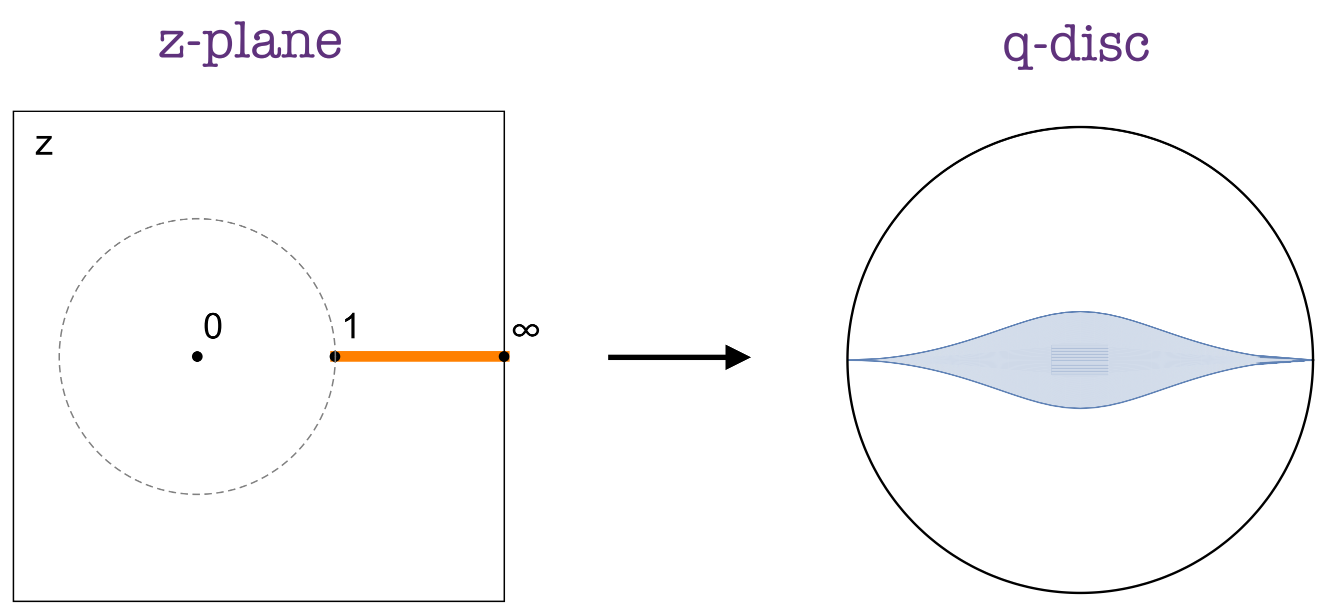




z-plane
q-disc
z
0
1
∞

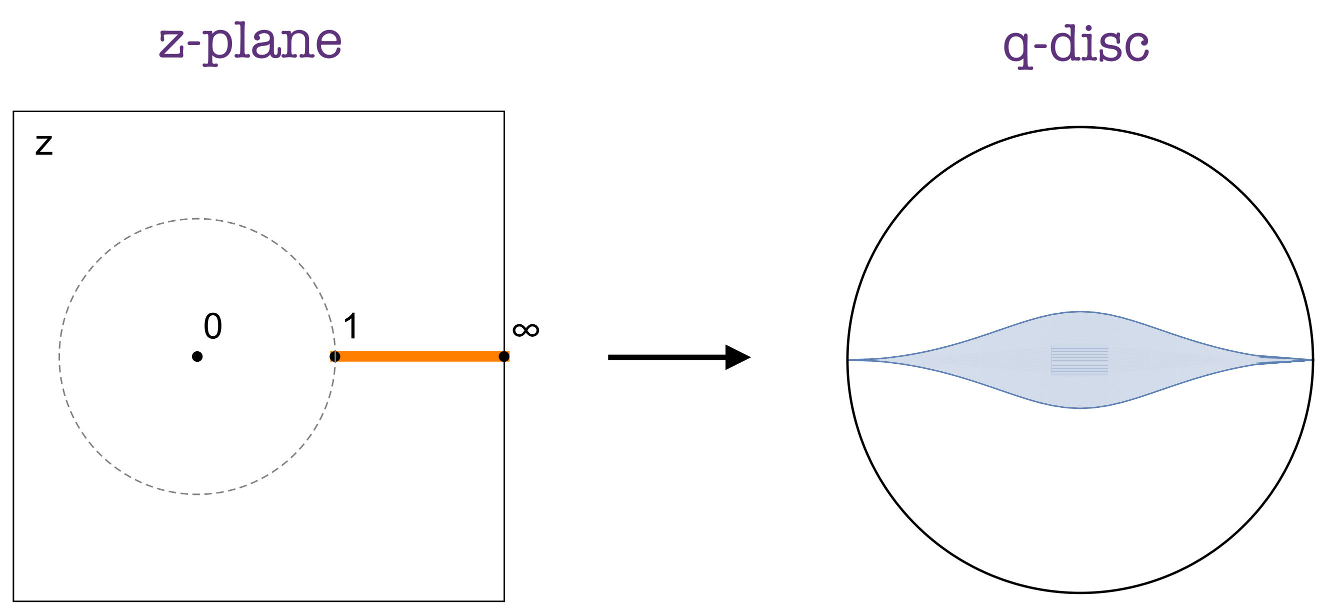


It is useful to work with the q-expansion of the Virasoro conformal block, which makes manifest various analyticity and positivity properties, and allows for efficient numerical evaluation.



It is useful to work with the q-expansion of the Virasoro conformal block, which makes manifest various analyticity and positivity properties, and allows for efficient numerical evaluation.

[Zamolodchikov ’87, Maldacena-Simmons-Duffin-Zhiboedov ’15]

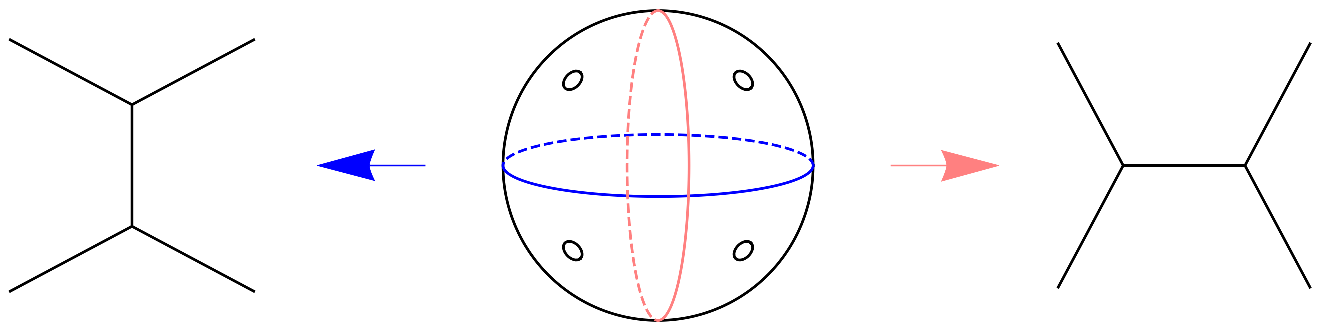


It is useful to work with the q-expansion of the Virasoro conformal block, which makes manifest various analyticity and positivity properties, and allows for efficient numerical evaluation.

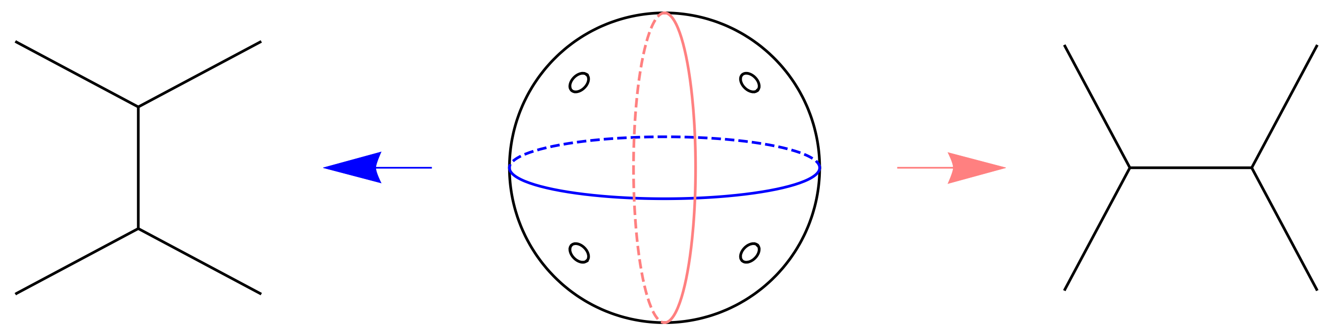
[Zamolodchikov '87, Maldacena-Simmons-Duffin-Zhiboedov '15]

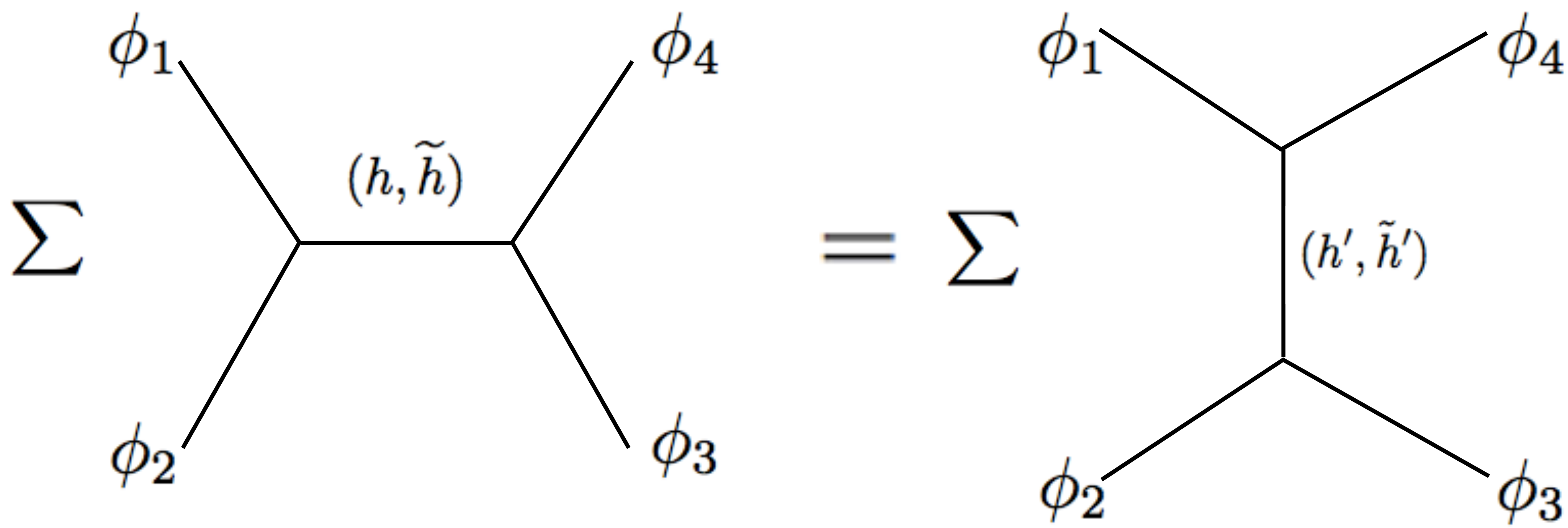
We perform practical computations using Zamolodchikov's recurrence relations, in which the Virasoro blocks are expressed in terms of residue contributions from poles in its analytic continuation in the weights or in the central charge.

The Crossing equation

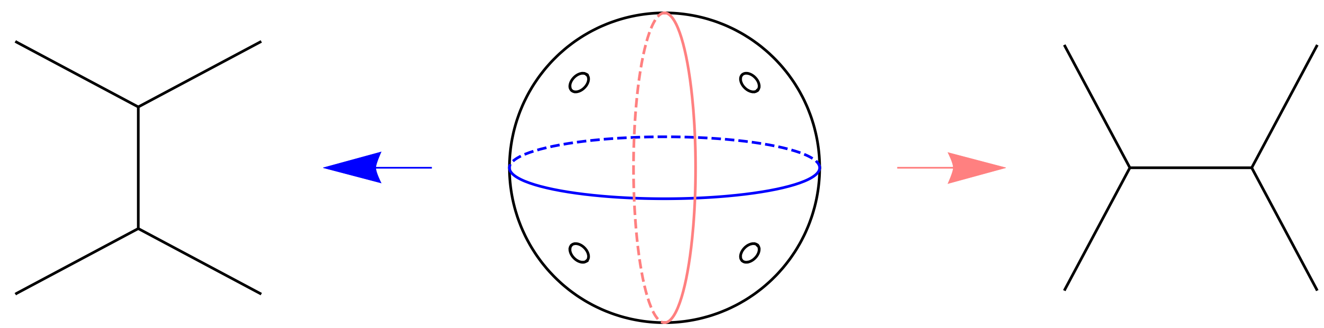


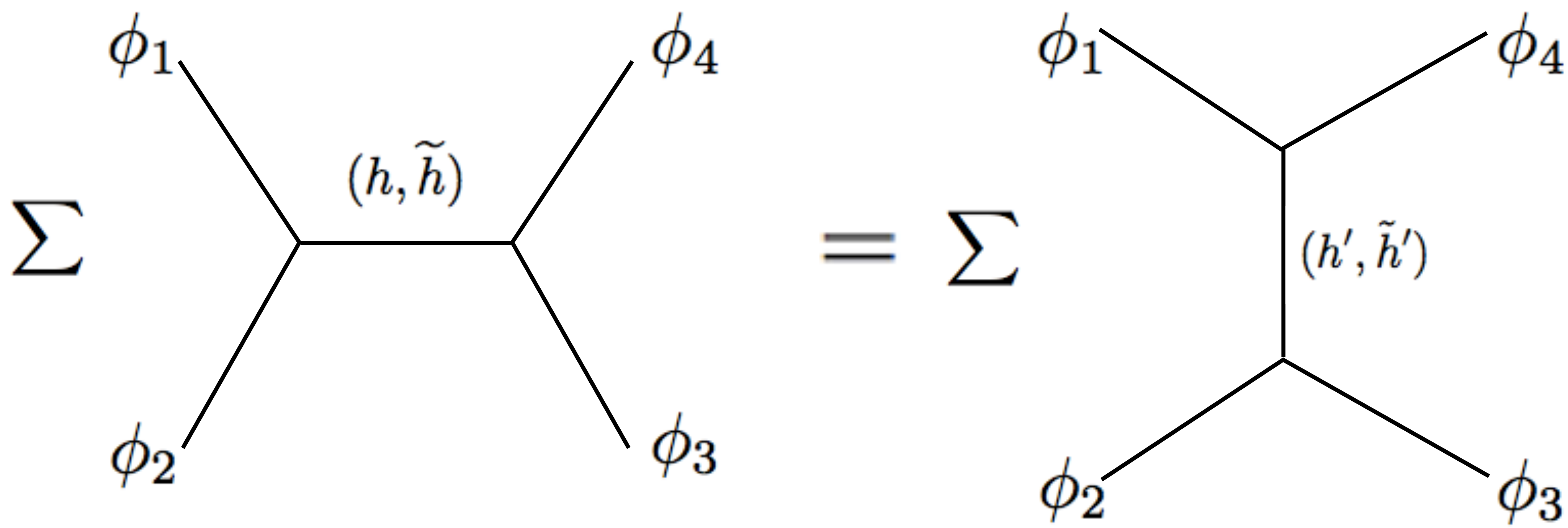
The Crossing equation





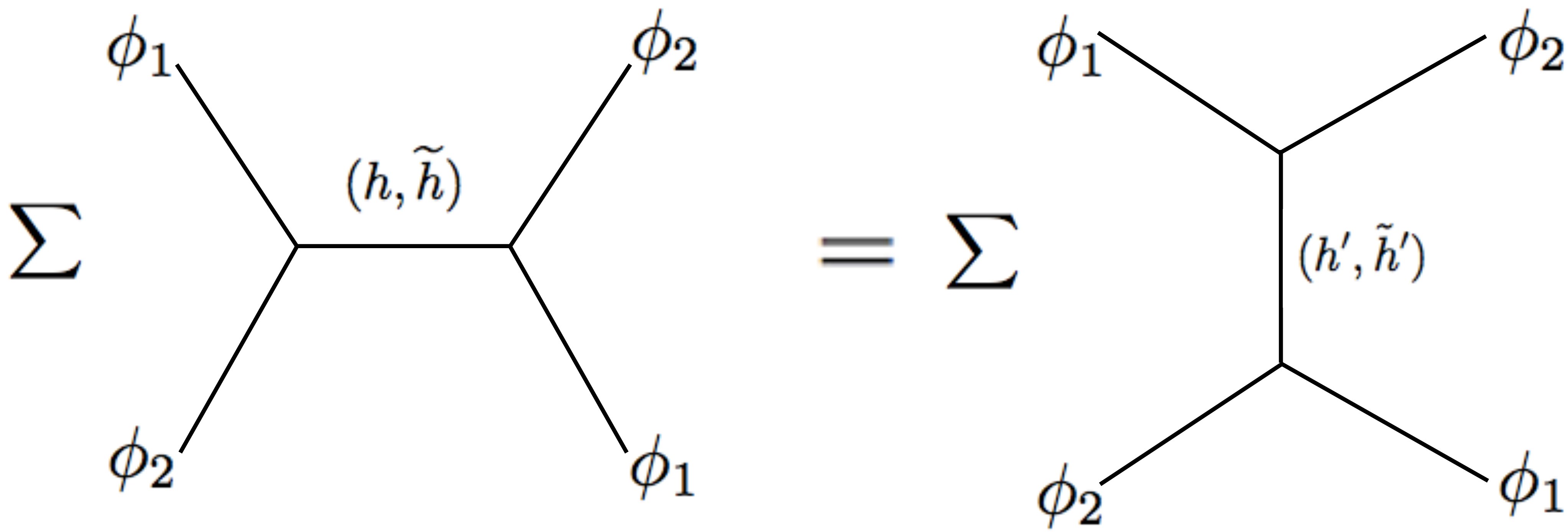
The Crossing equation



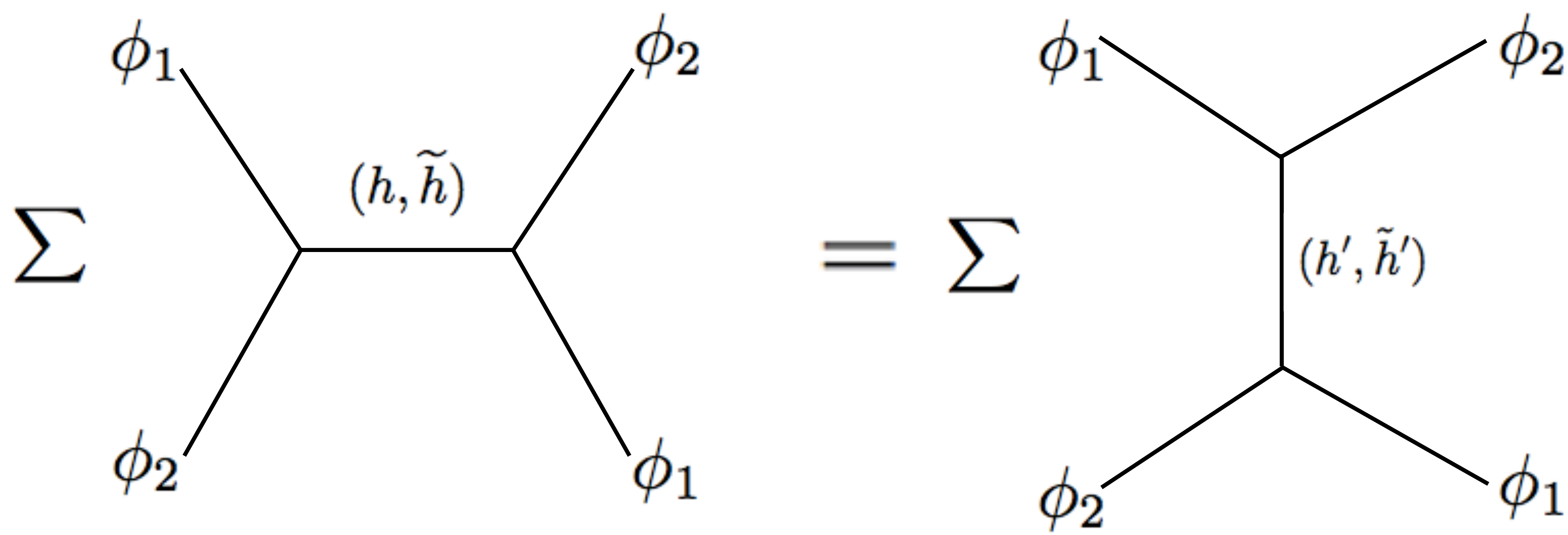


$$\sum_i C_{12i}C_{34i}F_{12|i|34}(z,\bar{z}) = \sum_i C_{14i}C_{32i}F_{14|i|32}(1-z,1-\bar{z})$$

For simplicity, restrict to a pair of primaries

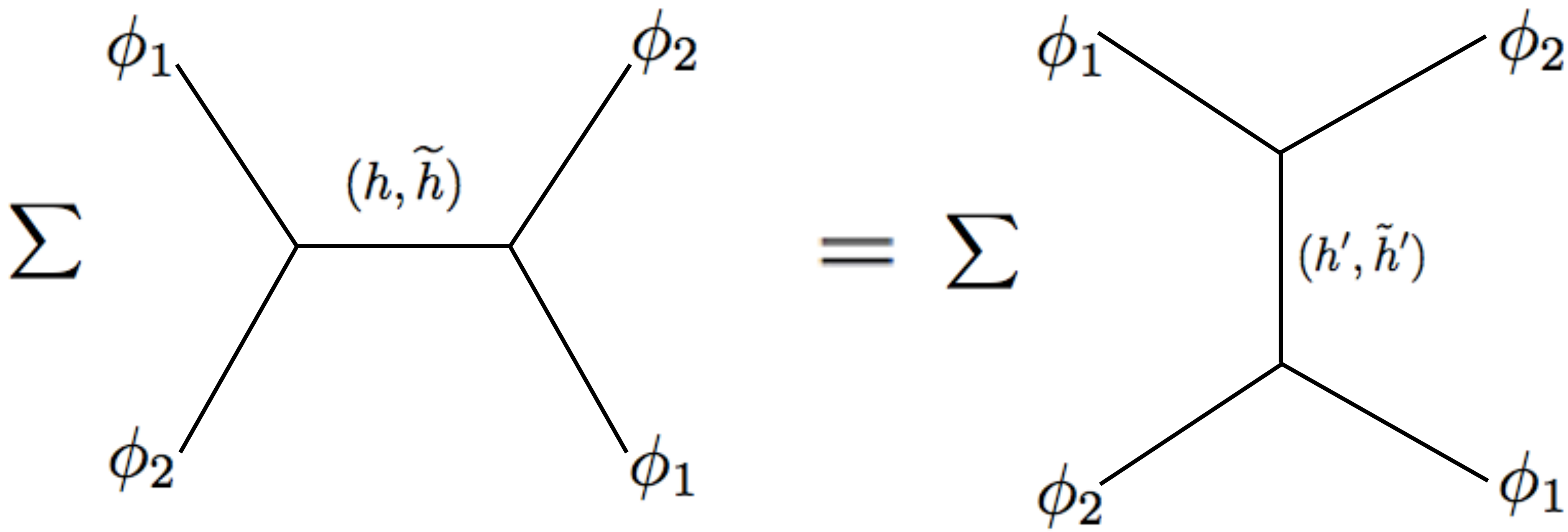


For simplicity, restrict to a pair of primaries



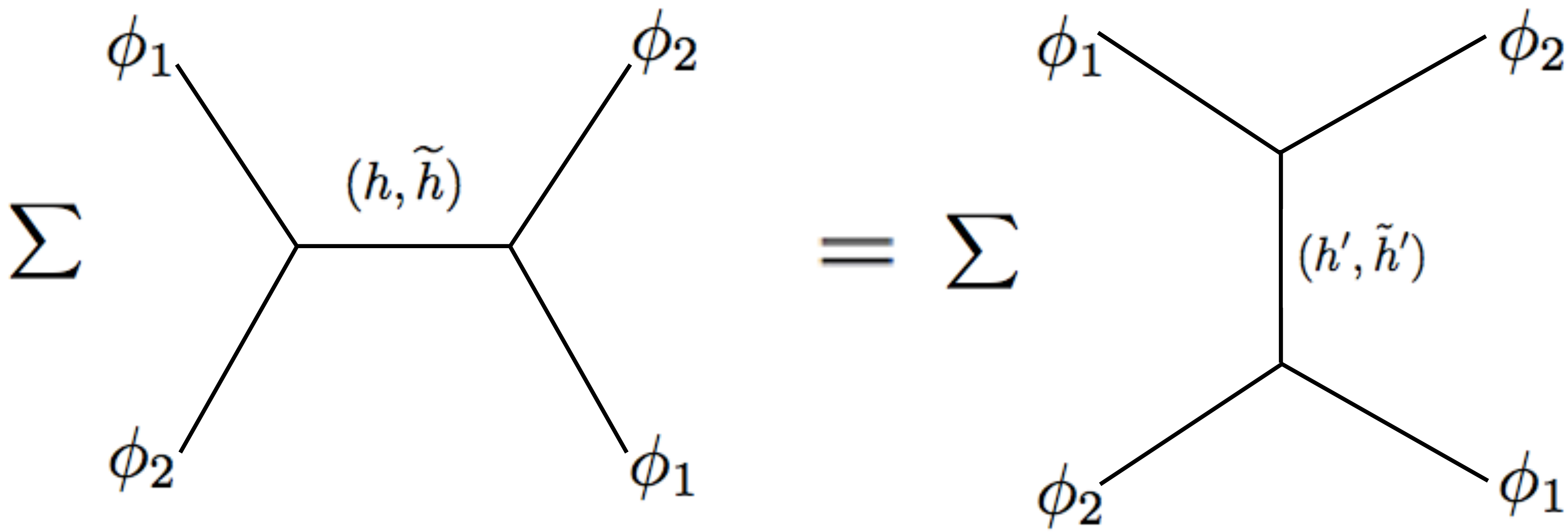
$$\sum_i C_{12i}^2 \left[F_{12|i|12}(z, \bar{z}) - F_{12|i|12}(1 - z, 1 - \bar{z}) \right] = 0$$

For simplicity, restrict to a pair of primaries



$$\sum_i C_{12i}^2 \left[F_{12|i|12}(z, \bar{z}) - F_{12|i|12}(1 - z, 1 - \bar{z}) \right] = 0$$

For simplicity, restrict to a pair of primaries



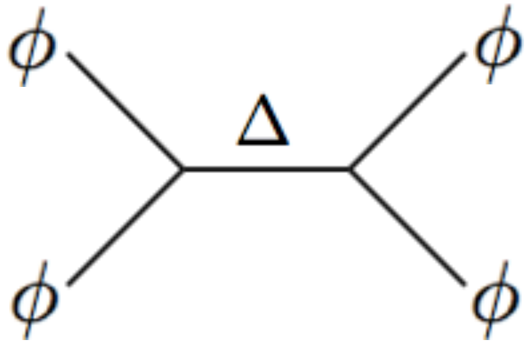
$$\sum_i C_{12i}^2 \left[F_{12|i|12}(z, \bar{z}) - F_{12|i|12}(1 - z, 1 - \bar{z}) \right] = 0$$

In a unitarity CFT, the OPE coefficients are real. We can again exploit the positivity of the coefficients of the conformal block expansion using semidefinite programming.

