Conformal Bootstrap in Two Dimensions

Xi Yin
Harvard University

based on works with
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Minjae Cho (Harvard)
Scott Collier (Harvard)
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Victor Rodriguez (Harvard)
Shu-Heng Shao (IAS)
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Yifan Wang (Princeton)
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- David Simmons-Duffin (IAS & Caltech) [1705.07151]
- Yifan Wang (Princeton) [1705.07151]
1. Motivations and questions

2. Modular constraints

3. Crossing equation and spectral function

4. Comments on superconformal theories

5. Genus two modular bootstrap
Defining properties of Conformal Field Theory in Two Dimensions

[Belavin-Polyakov-Zamolodchikov ’84, Friedan-Shenker ’87, Segal ’87, Moore-Seiberg ’88]
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Non-local operators and boundary states are beyond the scope of this talk.
2D CFTs: the known knowns
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5. Superconformal theories: much known about BPS sector and conformal manifold, not much beyond
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   arise on the worldsheet of superstrings in AdS
   [Bershadsky, Zhukov, Vaintrob ’99, Berkovits, Vafa, Witten ’99, Berkovits ’00, ’04]
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