Example of a bound on the OPE gap:

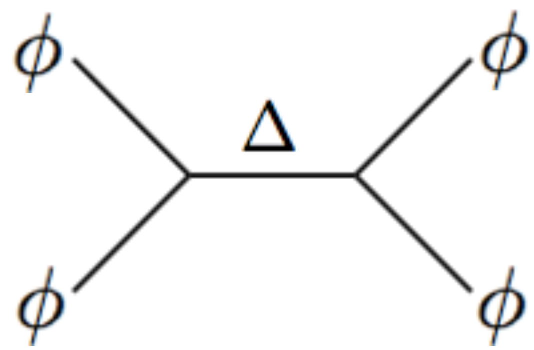
Example of a bound on the OPE gap:

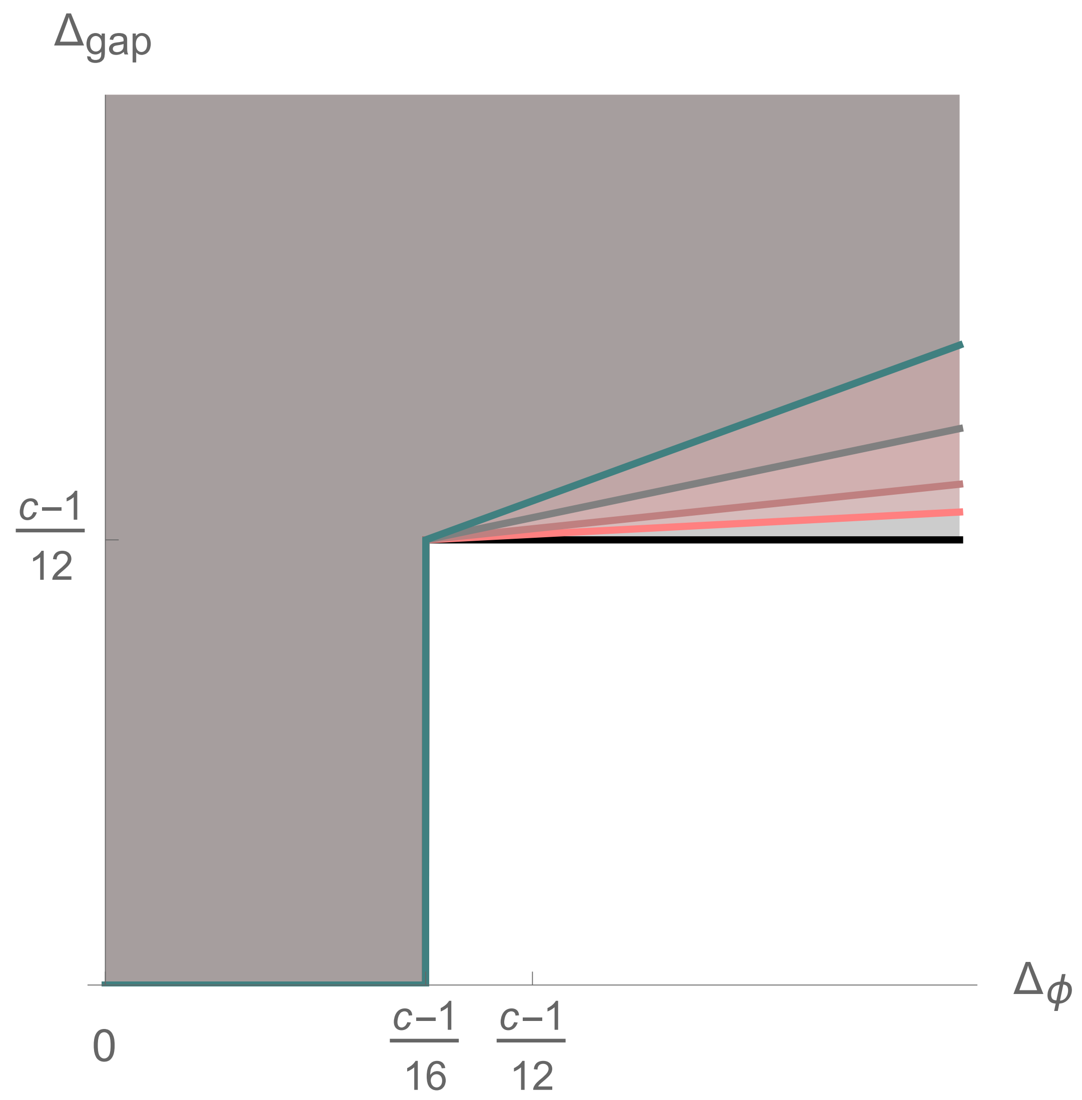
Assuming **only scalar** Virasoro primaries in OPE



Example of a bound on the OPE gap:

Assuming **only scalar** Virasoro primaries in OPE





[van Rees, unpublished; Collier, Lin, XY, unpublished]

Can do better than just bounding the gaps and OPE coefficients of the first few operators!

Can do better than just bounding the gaps and OPE coefficients of the first few operators!

4-point function $$G(z,\bar{z}) = \sum_i C_{12i}^2 F_{12|i|12}(z,\bar{z})$$

Can do better than just bounding the gaps and OPE coefficients of the first few operators!

4-point function $G(z,\bar{z}) = \sum_i C_{12i}^2 F_{12|i|12}(z,\bar{z})$

The spectral function

$$f(\Delta_*) = \frac{1}{G(\frac{1}{2},\frac{1}{2})} \sum_{\Delta_i < \Delta_*} C_{12i}^2 F_{12|i|12}(1/2, 1/2)$$

Can do better than just bounding the gaps and OPE coefficients of the first few operators!

4-point function $G(z,\bar{z}) = \sum_i C_{12i}^2 F_{12|i|12}(z,\bar{z})$

The spectral function

$$f(\Delta_*) = \frac{1}{G(\frac{1}{2},\frac{1}{2})} \sum_{\Delta_i < \Delta_*} C_{12i}^2 F_{12|i|12}(1/2,1/2)$$

captures distribution of OPE coefficients in scaling dimension.

[Kim, Kravchuk, Ooguri, '15; Collier, Kravchuk, Lin, XY, '17]

Can do better than just bounding the gaps and OPE coefficients of the first few operators!

4-point function $G(z,\bar{z}) = \sum_i C_{12i}^2 F_{12|i|12}(z,\bar{z})$

The spectral function

$$f(\Delta_*) = \frac{1}{G(\frac{1}{2},\frac{1}{2})} \sum_{\Delta_i<\Delta_*} C_{12i}^2 F_{12|i|12}(1/2,1/2)$$

captures distribution of OPE coefficients in scaling dimension.

[Kim, Kravchuk, Ooguri, '15; Collier, Kravchuk, Lin, XY, '17]

Can bound spectral function using semidefinite programming.

Can do better than just bounding the gaps and OPE coefficients of the first few operators!

4-point function $\quad G(z,\bar{z}) = \sum_i C_{12i}^2 F_{12|i|12}(z,\bar{z})$

The spectral function

$$f(\Delta_*) = \frac{1}{G(\frac{1}{2},\frac{1}{2})} \sum_{\Delta_i<\Delta_*} C_{12i}^2 F_{12|i|12}(1/2,1/2)$$

captures distribution of OPE coefficients in scaling dimension.

[Kim, Kravchuk, Ooguri, '15; Collier, Kravchuk, Lin, XY, '17]

Can bound spectral function using semidefinite programming.

Recall crossing equation, in the schematic form

$$\sum_\Delta C_\Delta^2 F_\Delta^{(m,n)} = 0, \qquad F_\Delta^{(m,n)} \equiv \partial_z^m \partial_{\bar{z}}^n F_\Delta \big|_{z=\bar{z}=\frac{1}{2}}, \quad m+n \text{ odd}$$

Now consider the inequality

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \geq 0, \quad \forall\, \Delta \in \mathcal{I}. \qquad (\star)$$

Now consider the inequality

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \geq 0, \quad \forall\, \Delta \in \mathcal{I}. \qquad (\star)$$

If $(\star)$ is obeyed by a set of coefficients $y_{0,0}$, $y_{m,n}$ $(m+n \text{ odd})$

Now consider the inequality

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \geq 0, \quad \forall \, \Delta \in \mathcal{I}. \qquad (\star)$$

If $(\star)$ is obeyed by a set of coefficients $y_{0,0}, \quad y_{m,n} \quad (m+n \text{ odd})$

then multiply by C_Δ^2, and sum over Δ, the coefficients $y_{m,n}$ drop out by virtue of crossing equation, and we end up with

$$f(\Delta_*) \geq y_{0,0}$$

Now consider the inequality

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \geq 0, \quad \forall\, \Delta \in \mathcal{I}. \qquad (\star)$$

If $(\star)$ is obeyed by a set of coefficients $\quad y_{0,0}, \quad y_{m,n} \quad (m+n \text{ odd})$

then multiply by C_Δ^2, and sum over Δ, the coefficients $y_{m,n}$ drop out by virtue of crossing equation, and we end up with

$$f(\Delta_*) \geq y_{0,0}$$

i.e. $y_{0,0}$ is a lower bound on the spectral function.

Now consider the inequality

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \geq 0, \quad \forall\, \Delta \in \mathcal{I}. \qquad (\star)$$

If $(\star)$ is obeyed by a set of coefficients $y_{0,0}, \quad y_{m,n} \quad (m+n \text{ odd})$

then multiply by C_Δ^2, and sum over Δ, the coefficients $y_{m,n}$ drop out by virtue of crossing equation, and we end up with

$$f(\Delta_*) \geq y_{0,0}$$

i.e. $y_{0,0}$ is a lower bound on the spectral function.

Optimal lower bound achieved by maximizing $y_{0,0}$ subject to $(\star)$.

Now consider the inequality

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \geq 0, \quad \forall\, \Delta \in \mathcal{I}. \qquad (\star)$$

If $(\star)$ is obeyed by a set of coefficients $y_{0,0}, \quad y_{m,n} \quad (m+n \text{ odd})$

then multiply by C_Δ^2, and sum over Δ, the coefficients $y_{m,n}$ drop out by virtue of crossing equation, and we end up with

$$f(\Delta_*) \geq y_{0,0}$$

i.e. $y_{0,0}$ is a lower bound on the spectral function.

Optimal lower bound achieved by maximizing $y_{0,0}$ subject to $(\star)$.

Likewise, optimal upper bound obtained by minimizing $y_{0,0}$ subject to

$$\theta(\Delta_* - \Delta) F_\Delta^{(0,0)} - y_{0,0} F_\Delta^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n} F_\Delta^{(m,n)} \leq 0, \quad \forall\, \Delta \in \mathcal{I}.$$

Crossing invariance of sphere 4-point function

Crossing invariance of sphere 4-point function

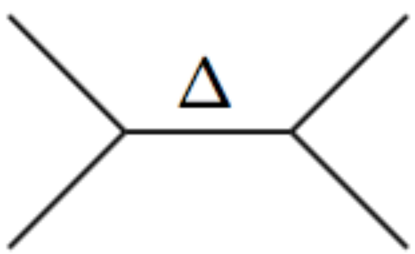
Assuming **only scalar** Virasoro primaries (c>1) [Collier-Kravchuk-Lin-XY '17]

Crossing invariance of sphere 4-point function

Assuming **only scalar** Virasoro primaries (c>1) [Collier-Kravchuk-Lin-XY '17]

Upper and lower bounds on spectral function:

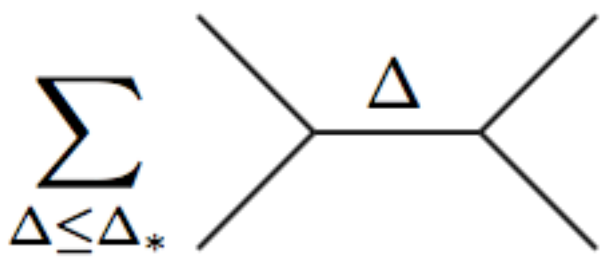
$$\sum_{\Delta \leq \Delta_*}$$

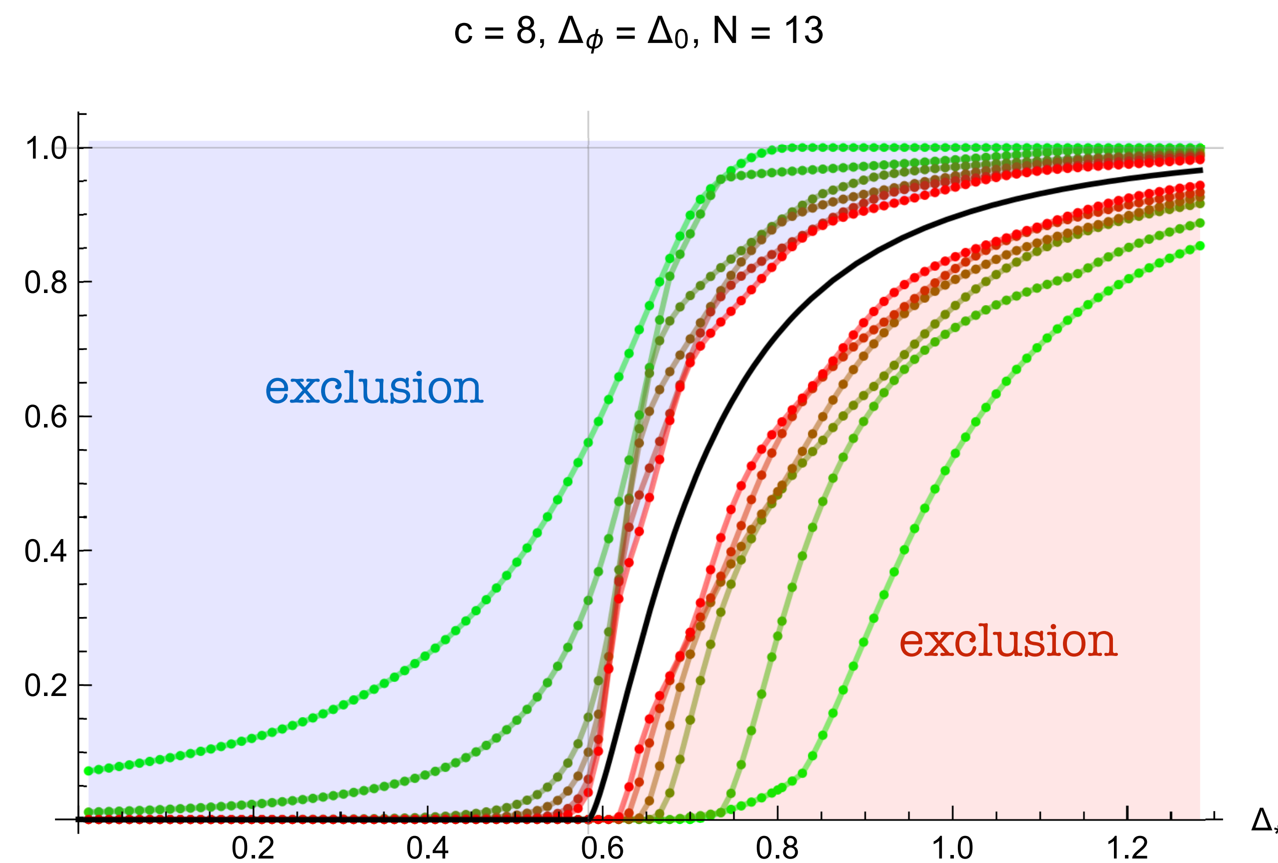


Crossing invariance of sphere 4-point function

Assuming **only scalar** Virasoro primaries (c>1) [Collier-Kravchuk-Lin-XY '17]

Upper and lower bounds on spectral function:

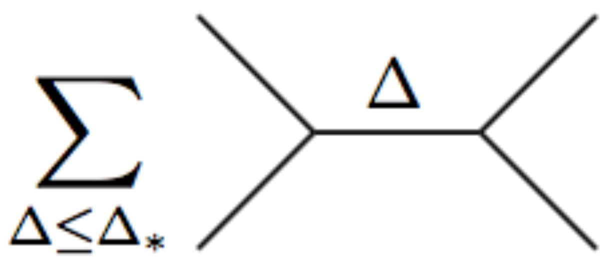


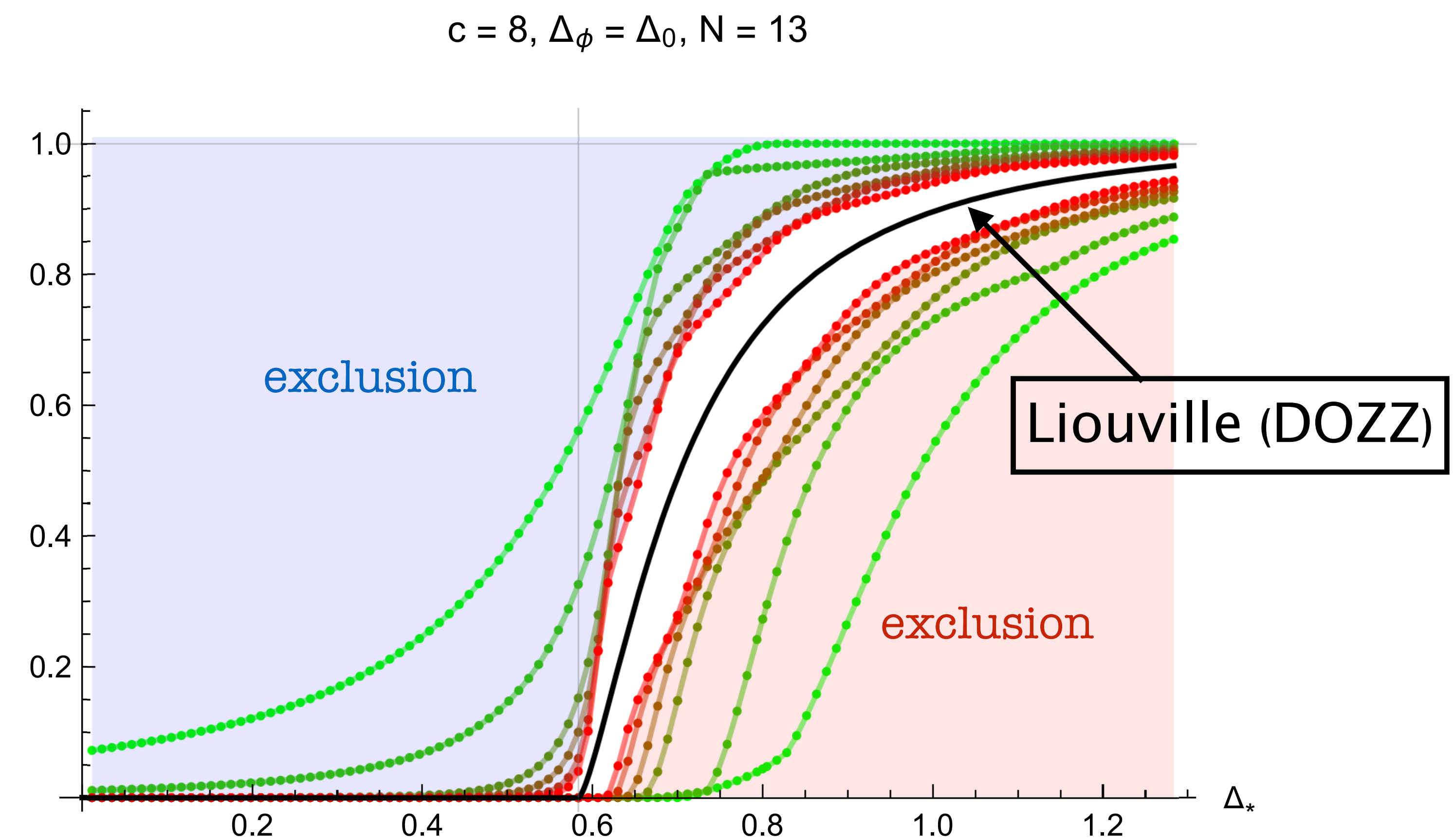


Crossing invariance of sphere 4-point function

Assuming **<u>only scalar</u>** Virasoro primaries (c>1) [Collier-Kravchuk-Lin-XY '17]

Upper and lower bounds on spectral function:

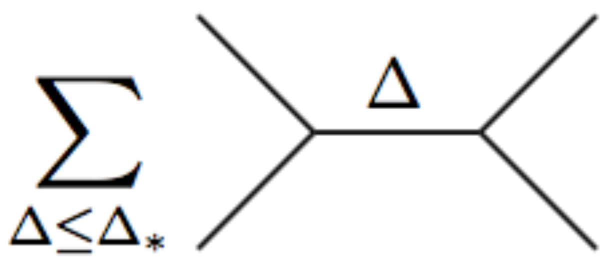


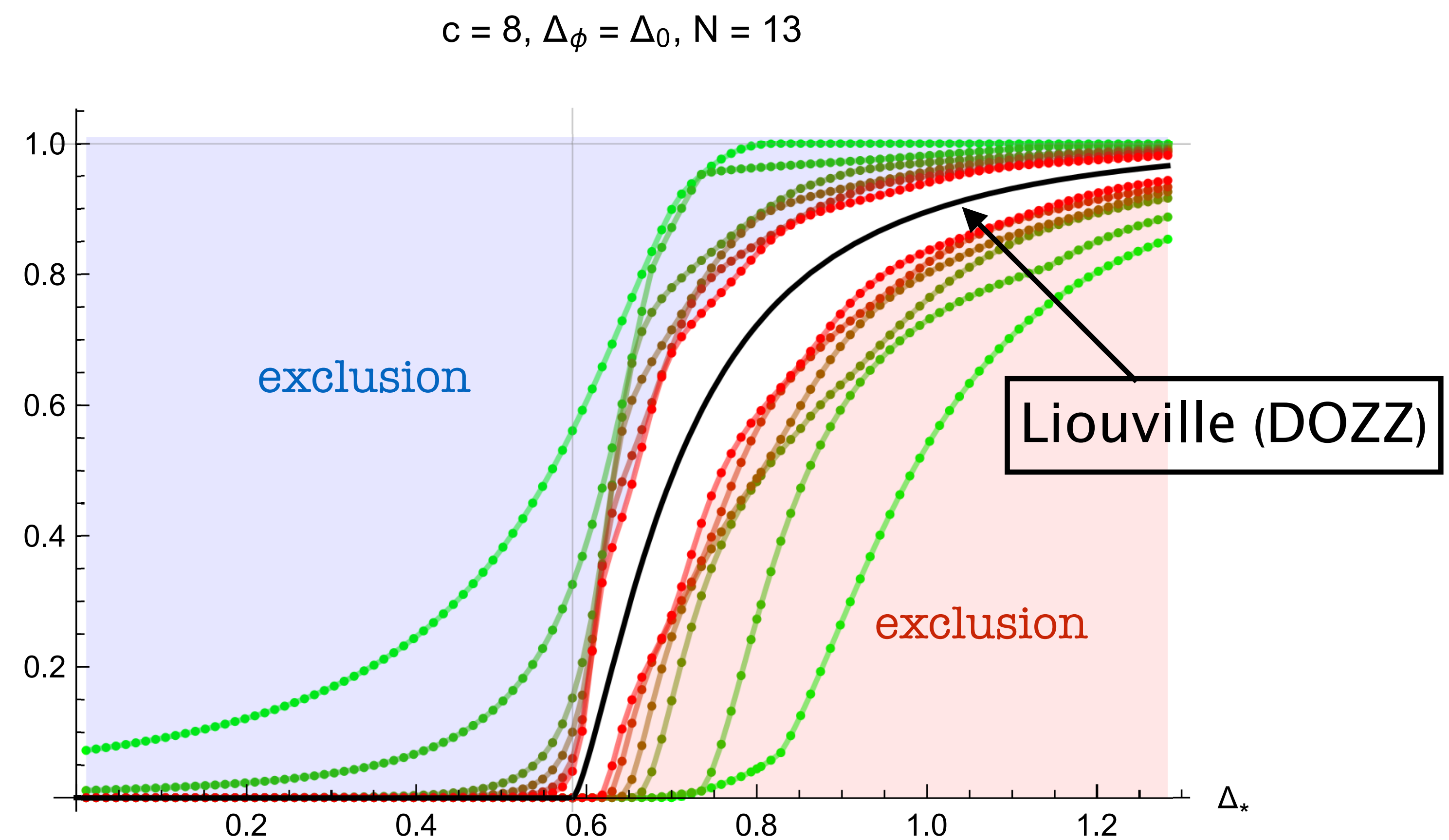


Crossing invariance of sphere 4-point function

Assuming **only scalar** Virasoro primaries (c>1) [Collier-Kravchuk-Lin-XY '17]

Upper and lower bounds on spectral function:





Conjecture: the bounds pin down Liouville CFT

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov^2, '95, Teschner '95, Ponsot-Teschner '99]

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov^2, '95, Teschner '95, Ponsot-Teschner '99]

$$S_{\text{Liouville}} = \frac{1}{4\pi}\int d^2z\sqrt{g}\left(g^{mn}\partial_m\phi\partial_n\phi + QR\phi + 4\pi\mu e^{2b\phi}\right)$$

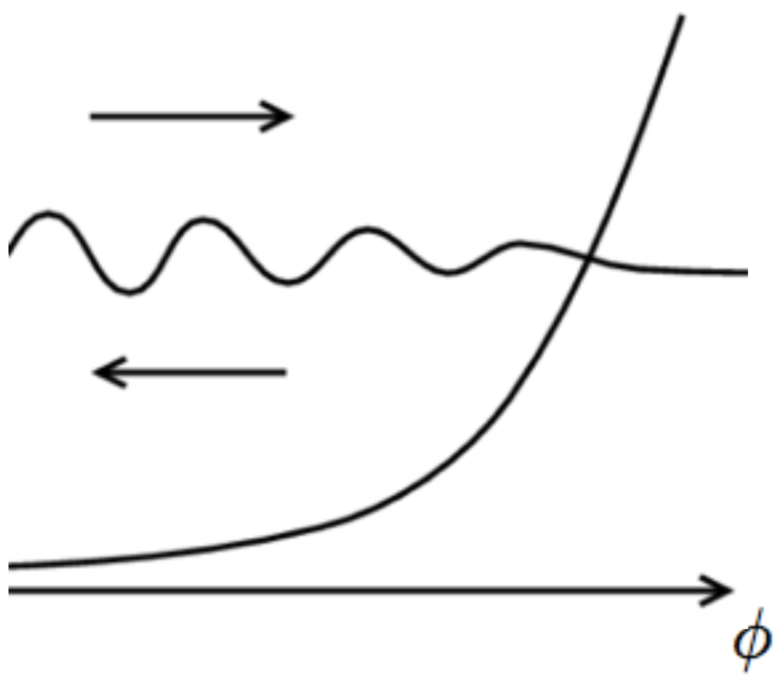
$$c = 1 + 6Q^2 \qquad\qquad Q = b + b^{-1}$$

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

$$S_{\text{Liouville}} = \frac{1}{4\pi} \int d^2 z \sqrt{g} \left(g^{mn} \partial_m \phi \partial_n \phi + Q R \phi + 4\pi \mu e^{2b\phi} \right)$$

$$c = 1 + 6Q^2 \qquad Q = b + b^{-1}$$

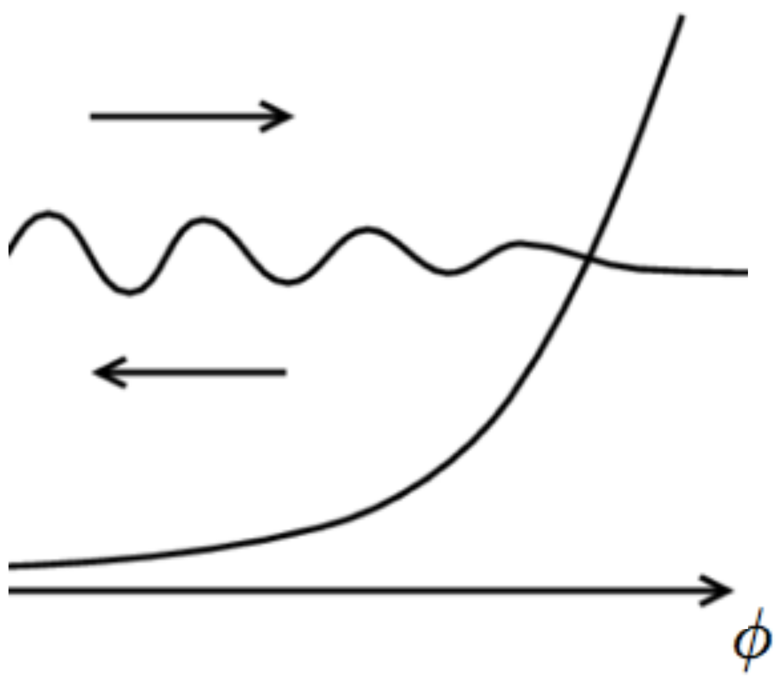


Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

$$S_{\text{Liouville}} = \frac{1}{4\pi}\int d^2z\sqrt{g}\left(g^{mn}\partial_m\phi\partial_n\phi + QR\phi + 4\pi\mu e^{2b\phi}\right)$$

$$c = 1 + 6Q^2 \qquad Q = b + b^{-1}$$



Virasoro primary operators take the form

$$\mathcal{V}_\alpha \sim S(\alpha)^{-\frac{1}{2}}e^{2\alpha\phi} + S(\alpha)^{\frac{1}{2}}e^{2(Q-\alpha)\phi}$$

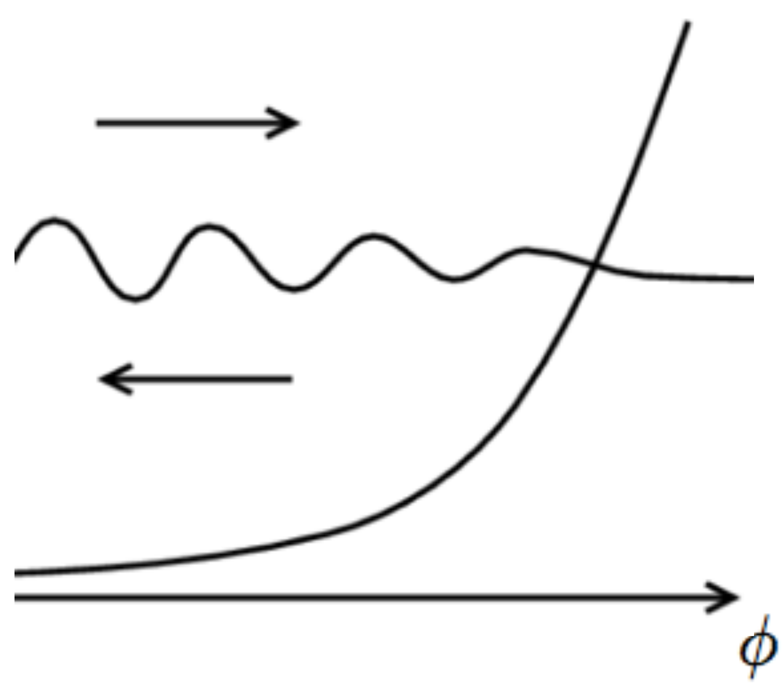
$$\phi \to -\infty$$

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

$$S_{\text{Liouville}} = \frac{1}{4\pi}\int d^2z\sqrt{g}\left(g^{mn}\partial_m\phi\partial_n\phi + QR\phi + 4\pi\mu e^{2b\phi}\right)$$

$$c = 1 + 6Q^2 \qquad Q = b + b^{-1}$$



Virasoro primary operators take the form

$$\mathcal{V}_\alpha \sim S(\alpha)^{-\frac{1}{2}}e^{2\alpha\phi} + S(\alpha)^{\frac{1}{2}}e^{2(Q-\alpha)\phi}$$

$$\phi \to -\infty$$

$$\alpha = \frac{Q}{2} + iP$$

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

Reflection coefficient:

$$S(\alpha) = -\left(\pi\mu\gamma(b^2)\right)^{(Q-2\alpha)/b}\frac{\Gamma(1-(Q-2\alpha)/b)\Gamma(1-(Q-2\alpha)b)}{\Gamma(1+(Q-2\alpha)/b)\Gamma(1+(Q-2\alpha)b)}$$

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

Reflection coefficient:

$$S(\alpha) = -\left(\pi\mu\gamma(b^2)\right)^{(Q-2\alpha)/b} \frac{\Gamma(1-(Q-2\alpha)/b)\Gamma(1-(Q-2\alpha)b)}{\Gamma(1+(Q-2\alpha)/b)\Gamma(1+(Q-2\alpha)b)}$$

DOZZ structure constants:

$$\langle \mathcal{V}_{\alpha_1}\mathcal{V}_{\alpha_2}\mathcal{V}_{\alpha_3}\rangle = \prod_{j=1}^{3} S(\alpha_i)^{-\frac{1}{2}} \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\sum\alpha_i}{b}}$$
$$\times \frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum\alpha_i - Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}$$

Brief recap of Liouville CFT

[Seiberg '91, Dorn-Otto '94, Zamolodchikov^2, '95, Teschner '95, Ponsot-Teschner '99]

Reflection coefficient:

$$S(\alpha) = -\left(\pi\mu\gamma(b^2)\right)^{(Q-2\alpha)/b}\frac{\Gamma(1-(Q-2\alpha)/b)\Gamma(1-(Q-2\alpha)b)}{\Gamma(1+(Q-2\alpha)/b)\Gamma(1+(Q-2\alpha)b)}$$

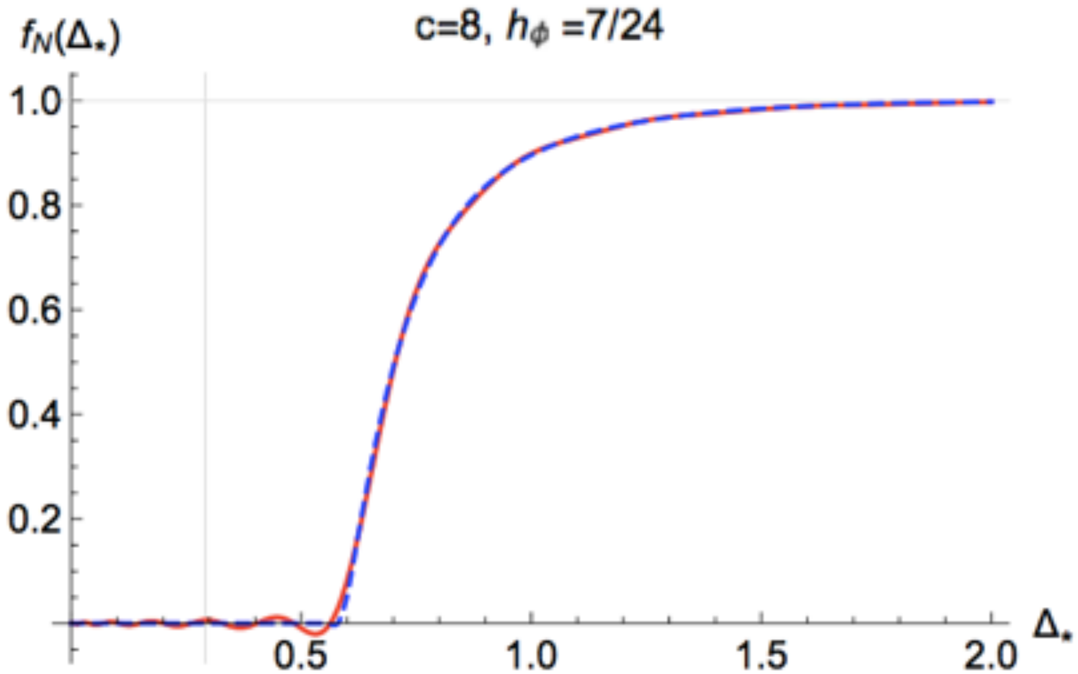
DOZZ structure constants:

$$\langle \mathcal{V}_{\alpha_1}\mathcal{V}_{\alpha_2}\mathcal{V}_{\alpha_3}\rangle = \prod_{j=1}^{3} S(\alpha_i)^{-\frac{1}{2}}\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\sum\alpha_i}{b}}$$
$$\times\frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum\alpha_i - Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$
$$\log\Upsilon_b(x) = \int_0^\infty dt\, t^{-1}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left[\left(\frac{Q}{2}-x\right)\frac{t}{2}\right]}{\sinh\frac{tb}{2}\sinh\frac{t}{2b}}\right],\ 0<\mathrm{Re}(x)<\mathrm{Re}(Q)$$

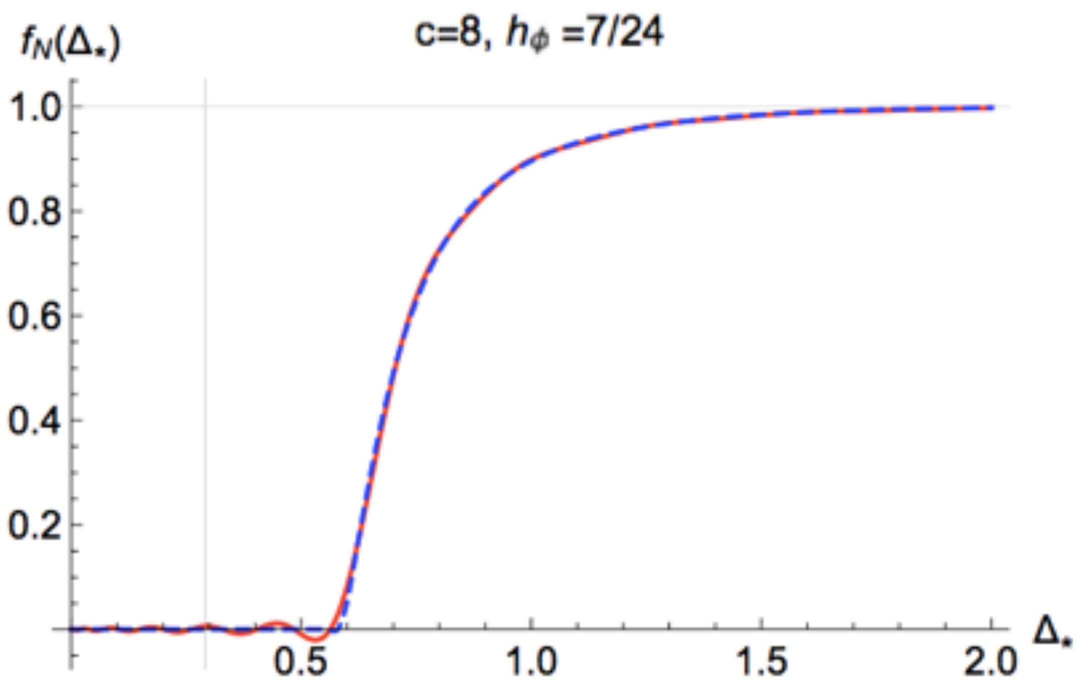
Direct numerical solution for the scalar-only spectral function from truncated crossing equation:

[Collier, Kravchuk, Lin, XY, '17]



Direct numerical solution for the scalar-only spectral function from truncated crossing equation:

[Collier, Kravchuk, Lin, XY, '17]

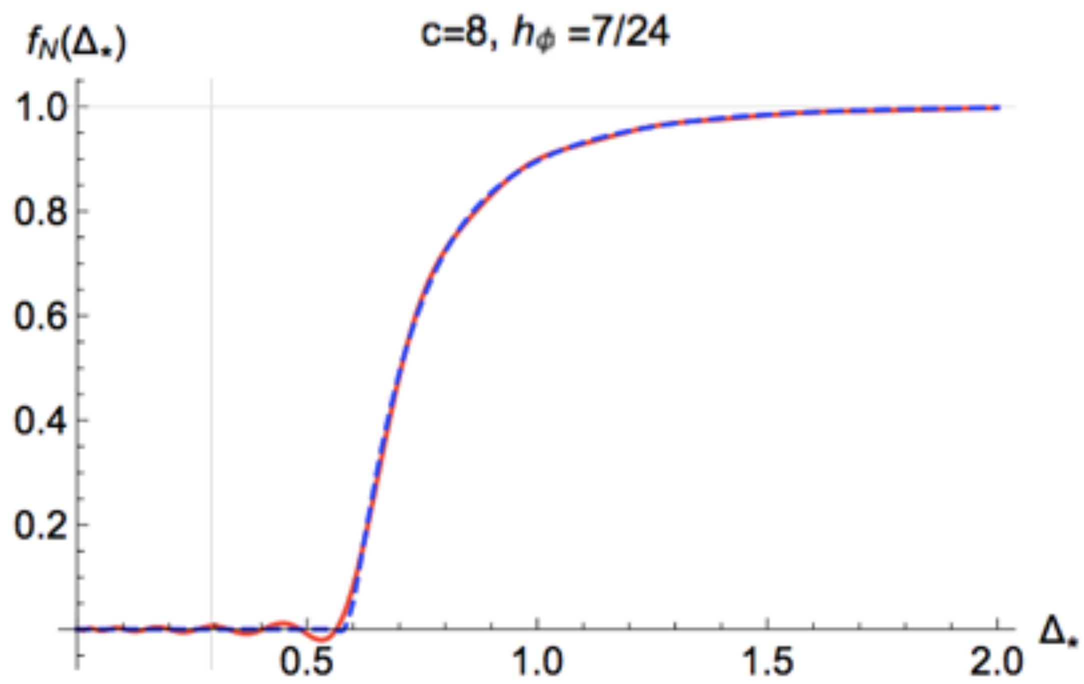


DOZZ

numerical solution from crossing

Direct numerical solution for the scalar-only spectral function from truncated crossing equation:

[Collier, Kravchuk, Lin, XY, '17]



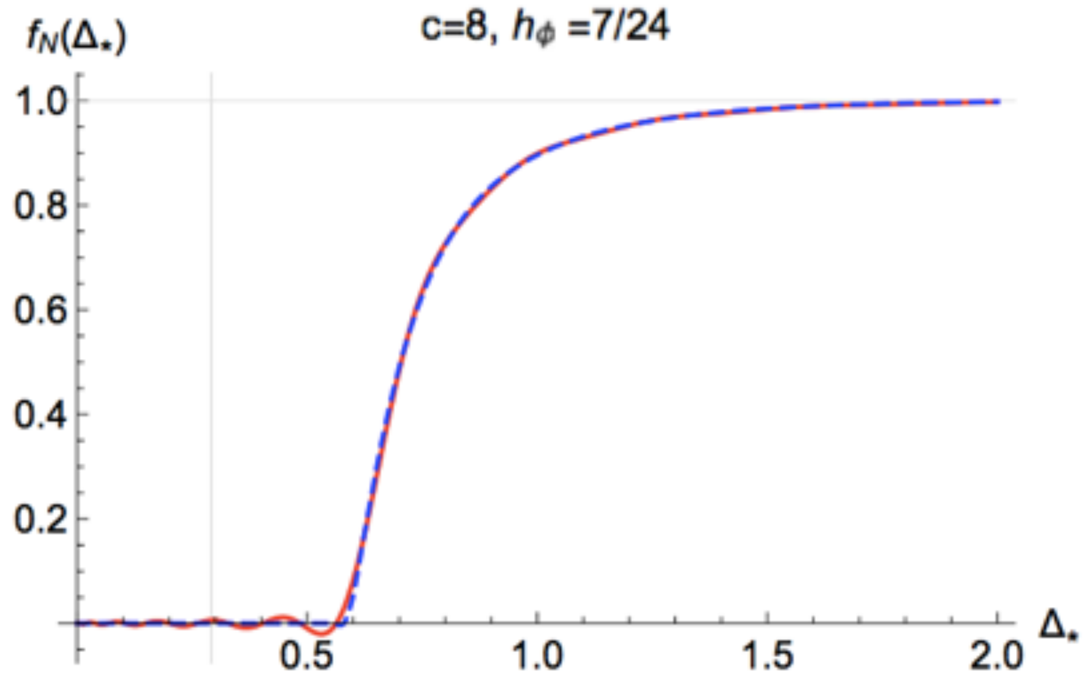
DOZZ

numerical solution from crossing

(No assumption of unitarity here!)

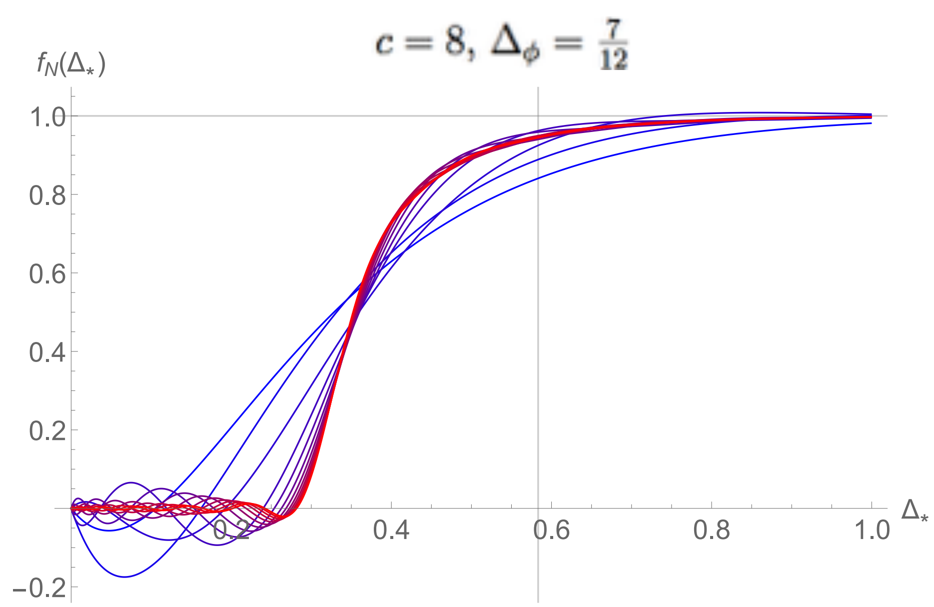
Direct numerical solution for the scalar-only spectral function from truncated crossing equation:

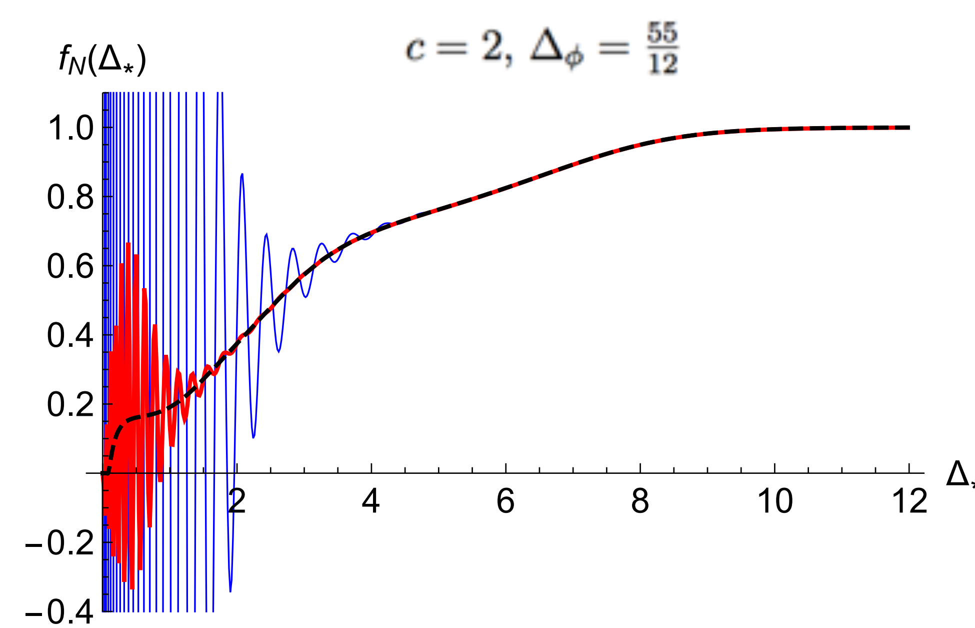
[Collier, Kravchuk, Lin, XY, '17]



(No assumption of unitarity here!)

Examples that demonstrate numerical convergence:





A few words on superconformal theories

A few words on superconformal theories

In (2,2) SCFT, marginality implies exactly marginality.
[Dixon '87, Green, Komargodski, Seiberg, Tachikawa '10]

A few words on superconformal theories

In (2,2) SCFT, marginality implies exactly marginality.
[Dixon '87, Green, Komargodski, Seiberg, Tachikawa '10]

They provide abundant examples of interacting, compact, irrational CFTs (along the conformal manifold).

A few words on superconformal theories

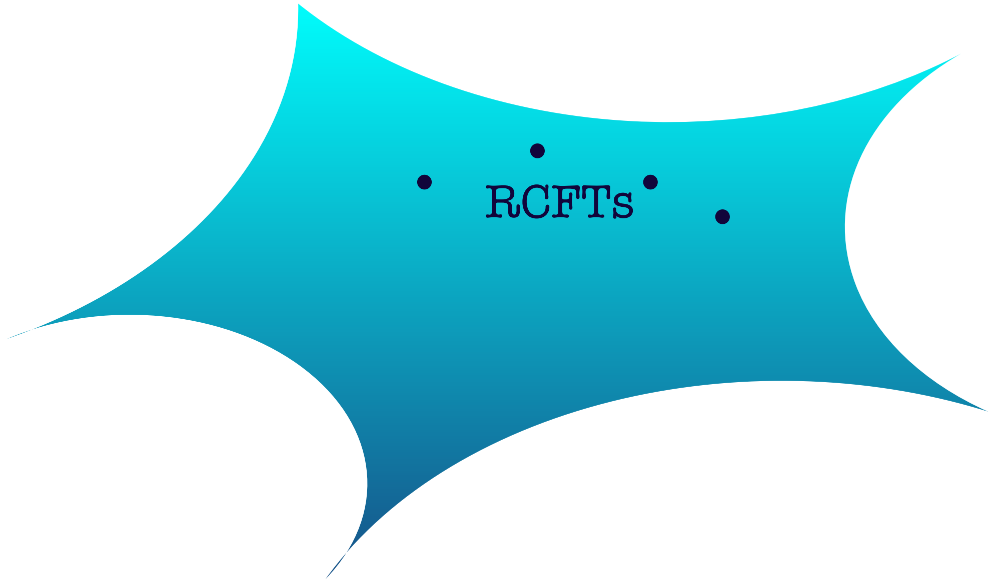
In (2,2) SCFT, marginality implies exactly marginality.
[Dixon '87, Green, Komargodski, Seiberg, Tachikawa '10]

They provide abundant examples of interacting, compact, irrational CFTs (along the conformal manifold).

A few words on superconformal theories

In (2,2) SCFT, marginality implies exactly marginality.
[Dixon '87, Green, Komargodski, Seiberg, Tachikawa '10]

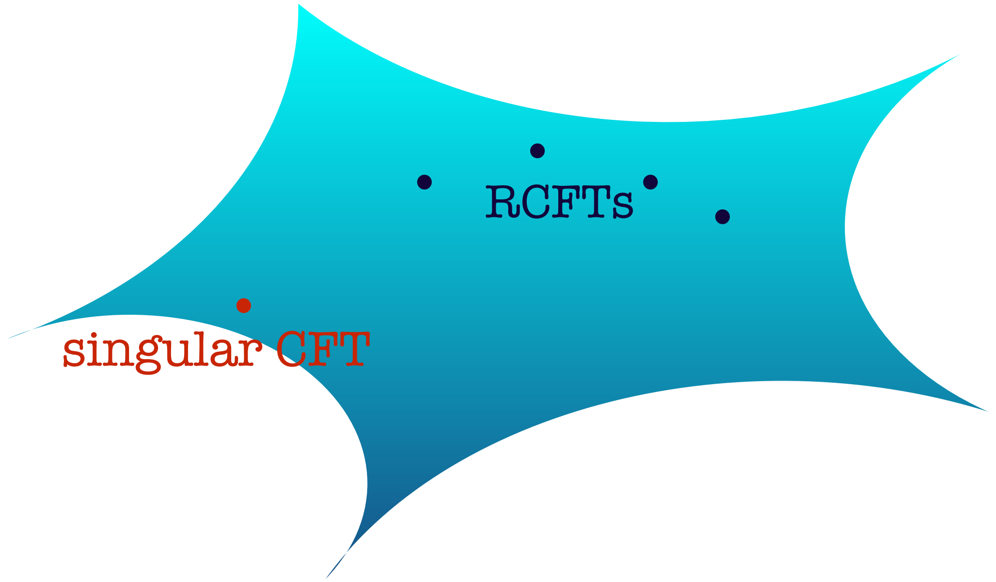
They provide abundant examples of interacting, compact, irrational CFTs (along the conformal manifold).



A few words on superconformal theories

In (2,2) SCFT, marginality implies exactly marginality.
[Dixon '87, Green, Komargodski, Seiberg, Tachikawa '10]

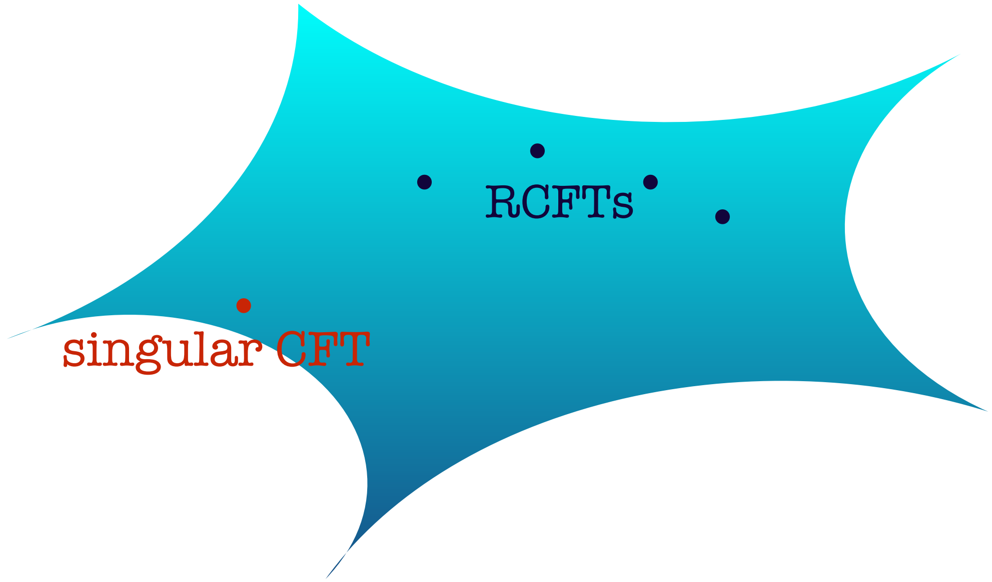
They provide abundant examples of interacting, compact, irrational CFTs (along the conformal manifold).



A few words on superconformal theories

In (2,2) SCFT, marginality implies exactly marginality.
[Dixon '87, Green, Komargodski, Seiberg, Tachikawa '10]

They provide abundant examples of interacting, compact, irrational CFTs (along the conformal manifold).



Bootstrap method allows us to get a handle on the non-BPS operators in SCFTs, by analyzing e.g. the OPE of BPS operators.

Some technical ingredients

Some technical ingredients

1. Moduli dependence is fed into the crossing equation through chiral ring relations and/or protected BPS correlators.

Some technical ingredients

1. Moduli dependence is fed into the crossing equation through chiral ring relations and/or protected BPS correlators.

e.g. in superconformal NLSM on K3, can determine integrated half BPS 4-point function [Kiritsis-Obers-Pioline, '00, Lin-Shao-Wang-XY, '15]

$$\int \frac{d^2 z}{|z(1-z)|}\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0)\mathcal{O}_k(1)\mathcal{O}_\ell(\infty)\rangle = \left.\frac{\partial^4}{\partial y^i \partial y^j \partial y^k \partial y^\ell}\right|_{y=0} \int_{\mathcal{F}} d^2\tau \frac{\Theta_\Lambda(y|\tau,\bar{\tau})}{\eta(\tau)^{24}}$$

Some technical ingredients

1. Moduli dependence is fed into the crossing equation through chiral ring relations and/or protected BPS correlators.

e.g. in superconformal NLSM on K3, can determine integrated half BPS 4-point function [Kiritsis-Obers-Pioline, '00, Lin-Shao-Wang-XY, '15]

$$\int \frac{d^2 z}{|z(1-z)|} \langle \mathcal{O}_i(z,\bar{z}) \mathcal{O}_j(0) \mathcal{O}_k(1) \mathcal{O}_\ell(\infty) \rangle = \frac{\partial^4}{\partial y^i \partial y^j \partial y^k \partial y^\ell}\bigg|_{y=0} \int_{\mathcal{F}} d^2\tau \frac{\Theta_\Lambda(y|\tau,\bar{\tau})}{\eta(\tau)^{24}}$$

2. N=2 Super-Virasoro conformal blocks known for BPS external operators.

Some technical ingredients

1. Moduli dependence is fed into the crossing equation through chiral ring relations and/or protected BPS correlators.

e.g. in superconformal NLSM on K3, can determine integrated half BPS 4-point function [Kiritsis-Obers-Pioline, ’00, Lin-Shao-Wang-XY, ‘15]

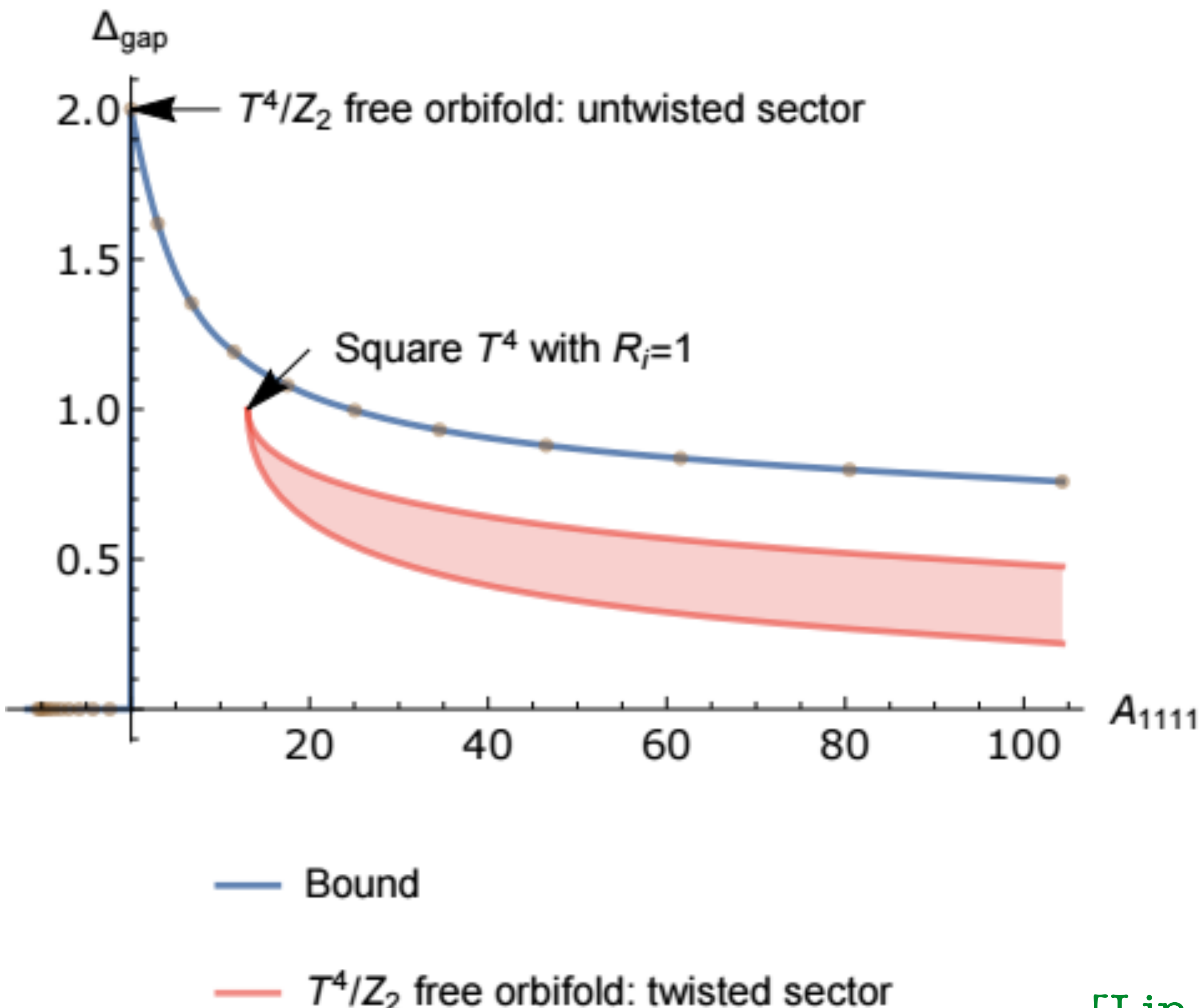
$$\int \frac{d^2 z}{|z(1-z)|} \langle \mathcal{O}_i(z,\bar{z}) \mathcal{O}_j(0) \mathcal{O}_k(1) \mathcal{O}_\ell(\infty) \rangle = \frac{\partial^4}{\partial y^i \partial y^j \partial y^k \partial y^\ell} \bigg|_{y=0} \int_{\mathcal{F}} d^2\tau \frac{\Theta_\Lambda(y|\tau,\bar{\tau})}{\eta(\tau)^{24}}$$

2. N=2 Super-Virasoro conformal blocks known for BPS external operators.

determined from BPS correlators in N=2 cigar SCFT, related to bosonic Virasoro blocks in simple ways [Chang, Lin, Shao, Wang, XY, ’14]

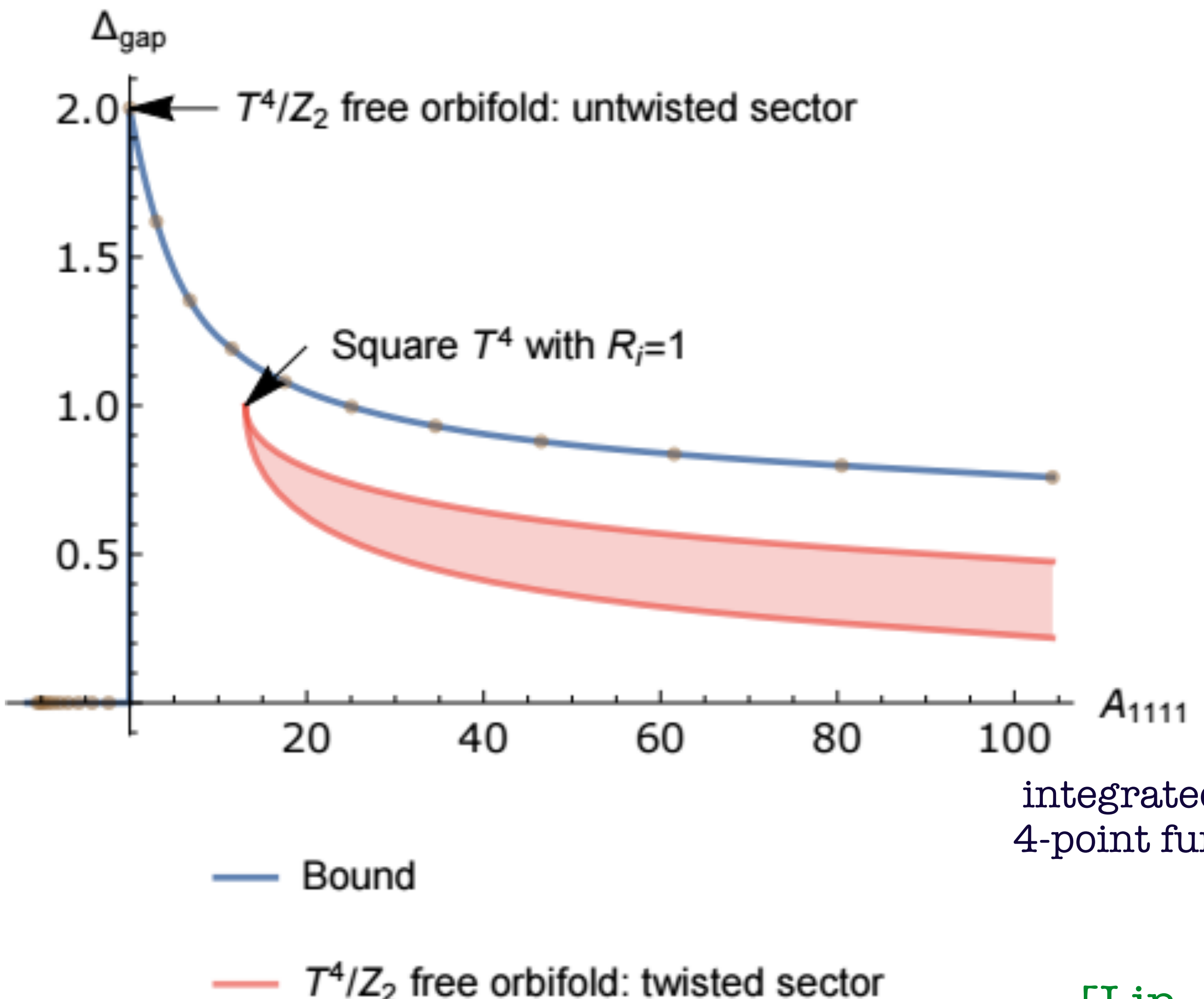
Example 1: for K3 CFT, bounding the gap of non-BPS primaries in the OPE of a pair of 1/2-BPS operators along the moduli space.

Example 1: for K3 CFT, bounding the gap of non-BPS primaries in the OPE of a pair of 1/2-BPS operators along the moduli space.



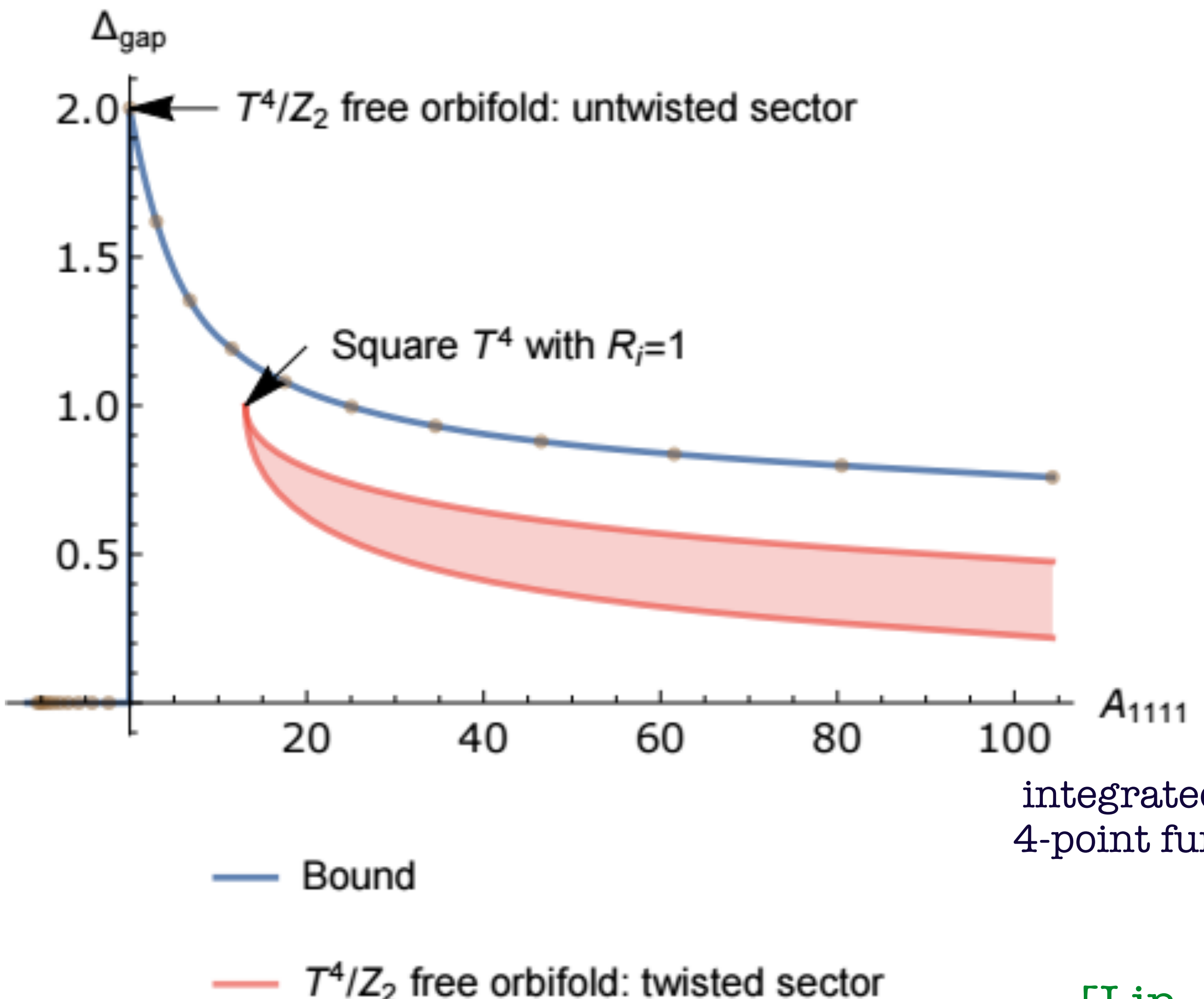
[Lin, Shao, Wang, Simmons-Duffin, XY '15]

Example 1: for K3 CFT, bounding the gap of non-BPS primaries in the OPE of a pair of 1/2-BPS operators along the moduli space.

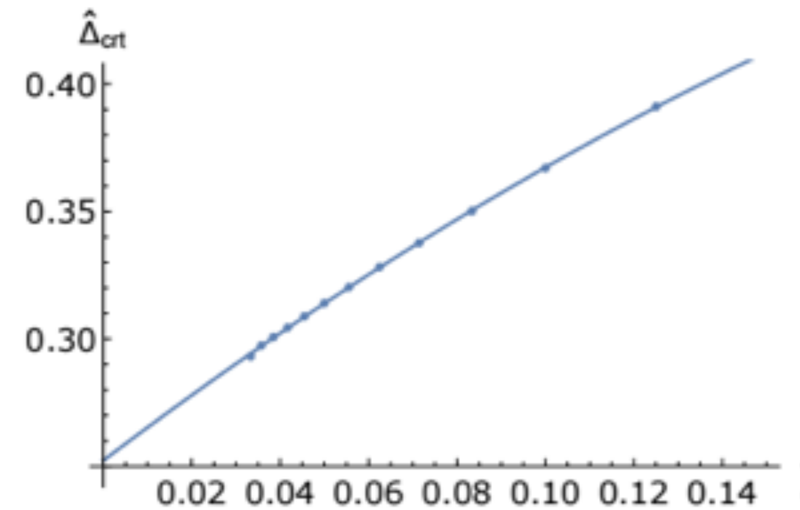


[Lin, Shao, Wang, Simmons-Duffin, XY '15]

Example 1: for K3 CFT, bounding the gap of non-BPS primaries in the OPE of a pair of 1/2-BPS operators along the moduli space.



integrated BPS
4-point function



Gap bound saturated by A_1 cigar CFT (c=6, N=4 Liouville)

[Lin, Shao, Wang, Simmons-Duffin, XY '15]

Example 2: (2,2) SCFTs with marginal deformation

[Lin, Shao, Wang, XY '16]

Example 2: (2,2) SCFTs with marginal deformation

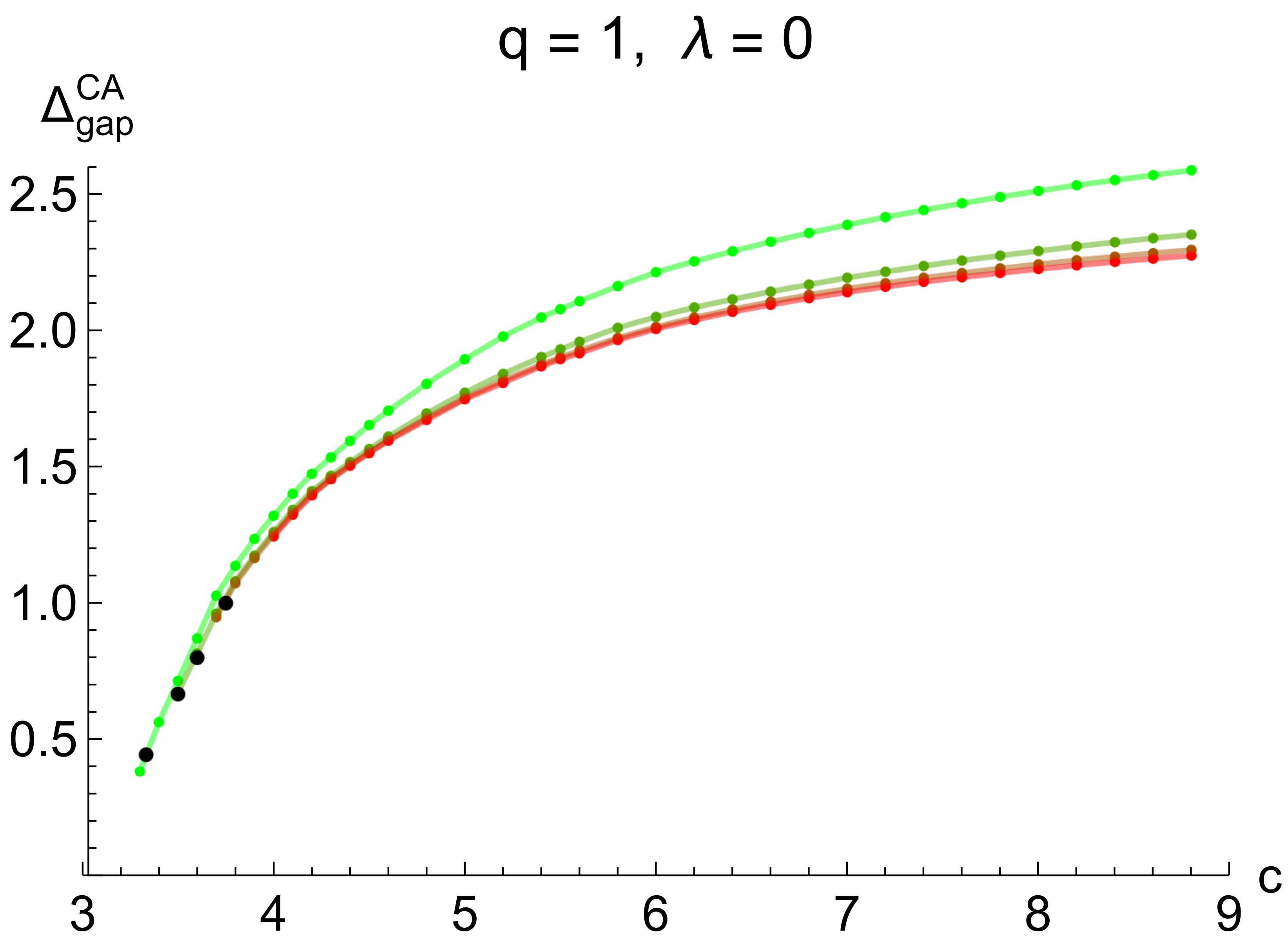
[Lin, Shao, Wang, XY '16]

Upper bound on the scaling dimension of the first non-BPS primary in the OPE of a pair of marginal chiral and anti-chiral primaries

Example 2: (2,2) SCFTs with marginal deformation

[Lin, Shao, Wang, XY '16]

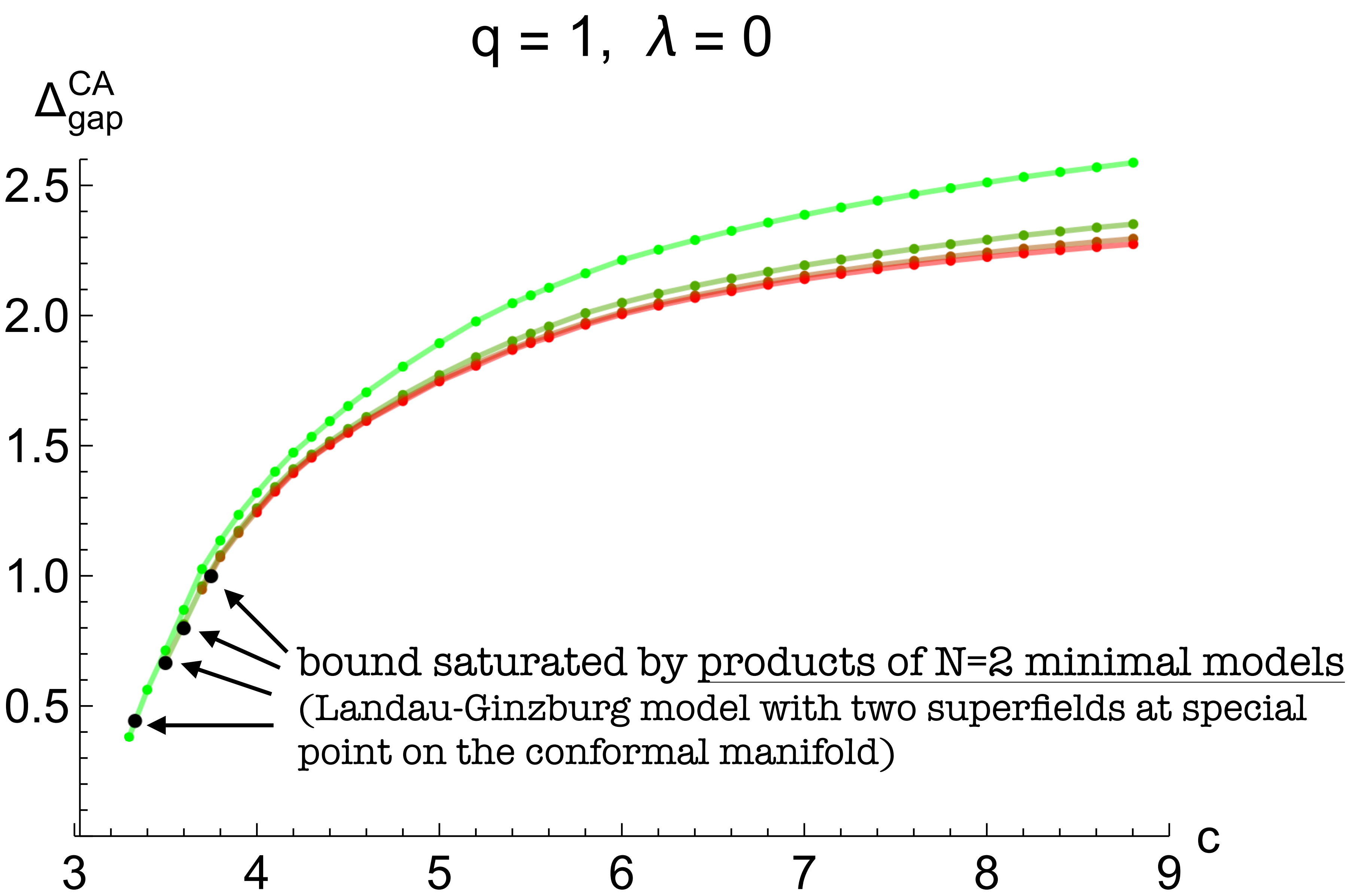
Upper bound on the scaling dimension of the first non-BPS primary in the OPE of a pair of marginal chiral and anti-chiral primaries



Example 2: (2,2) SCFTs with marginal deformation

[Lin, Shao, Wang, XY '16]

Upper bound on the scaling dimension of the first non-BPS primary in the OPE of a pair of marginal chiral and anti-chiral primaries



Combine OPE crossing and modular invariance?

Hard to work with a large set of crossing equations simultaneously
(one equation for each set of external primaries)

Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

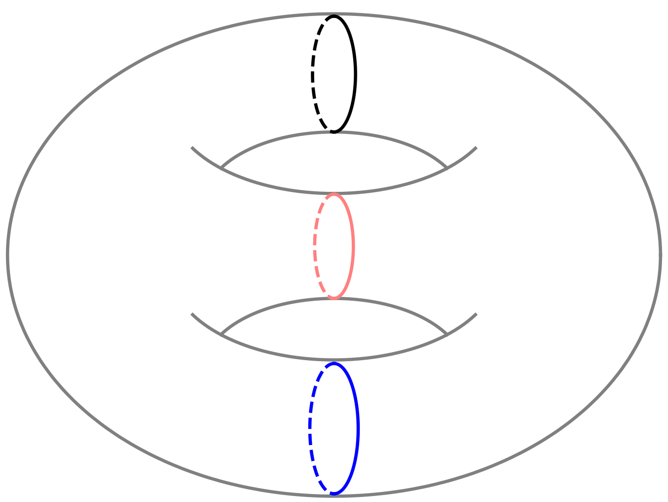
TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

A way out: go to higher genus!

Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

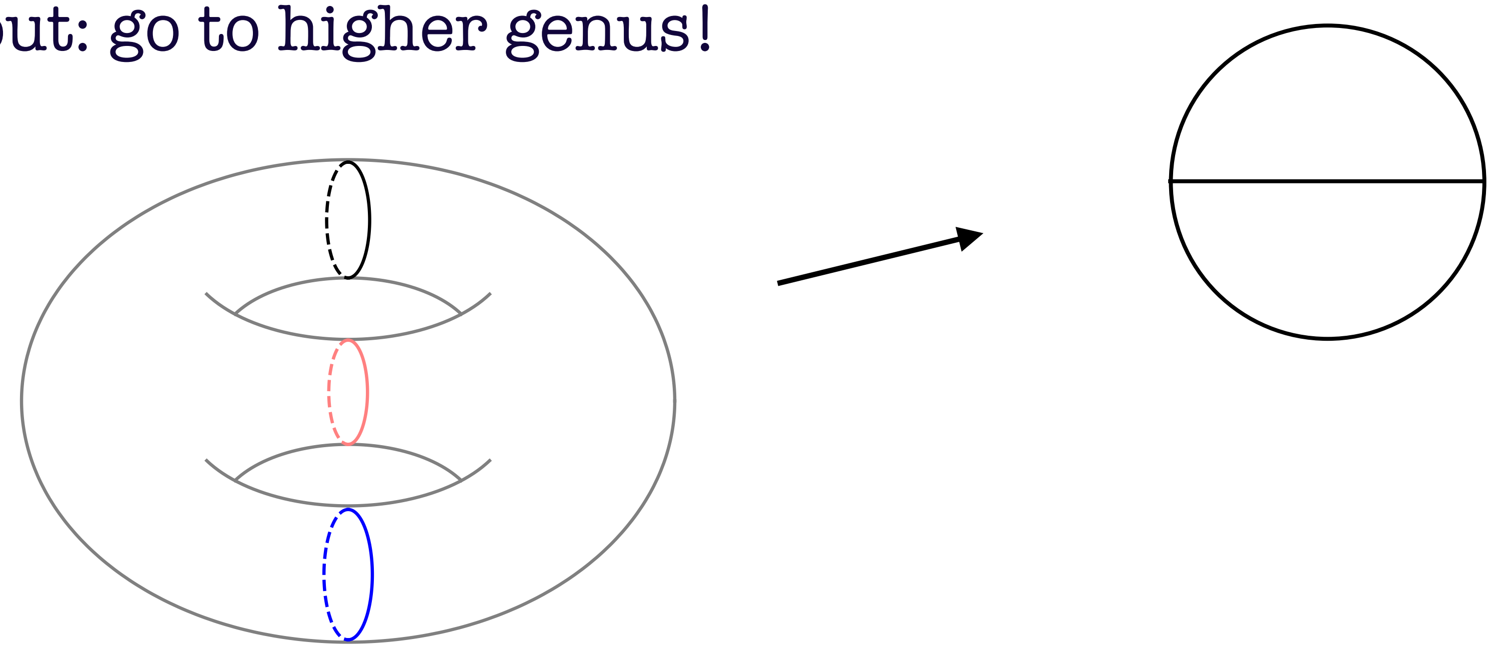
A way out: go to higher genus!



Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

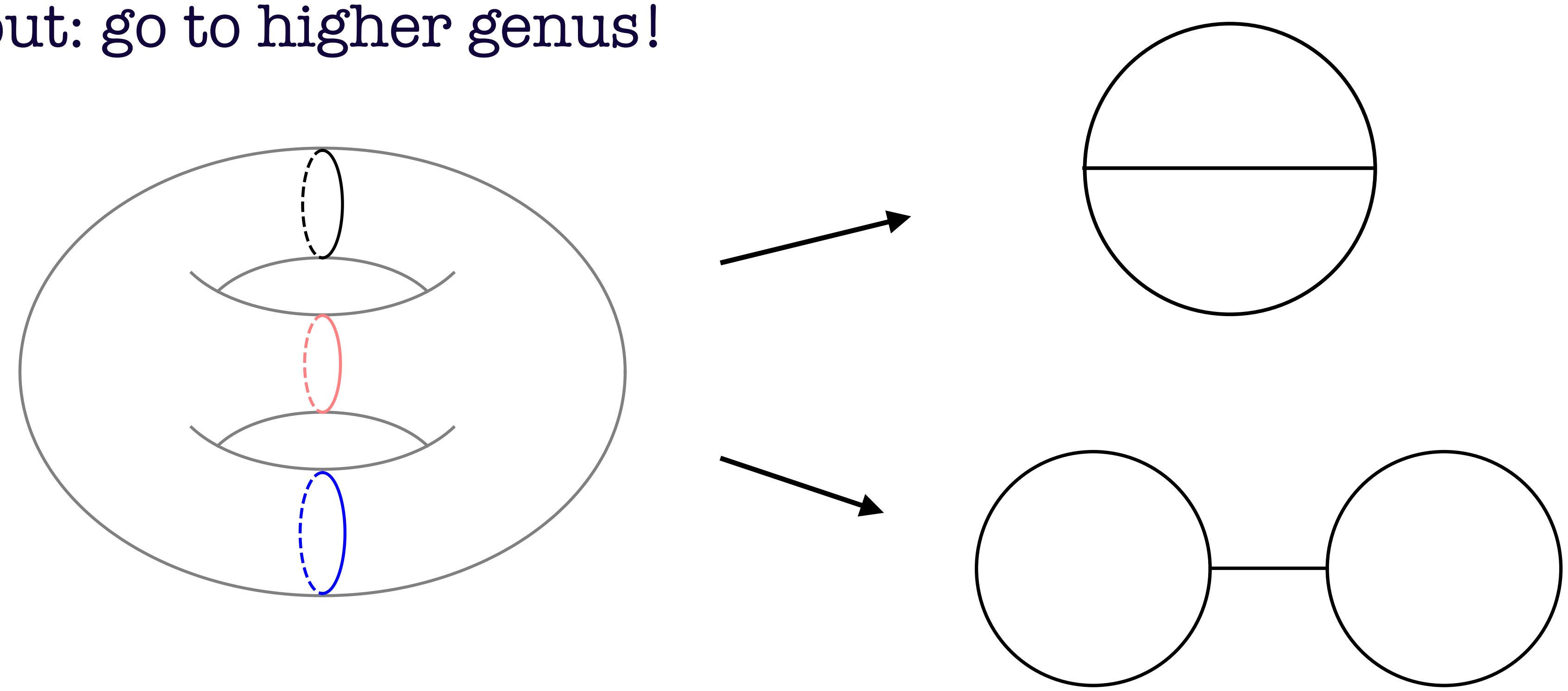
A way out: go to higher genus!



Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

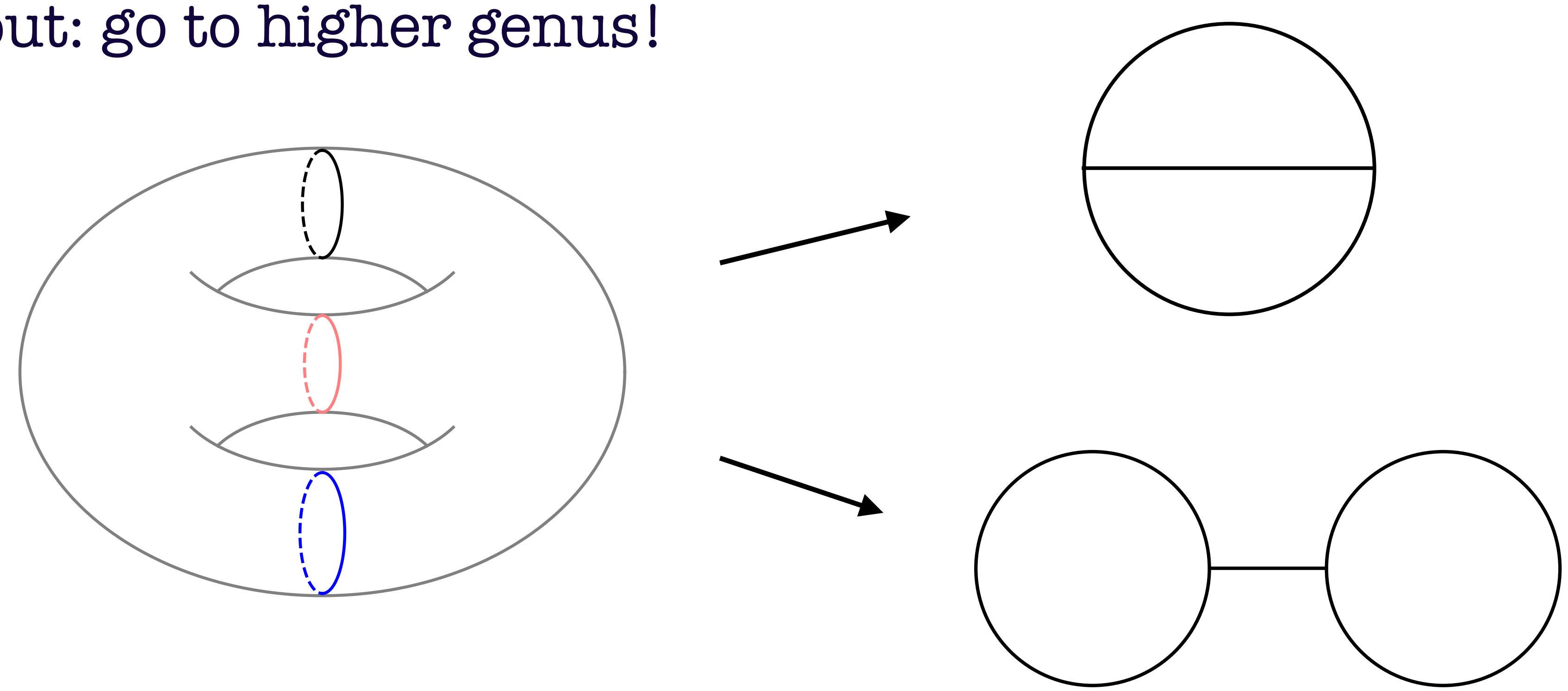
A way out: go to higher genus!



Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

A way out: go to higher genus!

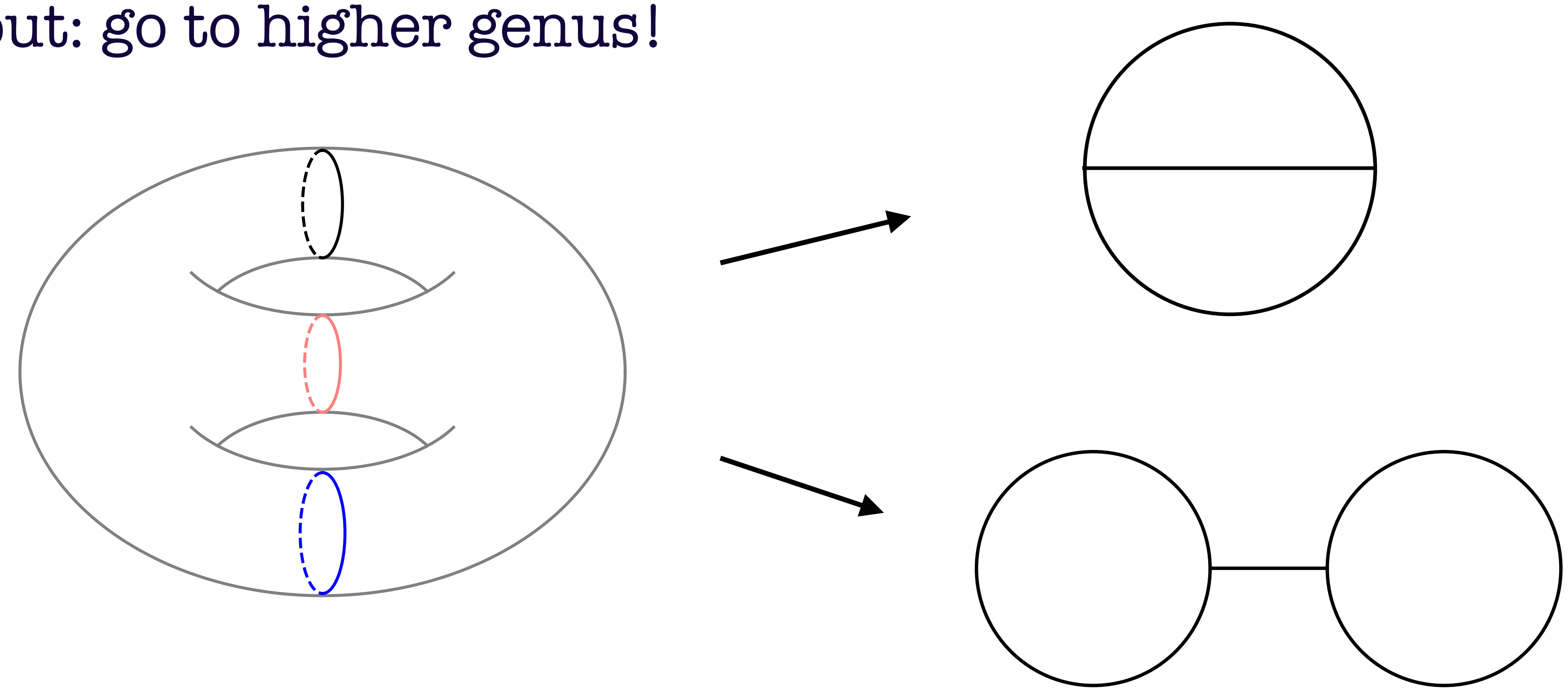


We will study genus two.

Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)

TMI: don't necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.

A way out: go to higher genus!



We will study genus two.

- rich enough to capture OPE *and* modular invariance.

A few technical challenges

A few technical challenges

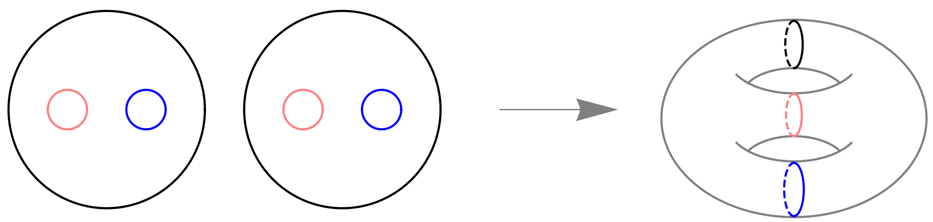
1. Conformal anomaly: choose a convenient conformal frame to make modular invariance manifest

A few technical challenges

1. Conformal anomaly: choose a convenient <u>conformal frame</u> to make modular invariance manifest

Plumbing frame:

modular invariance not manifest

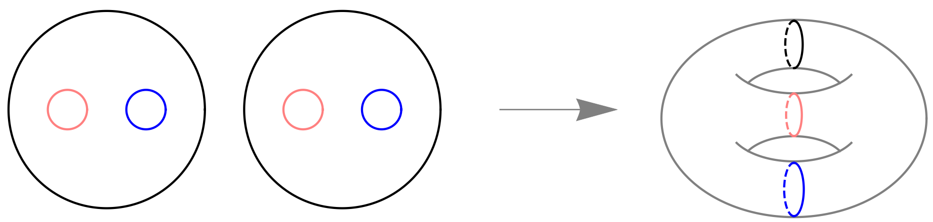


A few technical challenges

1. Conformal anomaly: choose a convenient conformal frame to make modular invariance manifest

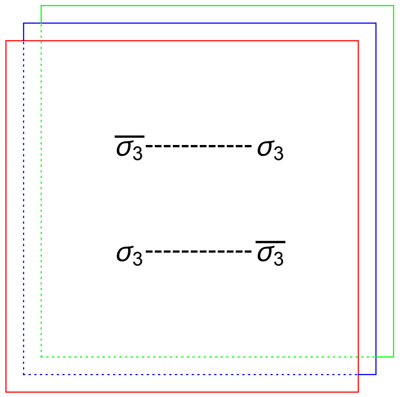
Plumbing frame:

modular invariance not manifest



Renyi frame:

modular invariance manifest

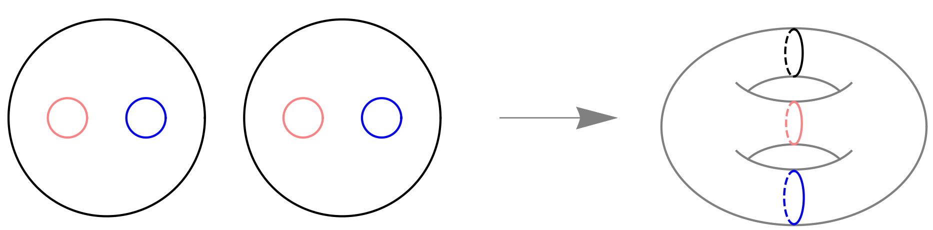


A few technical challenges

1. Conformal anomaly: choose a convenient conformal frame to make modular invariance manifest

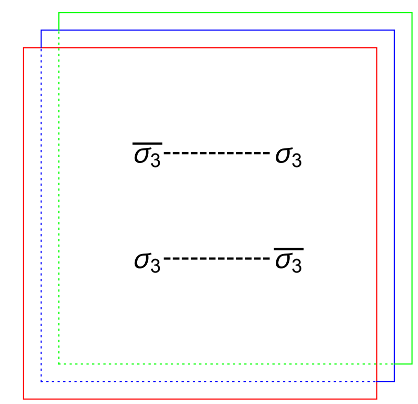
Plumbing frame:

modular invariance not manifest



Renyi frame:

modular invariance manifest



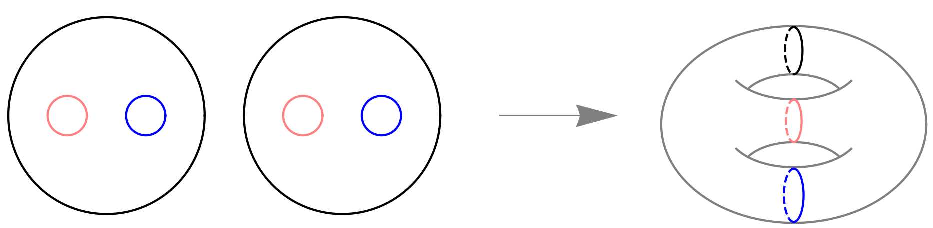
2. Need efficient method of computing genus two conformal blocks

A few technical challenges

1. Conformal anomaly: choose a convenient conformal frame to make modular invariance manifest

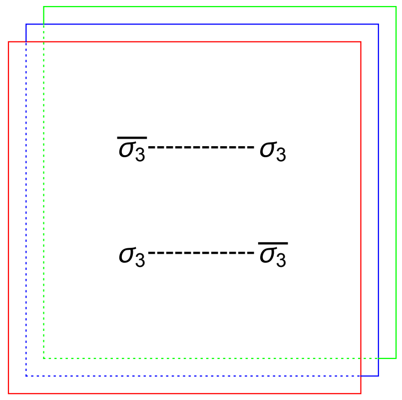
Plumbing frame:

modular invariance not manifest



Renyi frame:

modular invariance manifest



2. Need efficient method of computing genus two conformal blocks

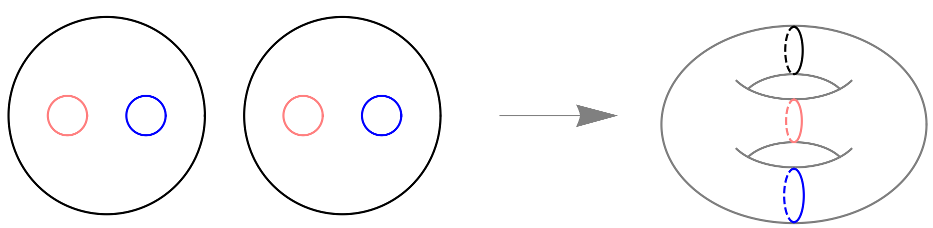
Recursion relations via analytic continuation in central charge.

A few technical challenges

1. Conformal anomaly: choose a convenient conformal frame to make modular invariance manifest

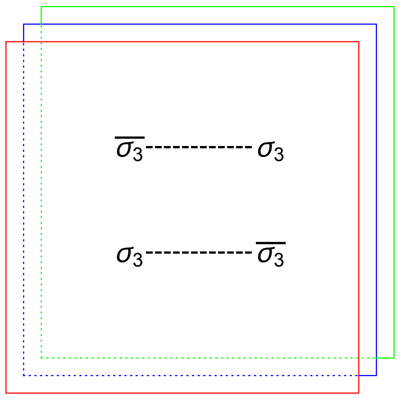
Plumbing frame:

modular invariance not manifest



Renyi frame:

modular invariance manifest



2. Need efficient method of computing genus two conformal blocks

Recursion relations via analytic continuation in central charge.

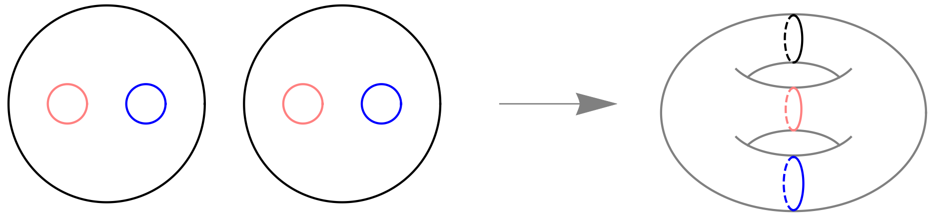
3. Need to handle semidefinite programming on functions of three internal weights

A few technical challenges

1. Conformal anomaly: choose a convenient <u>conformal frame</u> to make modular invariance manifest

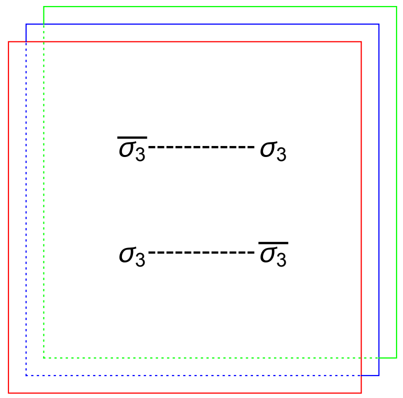
Plumbing frame:

modular invariance not manifest



Renyi frame:

modular invariance manifest



2. Need efficient method of computing genus two conformal blocks

Recursion relations via analytic continuation in central charge.

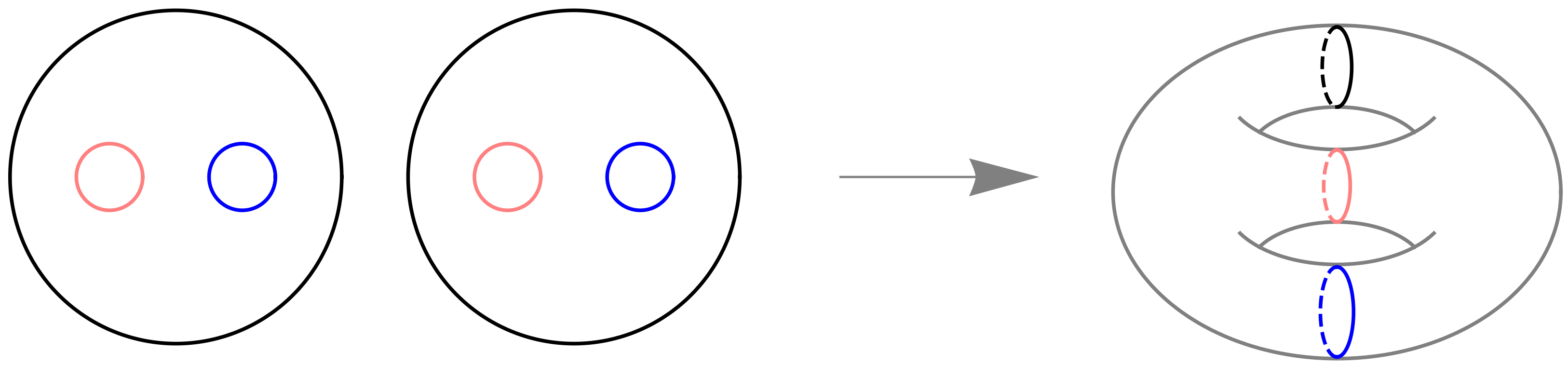
3. Need to handle semidefinite programming on functions of three internal weights

(Don't have the computer program to do this efficiently yet.)

General Virasoro conformal blocks

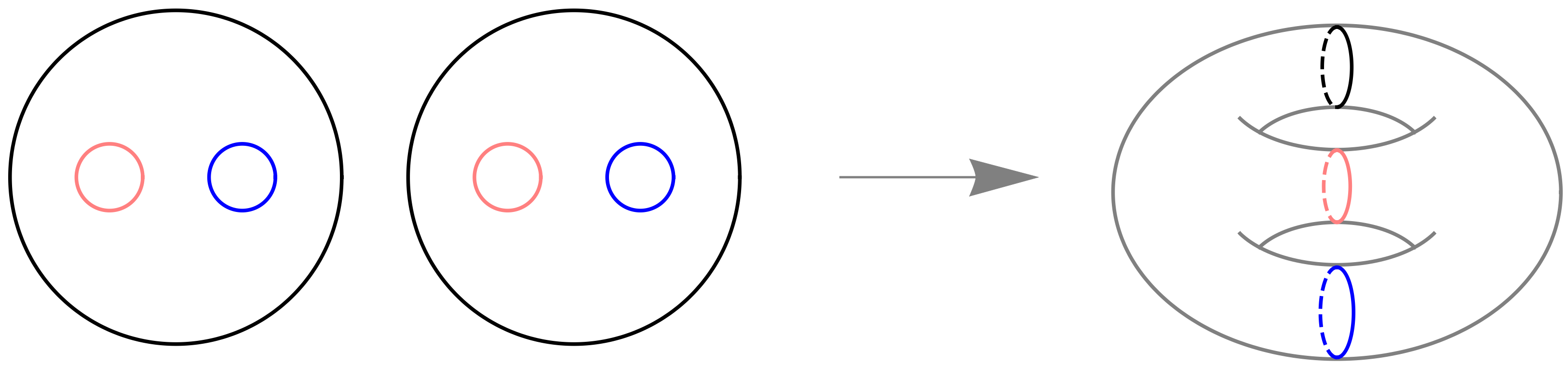
General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.



General Virasoro conformal blocks

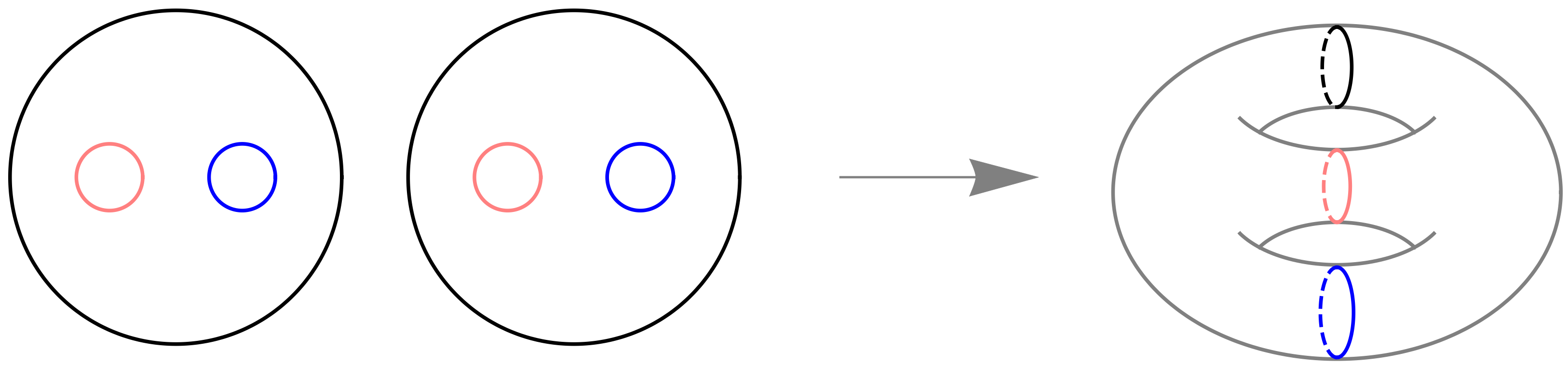
In the plumbing frame, the large c limit is finite.



$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\})$$
$$+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.

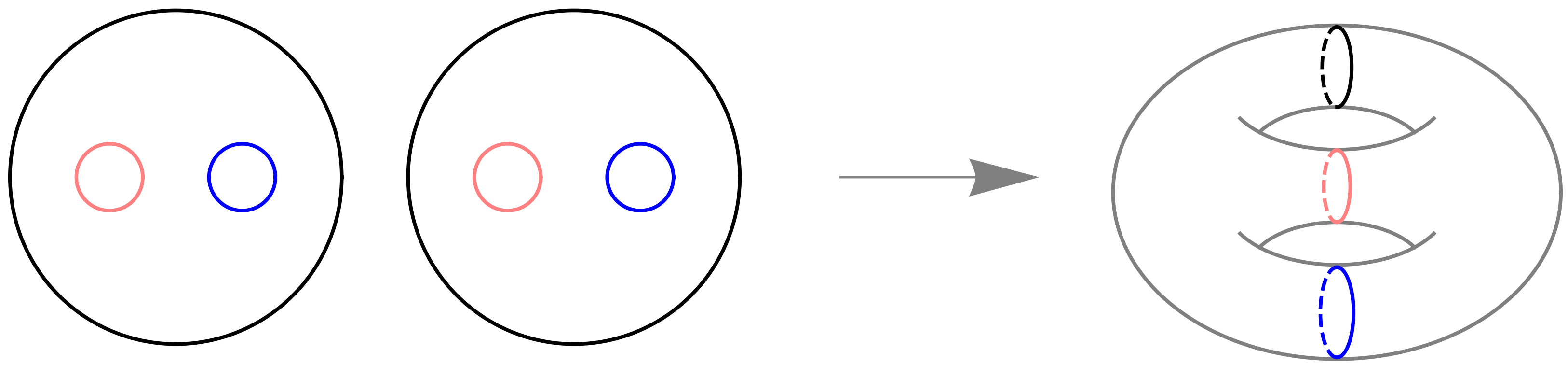


global SL(2) block

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\})$$
$$+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.



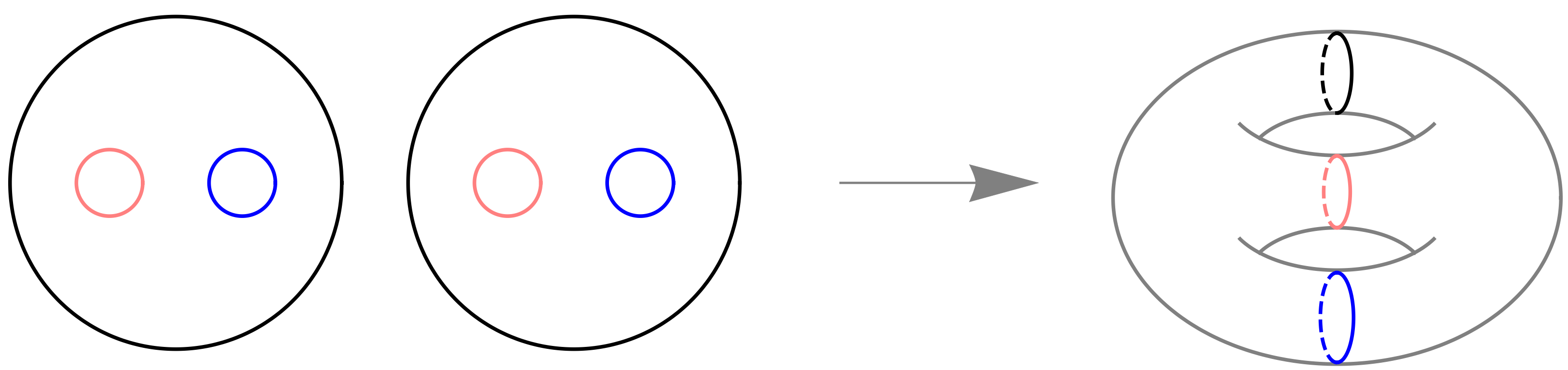
c=∞ vacuum block

global SL(2) block

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\})$$
$$+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c-c_{rs}(h_j)}\mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\};h_j\to h_j+rs;\{q_i\})$$

General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.



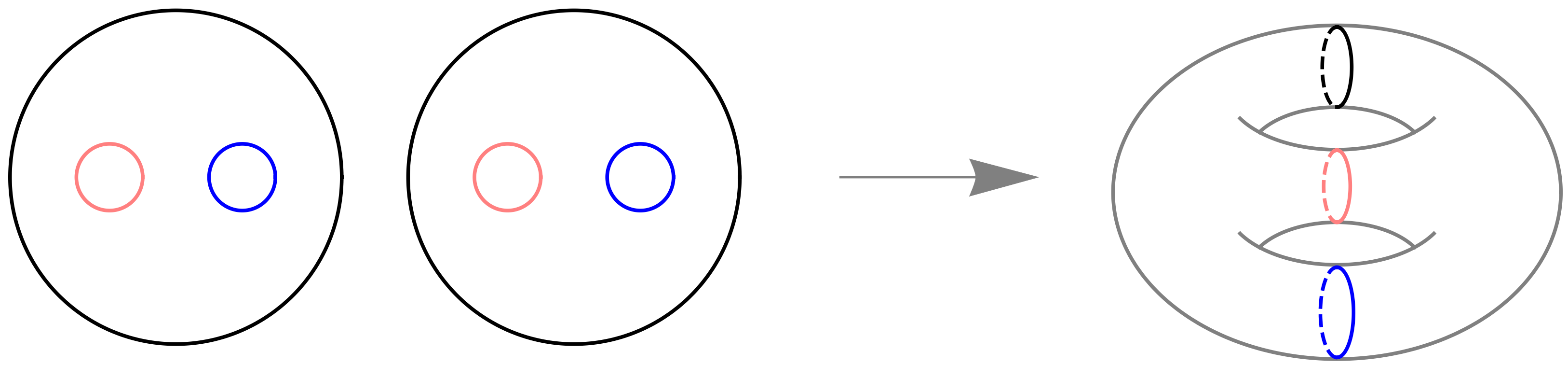
(captured by 1-loop partition function of 3D gravity on a handlebody)
[Giombi-Maloney-XY '08]

$\boxed{c=\infty \text{ vacuum block}}$ $\boxed{\text{global } SL(2) \text{ block}}$

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\})$$
$$+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c-c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.



(captured by 1-loop partition function of 3D gravity on a handlebody)
[Giombi-Maloney-XY '08]

c=∞ vacuum block

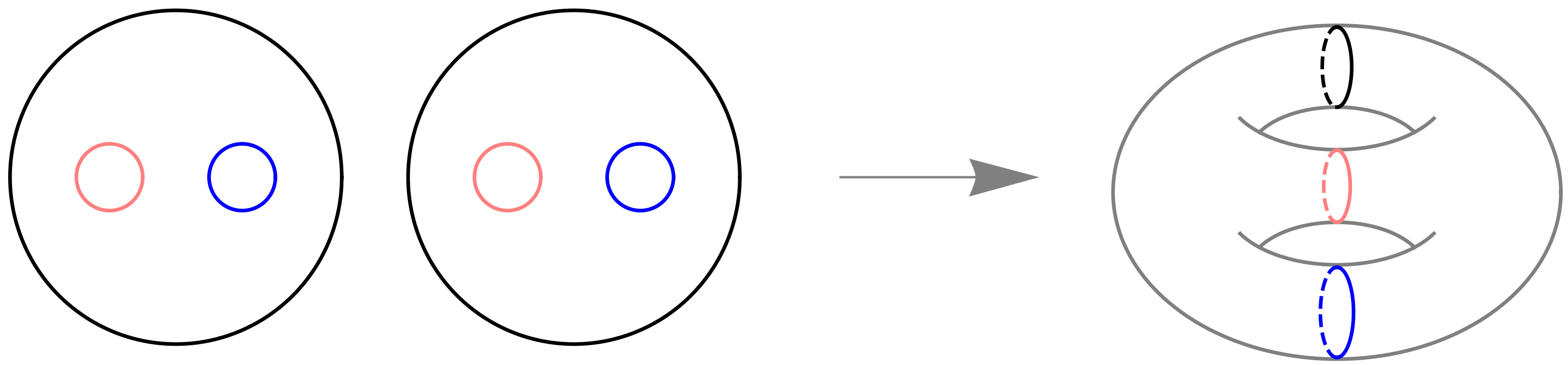
global SL(2) block

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

residues at poles are determined recursively

General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.



(captured by 1-loop partition function of 3D gravity on a handlebody)
[Giombi-Maloney-XY '08]

c=∞ vacuum block

global SL(2) block

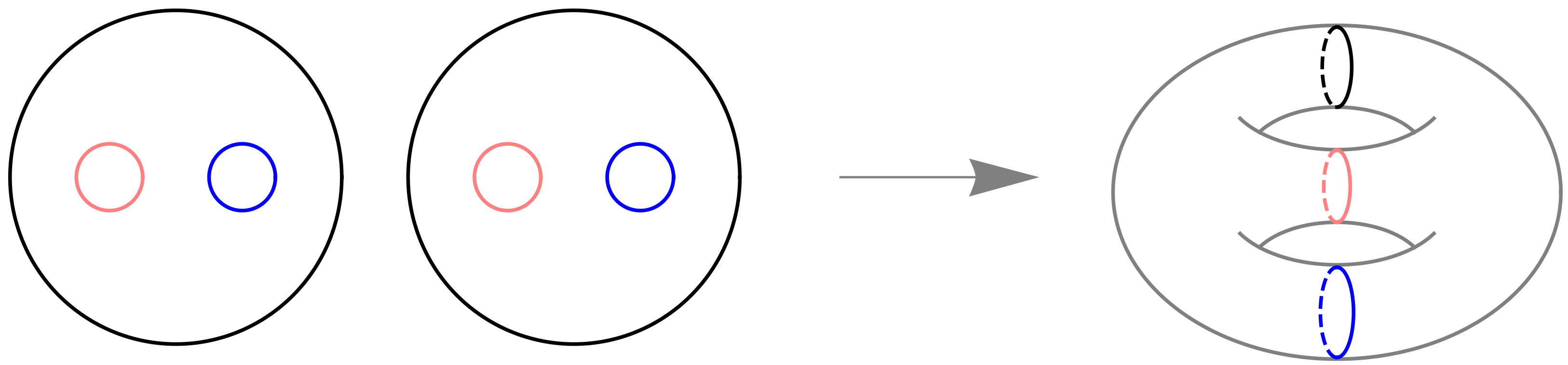
$$\mathcal{G}_c(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\rm ext}\}; h_j \to h_j + rs; \{q_i\})$$

residues at poles are determined recursively

[Zamolodchikov '84]

General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.



(captured by 1-loop partition function of 3D gravity on a handlebody)
[Giombi-Maloney-XY '08]

c=∞ vacuum block

global SL(2) block

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

residues at poles are determined recursively

[Zamolodchikov '84]

generalizations by [Hadasz, Jaskolski, Suchanek '09] [Cho, Collier, XY '17]

General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\})$$
$$+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

Allows for efficient computation of arbitrary Virasoro conformal blocks.

General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\rm ext}\}; h_j \to h_j + rs; \{q_i\})$$

Allows for efficient computation of arbitrary Virasoro conformal blocks.

A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

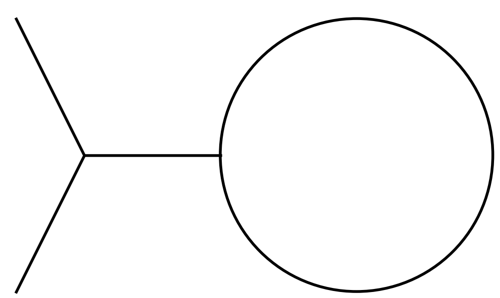
General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\rm ext}\};\{h_i\};\{q_i\})
+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c-c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\rm ext}\}; h_j \to h_j + rs; \{q_i\})$$

Allows for efficient computation of arbitrary Virasoro conformal blocks.

A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

OPE
channel



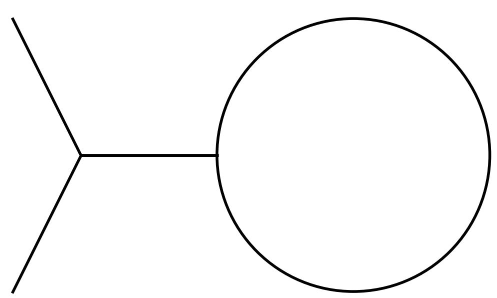
General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\rm ext}\};\{h_i\};\{q_i\}) \\ + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\rm ext}\}; h_j \to h_j + rs; \{q_i\})$$

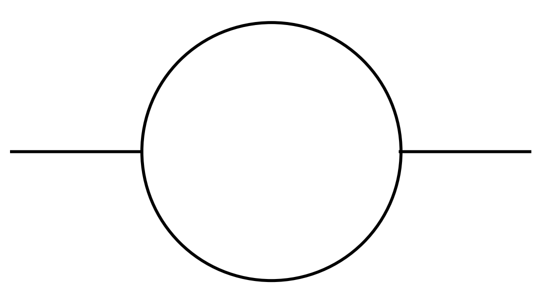
Allows for efficient computation of arbitrary Virasoro conformal blocks.

A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

OPE channel



Necklace channel



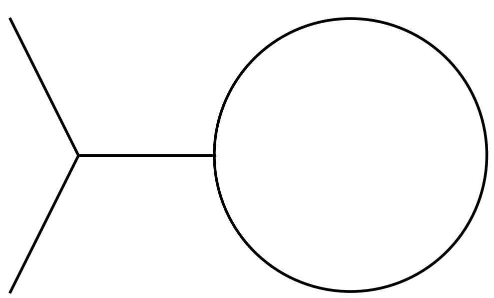
General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\})$$
$$+\sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c-c_{rs}(h_j)}\mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

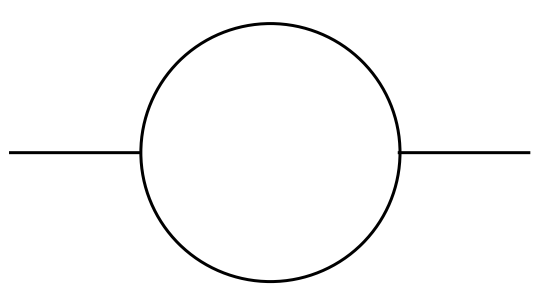
Allows for efficient computation of arbitrary Virasoro conformal blocks.

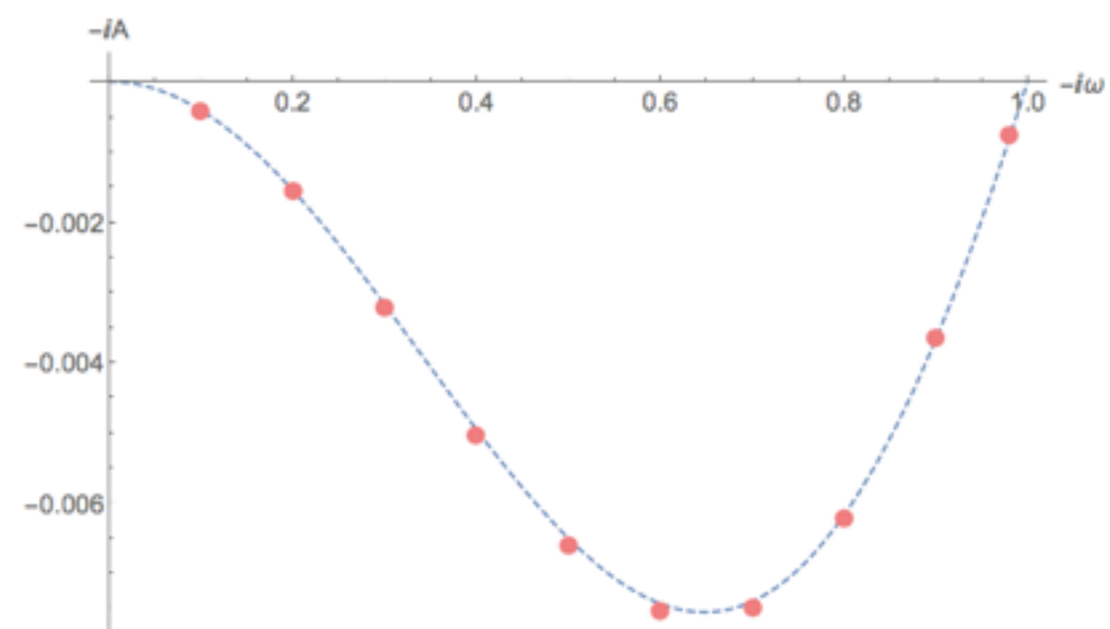
A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

OPE channel



Necklace channel





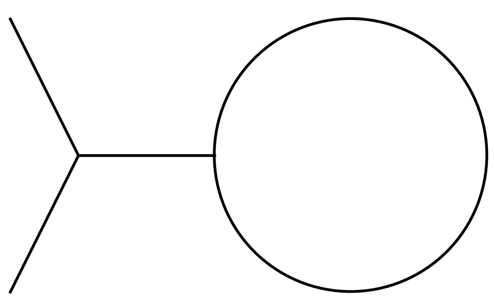
General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

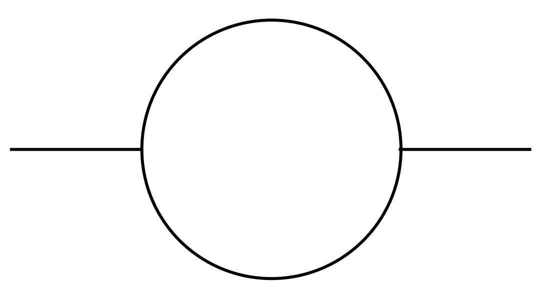
Allows for efficient computation of arbitrary Virasoro conformal blocks.

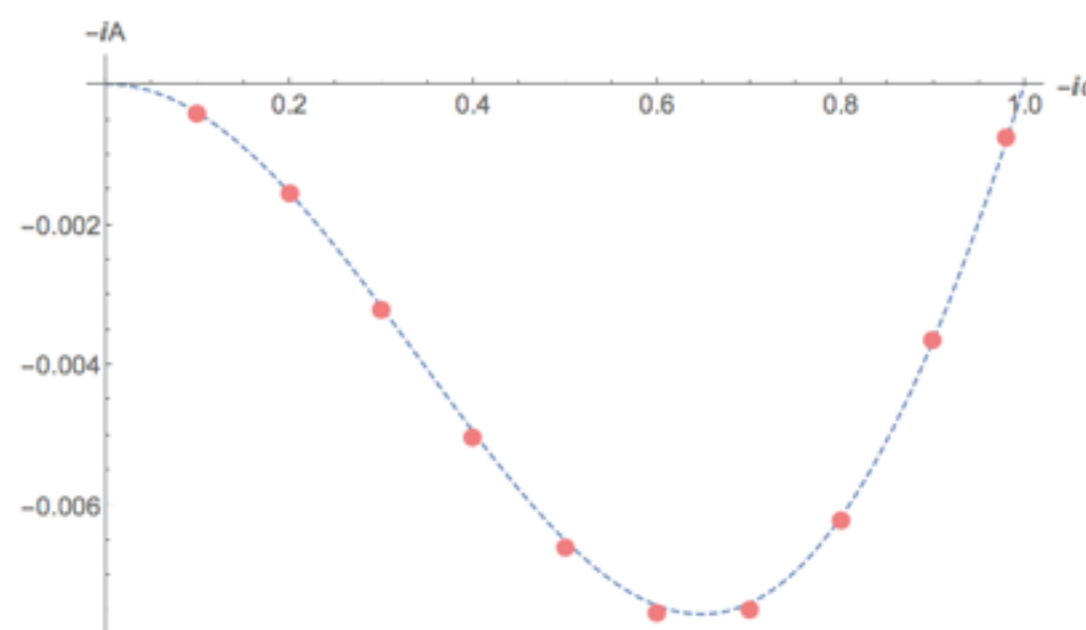
A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

OPE channel



Necklace channel





Red: numerical result for genus one reflection amplitude in c=1 string theory from the worldsheet

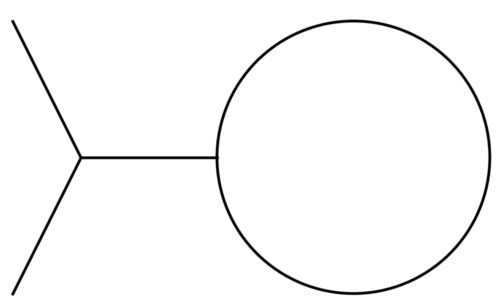
General Virasoro conformal blocks

$$\mathcal{G}_c(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) = \mathcal{G}_\infty(0;0;\{q_j\})\mathcal{G}_{SL(2)}(\{h_a^{\text{ext}}\};\{h_i\};\{q_i\}) + \sum_j \sum_{r\geq 2, s\geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} \mathcal{G}_{c_{rs}(h_j)}(\{h_a^{\text{ext}}\}; h_j \to h_j + rs; \{q_i\})$$

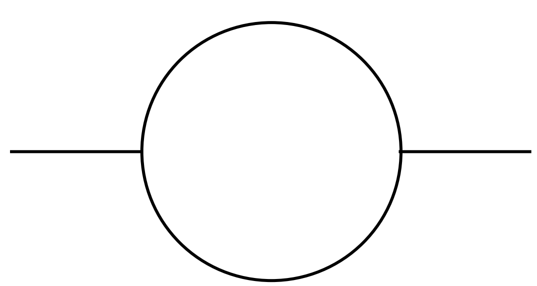
Allows for efficient computation of arbitrary Virasoro conformal blocks.

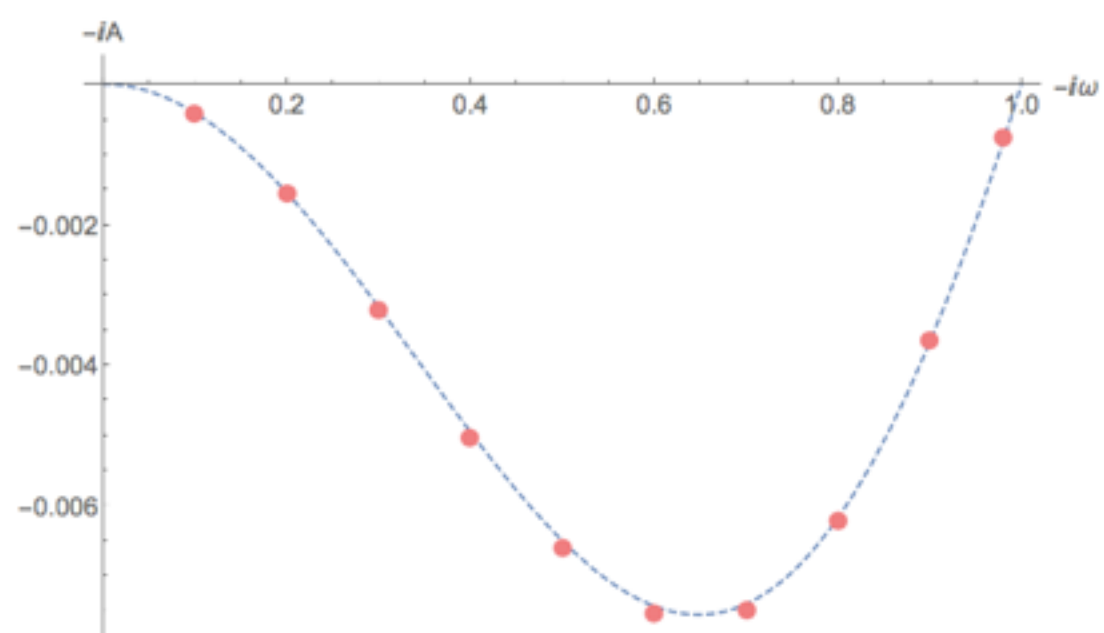
A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

OPE channel



Necklace channel





Red: numerical result for genus one reflection amplitude in c=1 string theory from the worldsheet

Blue: matrix model result

Genus two Renyi surface

To make modular invariance manifest, work in a different conformal frame. A convenience choice is the “Renyi frame”.

Genus two Renyi surface

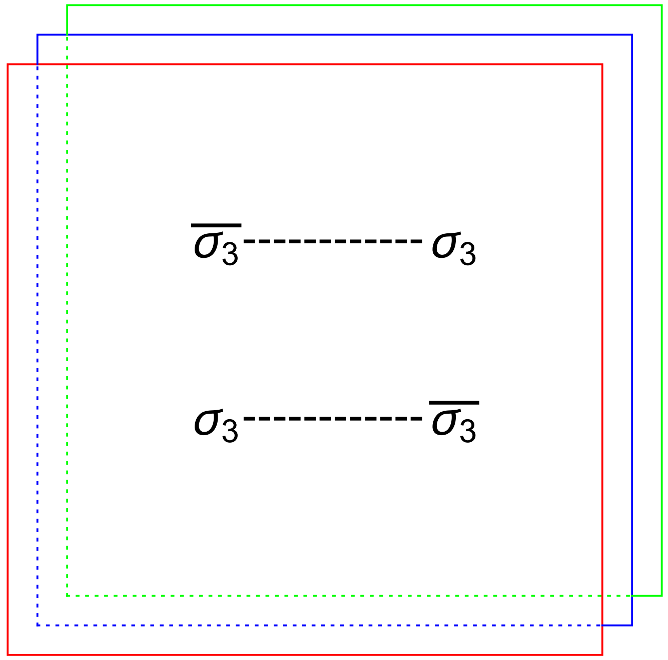
To make modular invariance manifest, work in a different conformal frame. A convenience choice is the "Renyi frame".

To begin with, consider the Z_3 invariant Renyi surface (a genus two surface that is a 3-fold cover of the Riemann sphere branched at four points):

Genus two Renyi surface

To make modular invariance manifest, work in a different conformal frame. A convenience choice is the “Renyi frame”.

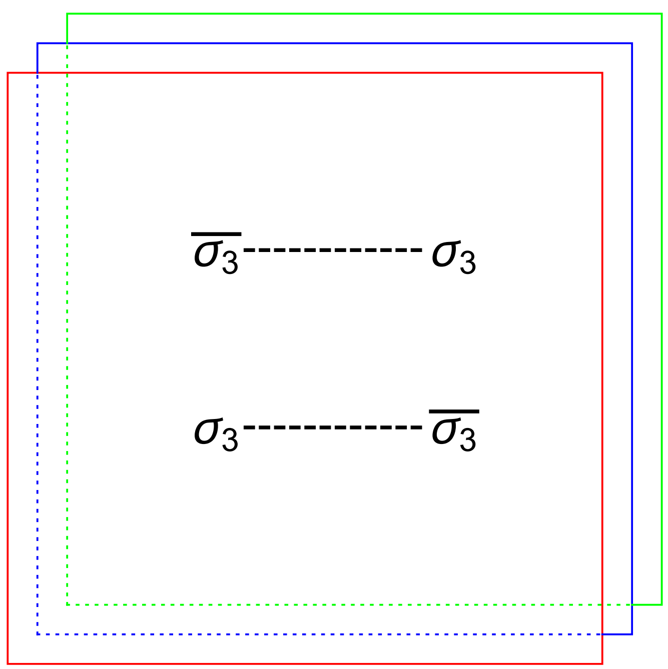
To begin with, consider the Z_3 invariant Renyi surface (a genus two surface that is a 3-fold cover of the Riemann sphere branched at four points):



Genus two Renyi surface

To make modular invariance manifest, work in a different conformal frame. A convenience choice is the “Renyi frame”.

To begin with, consider the Z_3 invariant Renyi surface (a genus two surface that is a 3-fold cover of the Riemann sphere branched at four points):



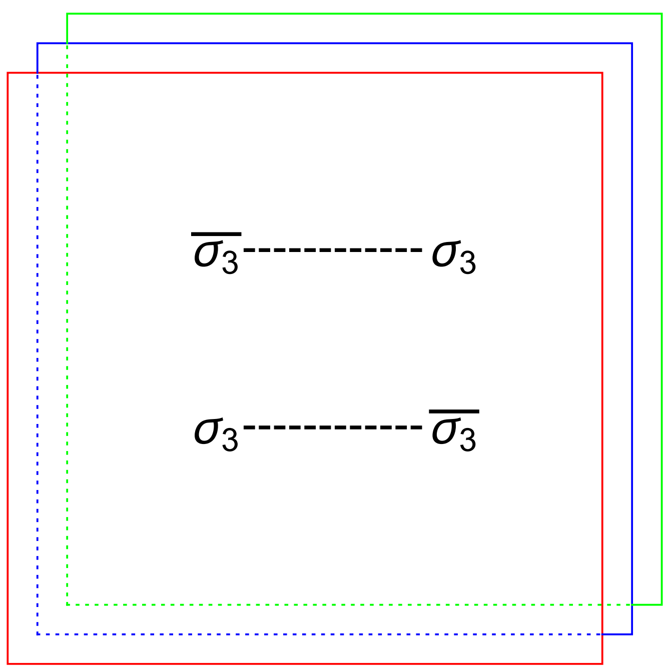
The Renyi surfaces occupy a 1 complex dimensional locus of the moduli space of genus two Riemann surfaces.

$$\Omega = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \frac{i\ {}_2F_1(\frac{2}{3}, \frac{1}{3}, 1|1-z)}{\sqrt{3}\ {}_2F_1(\frac{2}{3}, \frac{1}{3}, 1|z)}$$

Genus two Renyi surface

To make modular invariance manifest, work in a different conformal frame. A convenience choice is the “Renyi frame”.

To begin with, consider the Z_3 invariant Renyi surface (a genus two surface that is a 3-fold cover of the Riemann sphere branched at four points):

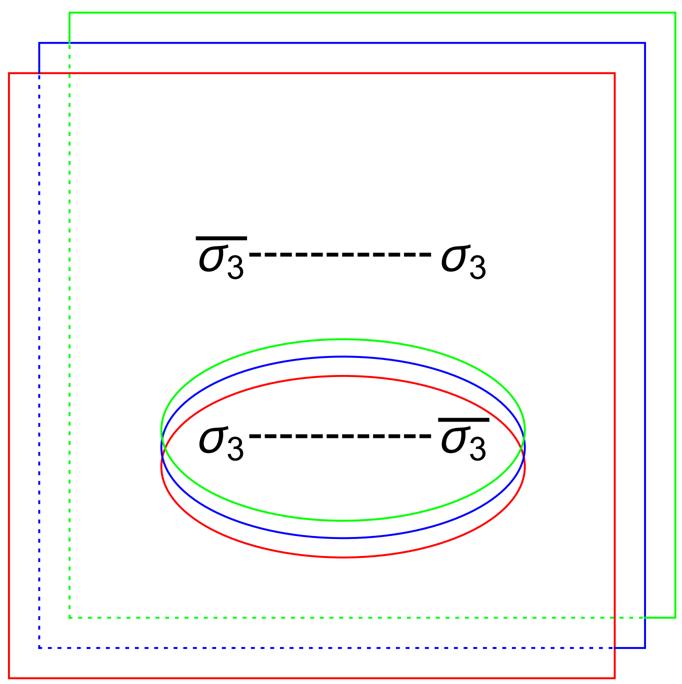


The Renyi surfaces occupy a 1 complex dimensional locus of the moduli space of genus two Riemann surfaces.

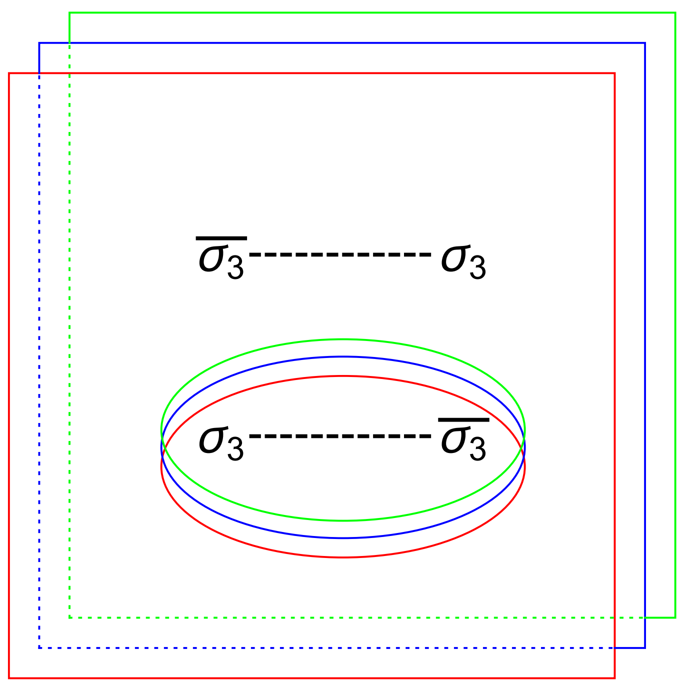
$$\Omega = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \frac{i \ {}_2F_1(\frac{2}{3}, \frac{1}{3}, 1|1-z)}{\sqrt{3}\ {}_2F_1(\frac{2}{3}, \frac{1}{3}, 1|z)}$$

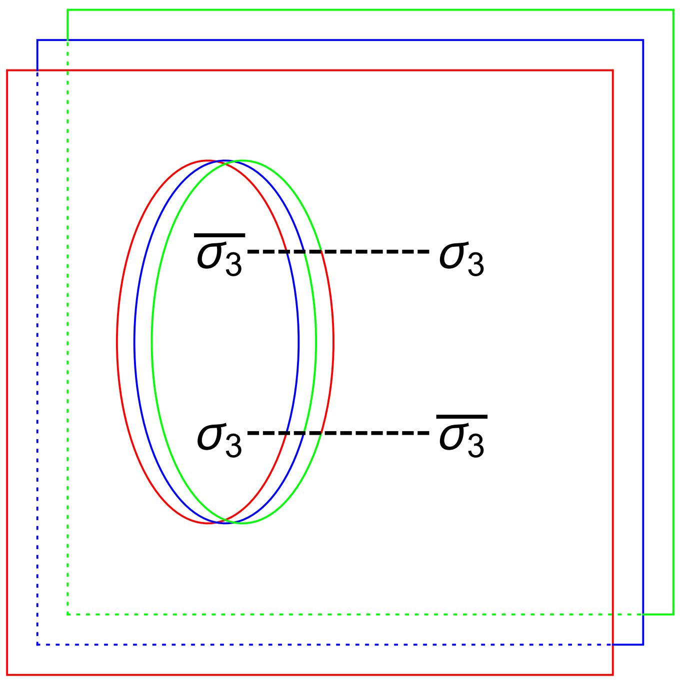
The parameter **z** is the cross ratio of the four branch points on the sphere.

Genus two crossing

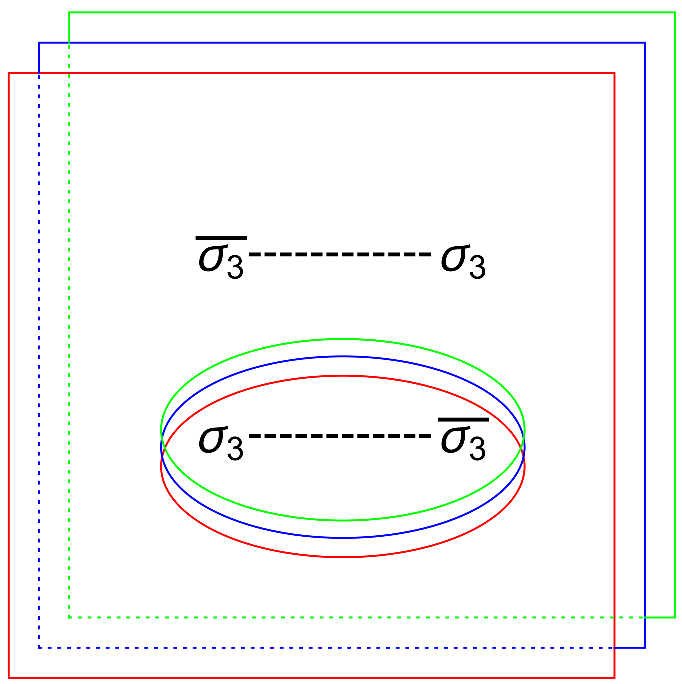


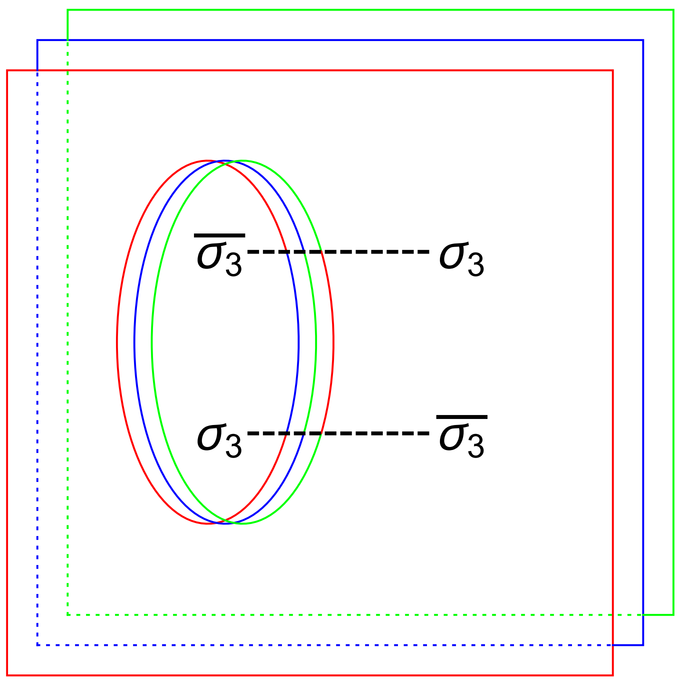
Genus two crossing





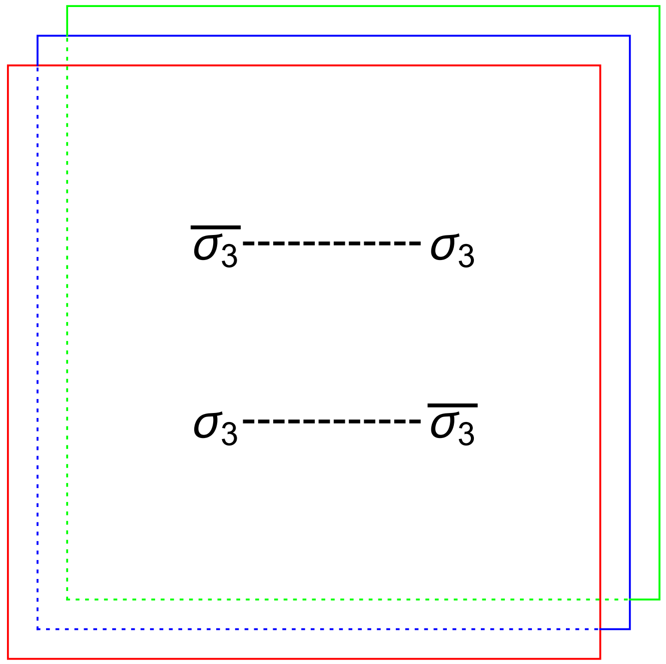
Genus two crossing





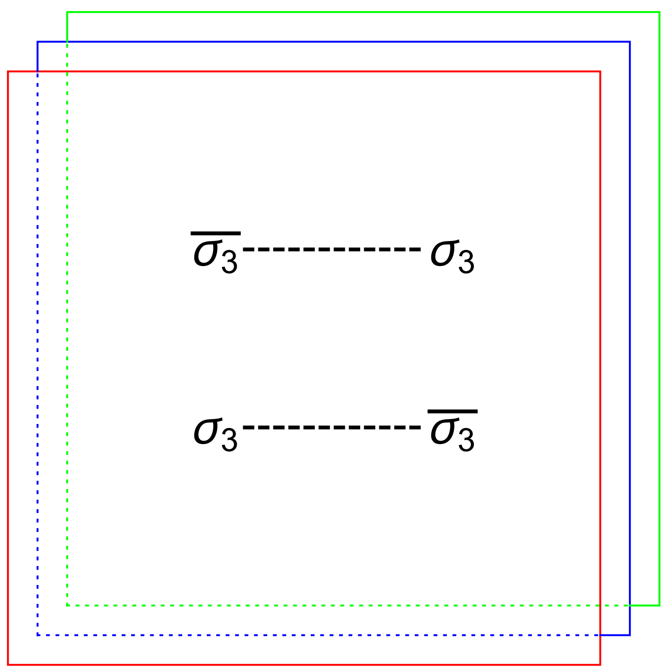
A nontrivial generator of the genus two modular group Sp(4,Z) is the crossing transformation of the four-point function of Z_3 twist fields.

Genus two conformal block



$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar{z})\sigma_3(1)\overline{\sigma}'_3(\infty)\rangle = \sum_{i,j,k} C^2_{ijk}\mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde{h}_i,\tilde{h}_j,\tilde{h}_k;\bar{z})$$

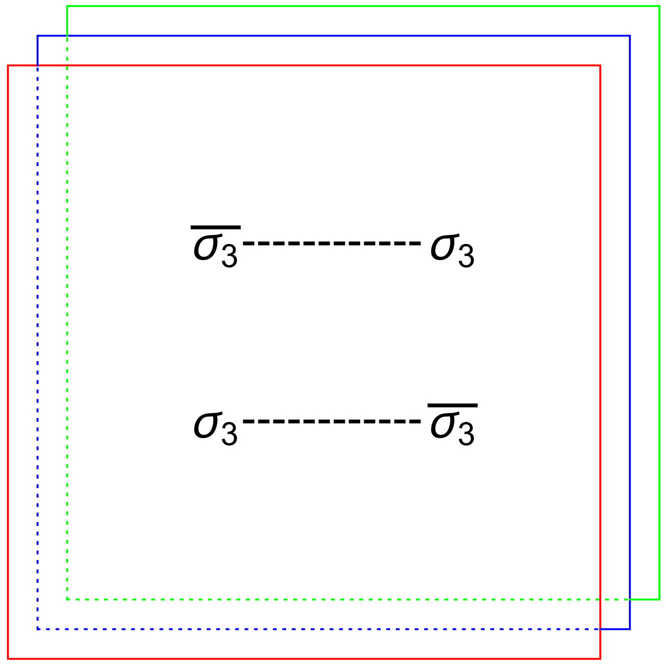
Genus two conformal block



$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar{z})\sigma_3(1)\overline{\sigma}'_3(\infty)\rangle = \sum_{i,j,k} C^2_{ijk}\mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde{h}_i,\tilde{h}_j,\tilde{h}_k;\bar{z})$$

$$\mathcal{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathcal{F}^{cl}(z)\right]\mathcal{G}_c(h_1,h_2,h_3;z)$$

Genus two conformal block

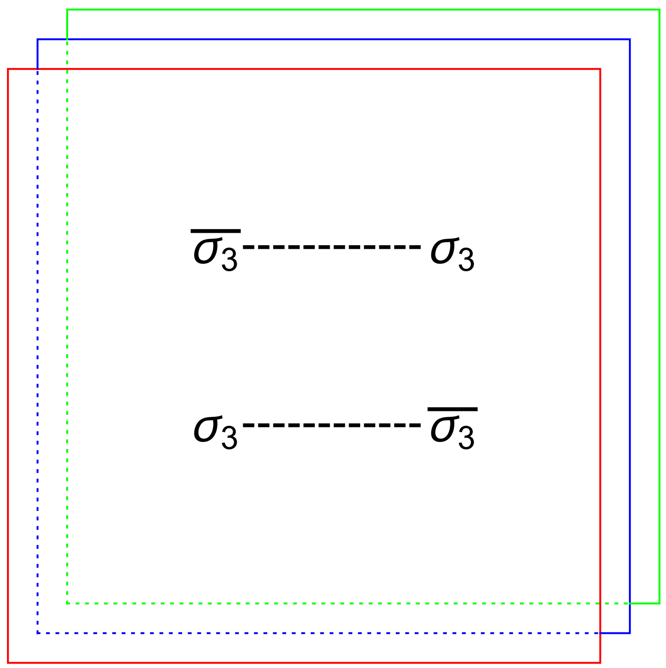


$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar{z})\sigma_3(1)\overline{\sigma}_3'(\infty)\rangle = \sum_{i,j,k} C_{ijk}^2 \mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde{h}_i,\tilde{h}_j,\tilde{h}_k;\bar{z})$$

$$\mathcal{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathcal{F}^{cl}(z)\right]\mathcal{G}_c(h_1,h_2,h_3;z)$$

plumbing frame block

Genus two conformal block



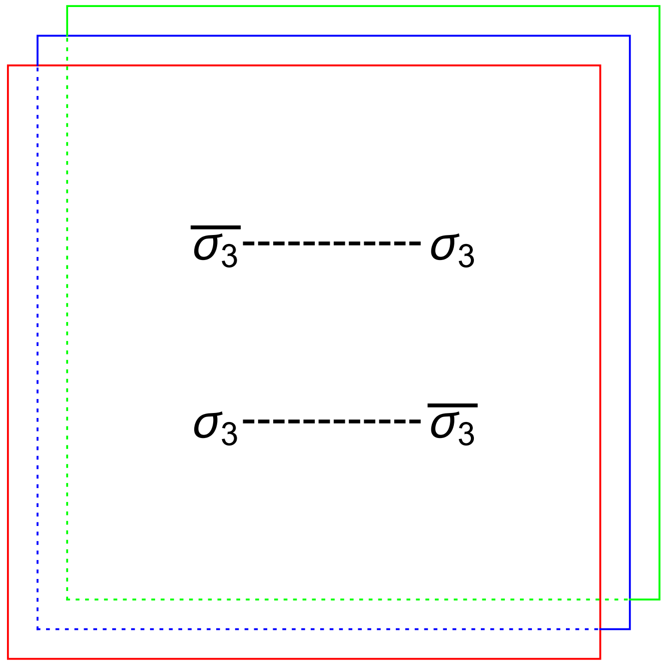
$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar{z})\sigma_3(1)\overline{\sigma}_3'(\infty)\rangle = \sum_{i,j,k} C_{ijk}^2 \mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde{h}_i,\tilde{h}_j,\tilde{h}_k;\bar{z})$$

$$\mathcal{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathcal{F}^{cl}(z)\right]\mathcal{G}_c(h_1,h_2,h_3;z)$$

conformal anomaly

plumbing frame block

Genus two conformal block



$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar z)\sigma_3(1)\overline{\sigma}'_3(\infty)\rangle = \sum_{i,j,k} C^2_{ijk}\mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde h_i,\tilde h_j,\tilde h_k;\bar z)$$

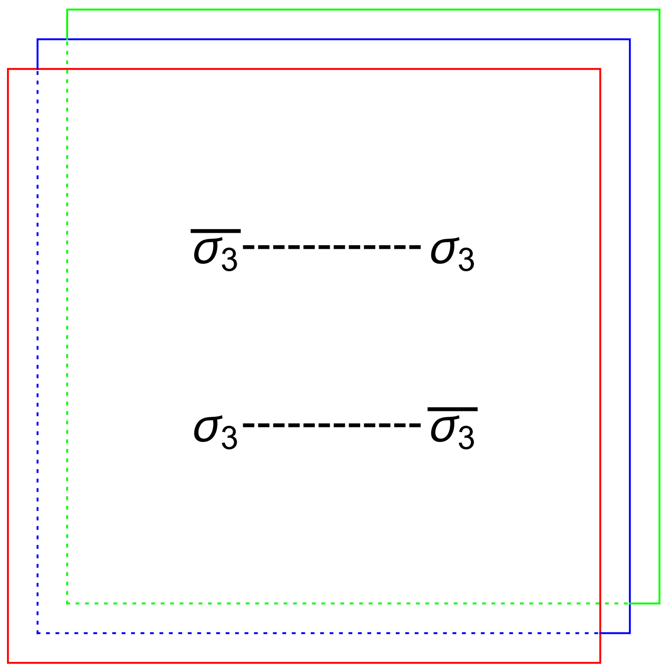
$$\mathcal{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathcal{F}^{cl}(z)\right]\mathcal{G}_c(h_1,h_2,h_3;z)$$

conformal anomaly

plumbing frame block

$$\mathcal{F}^{cl}(z) = -\frac{2}{9}\log(z) + 6\left(\frac{z}{27}\right)^2 + 162\left(\frac{z}{27}\right)^3 + 3975\left(\frac{z}{27}\right)^4 + 96552\left(\frac{z}{27}\right)^5 + 2356039\left(\frac{z}{27}\right)^6 + \cdots$$

Genus two conformal block



$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar{z})\sigma_3(1)\overline{\sigma}_3'(\infty)\rangle = \sum_{i,j,k} C_{ijk}^2 \mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde{h}_i,\tilde{h}_j,\tilde{h}_k;\bar{z})$$

$$\mathcal{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathcal{F}^{cl}(z)\right]\mathcal{G}_c(h_1,h_2,h_3;z)$$

conformal anomaly

plumbing frame block

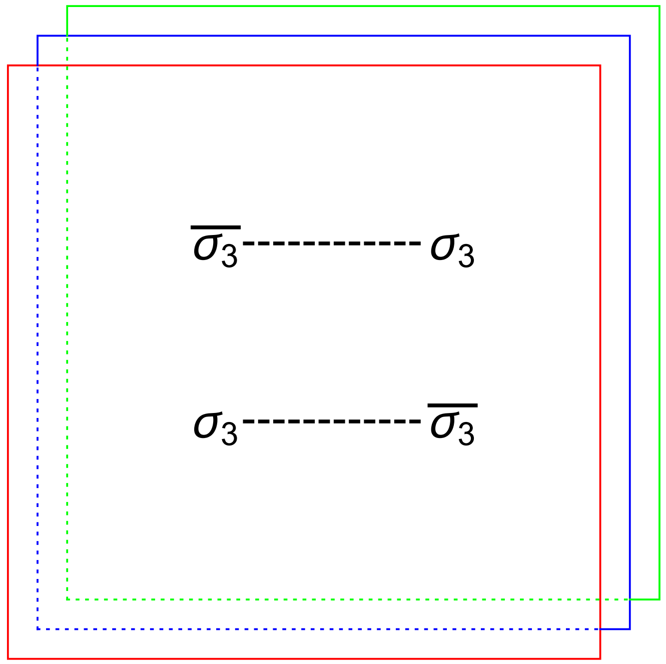
$$\mathcal{F}^{cl}(z) = -\frac{2}{9}\log(z) + 6\left(\frac{z}{27}\right)^2 + 162\left(\frac{z}{27}\right)^3 + 3975\left(\frac{z}{27}\right)^4 + 96552\left(\frac{z}{27}\right)^5 + 2356039\left(\frac{z}{27}\right)^6 + \cdots$$

The infinite c limit of the plumbing frame block for the Renyi surface is

$$\mathcal{G}_\infty(h_1,h_2,h_3|z) = \left(\frac{z}{27}\right)^{h_1+h_2+h_3}\left\{1 + \left[\frac{h_1+h_2+h_3}{2} + \frac{(h_2-h_3)^2}{54h_1} + \frac{(h_3-h_1)^2}{54h_2} + \frac{(h_1-h_2)^2}{54h_3}\right]z \right.$$

$$\left. + [\text{illegible}] + \mathcal{O}(z^3)\right\}$$

Genus two conformal block



$$\langle \sigma_3(0)\overline{\sigma}_3(z,\bar{z})\sigma_3(1)\overline{\sigma}_3'(\infty)\rangle = \sum_{i,j,k} C_{ijk}^2 \mathcal{F}_c(h_i,h_j,h_k;z)\overline{\mathcal{F}}_c(\tilde{h}_i,\tilde{h}_j,\tilde{h}_k;\bar{z})$$

$$\mathcal{F}_c(h_1,h_2,h_3;z) = \exp\left[c\mathcal{F}^{cl}(z)\right]\mathcal{G}_c(h_1,h_2,h_3;z)$$

conformal anomaly

plumbing frame block

$$\mathcal{F}^{cl}(z) = -\frac{2}{9}\log(z) + 6\left(\frac{z}{27}\right)^2 + 162\left(\frac{z}{27}\right)^3 + 3975\left(\frac{z}{27}\right)^4 + 96552\left(\frac{z}{27}\right)^5 + 2356039\left(\frac{z}{27}\right)^6 + \cdots$$

The infinite c limit of the plumbing frame block for the Renyi surface is

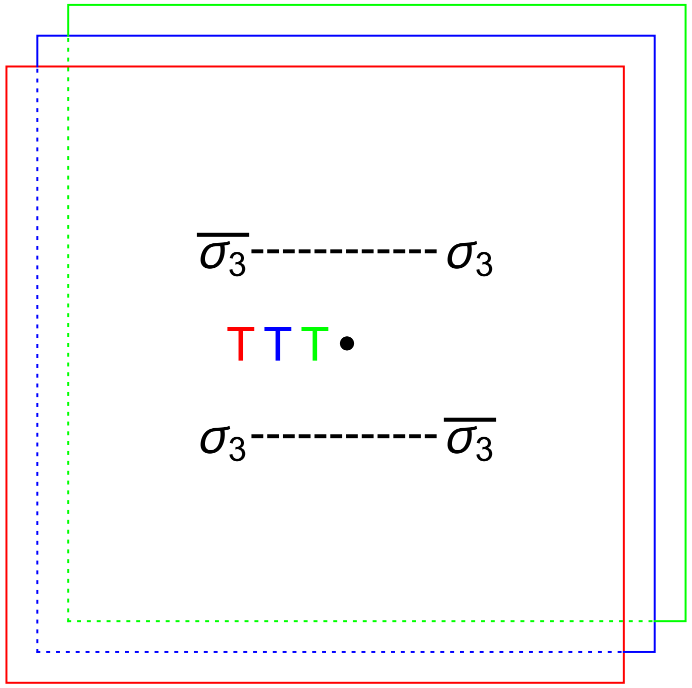
$$\mathcal{G}_\infty(h_1,h_2,h_3|z) = \left(\frac{z}{27}\right)^{h_1+h_2+h_3}\left\{1 + \left[\frac{h_1+h_2+h_3}{2} + \frac{(h_2-h_3)^2}{54h_1} + \frac{(h_3-h_1)^2}{54h_2} + \frac{(h_1-h_2)^2}{54h_3}\right]z \right.$$

$$\left. + \mathcal{O}(z^3)\right\}$$

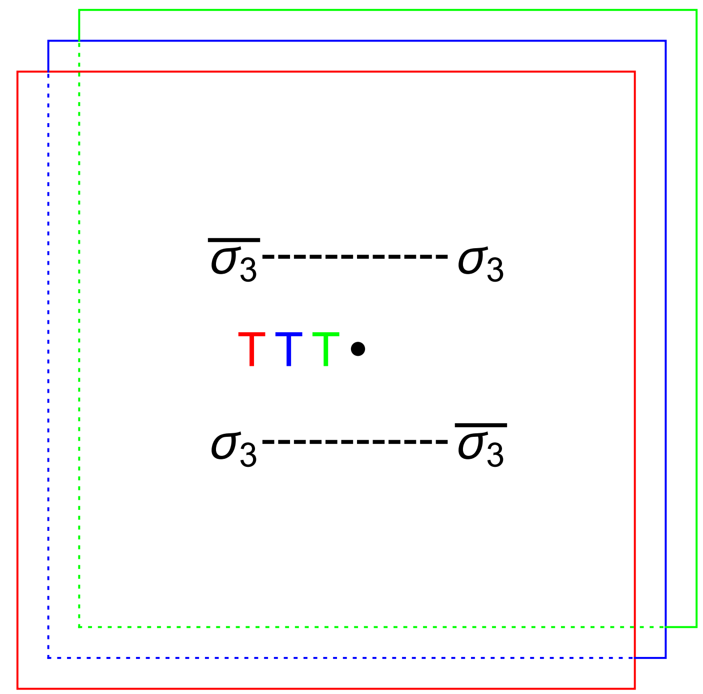
(Finite c result can be recovered by recursion formula.)

Genus two crossing equation beyond the Renyi surface

Genus two crossing equation beyond the Renyi surface



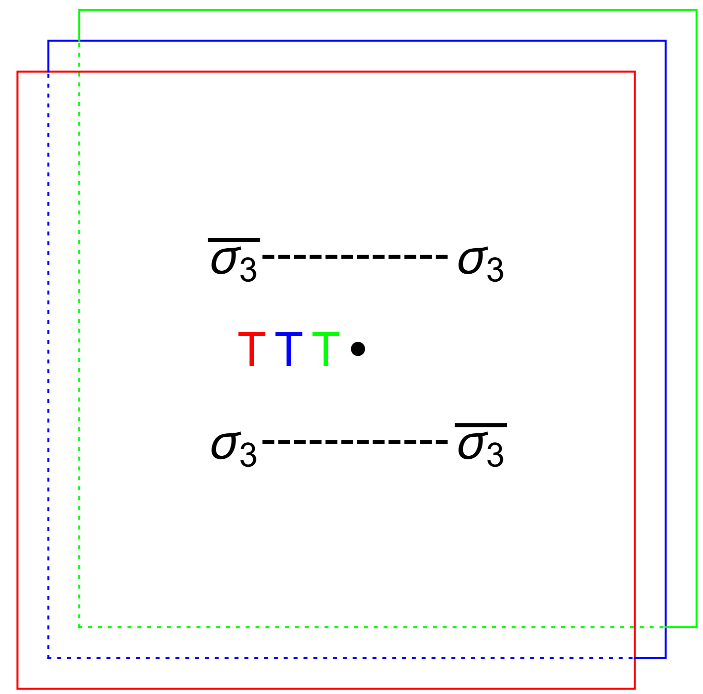
Genus two crossing equation beyond the Renyi surface



$$(-)^{\sum_{j=1}^{3}(|R_j|+|\tilde{R}_j|)} \sum_{(h_i,\tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3} \mathbb{F}(h_1,h_2,h_3;R_1,R_2,R_3;w|z)\mathbb{F}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3;\tilde{R}_1,\tilde{R}_2,\tilde{R}_3;\bar{w}|\bar{z})$$

$$= \sum_{(h_i,\tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3} \mathbb{F}(h_1,h_2,h_3;R_1,R_2,R_3;1-w|1-z)\mathbb{F}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3;\tilde{R}_1,\tilde{R}_2,\tilde{R}_3;1-\bar{w}|1-\bar{z}).$$

Genus two crossing equation beyond the Renyi surface

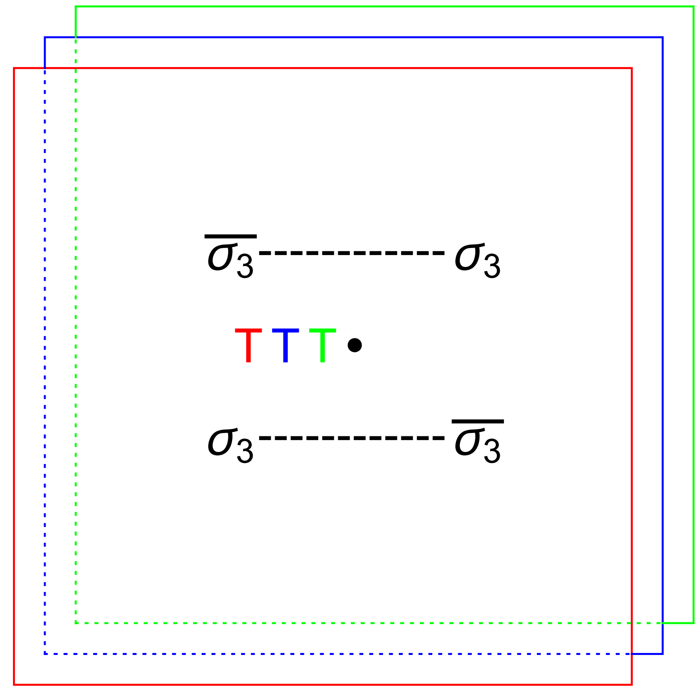


Triplet of Virasoro descendants of identity operator inserted on the three sheets

$$(-)^{\sum_{j=1}^{3}(|R_j|+|\tilde{R}_j|)}\sum_{(h_i,\tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3}\mathbb{F}(h_1,h_2,h_3;\boxed{R_1,R_2,R_3;w}|z)\mathbb{F}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3;\boxed{\tilde{R}_1,\tilde{R}_2,\tilde{R}_3;\bar{w}}|\bar{z})$$

$$=\sum_{(h_i,\tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3}\mathbb{F}(h_1,h_2,h_3;R_1,R_2,R_3;1-w|1-z)\mathbb{F}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3;\tilde{R}_1,\tilde{R}_2,\tilde{R}_3;1-\bar{w}|1-\bar{z}).$$

Genus two crossing equation beyond the Renyi surface



Triplet of Virasoro descendants of identity operator inserted on the three sheets

$$(-)^{\sum_{j=1}^{3}(|R_j|+|\tilde{R}_j|)} \sum_{(h_i,\tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3} \mathbb{F}(h_1,h_2,h_3;\boxed{R_1,R_2,R_3;w}|z)\mathbb{F}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3;\boxed{\tilde{R}_1,\tilde{R}_2,\tilde{R}_3;\bar{w}}|\bar{z})$$

$$= \sum_{(h_i,\tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3} \mathbb{F}(h_1,h_2,h_3;R_1,R_2,R_3;1-w|1-z)\mathbb{F}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3;\tilde{R}_1,\tilde{R}_2,\tilde{R}_3;1-\bar{w}|1-\bar{z}).$$

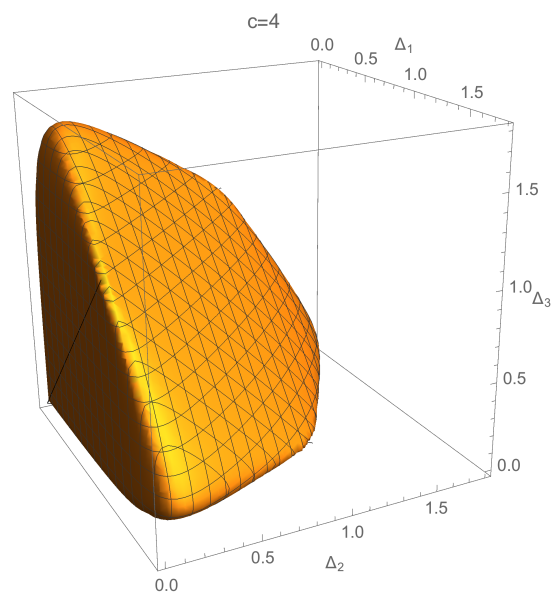
Modified genus two conformal blocks (with insertions of Virasoro descendants of id)

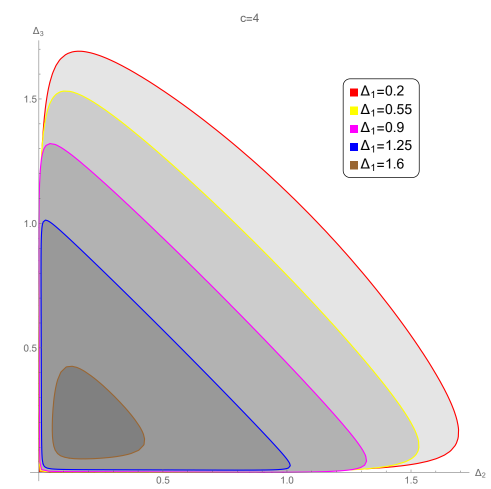
$$\begin{aligned}\mathbb{F}(h_1,h_2,h_3;R_1,R_2,R_3;w|z) = 3^{-3\sum_{i=1}^{3} h_i} &\sum_{\{N_i\},\{M_i\}} z^{-2h_\sigma+\sum_{i=1}^{3}(h_i+|N_i|)} w^{\sum_{k=1}^{3}(|M_k|-|N_k|-|R_k|)} \\ &\times \rho(\mathcal{L}^{\infty}_{-N_3}h_3, \mathcal{L}^{1}_{-N_2}h_2, \mathcal{L}^{0}_{-N_1}h_1)\rho(\mathcal{L}^{\infty *}_{-M_3}h_3, \mathcal{L}^{1*}_{-M_2}h_2, \mathcal{L}^{0*}_{-M_1}h_1) \\ &\times \sum_{|P_i|=|N_i|,\, |Q_i|=|M_i|} \prod_{k=1}^{3} G^{N_k P_k}_{h_k} G^{M_k Q_k}_{h_k} \rho(L_{-Q_k}h_k, L_{-R_k}\mathrm{id}, L_{-P_k}h_k)\end{aligned}$$

Some nontrivial bounds relating structure constants of small and large dimension operators can be derived by simply inspecting the first few orders of the expansion of the genus two modular crossing equation around $z=1/2$.

Some nontrivial bounds relating structure constants of small and large dimension operators can be derived by simply inspecting the first few orders of the expansion of the genus two modular crossing equation around z=1/2.

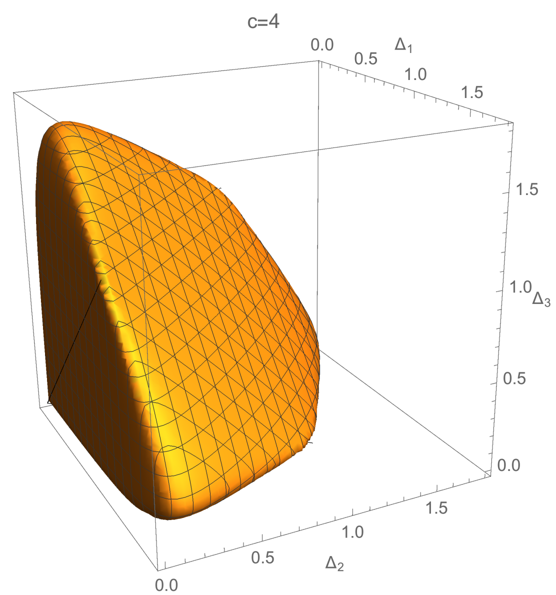
e.g. “critical domain” for structure constants in the space of weights

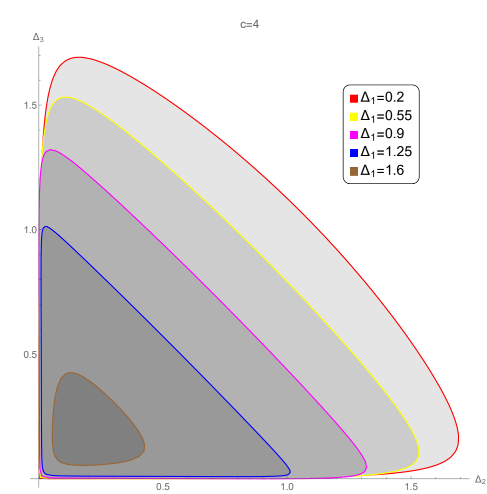




Some nontrivial bounds relating structure constants of small and large dimension operators can be derived by simply inspecting the first few orders of the expansion of the genus two modular crossing equation around z=1/2.

e.g. “critical domain” for structure constants in the space of weights





A systematic investigation of the consequences of the genus two modular crossing equation is yet to be performed.

Summary: we know very little about “generic” 2D CFTs.

Summary: we know very little about “generic” 2D CFTs.

(Embarrassingly so when it comes to CFTs with “nice” holography duals.)

Summary: we know very little about “generic” 2D CFTs.

(Embarrassingly so when it comes to CFTs with “nice” holography duals.)

At least the rule of the game is clear.

Summary: we know very little about “generic” 2D CFTs.

(Embarrassingly so when it comes to CFTs with “nice” holography duals.)

At least the rule of the game is clear.

Lots of work to do for physicists and mathematicians!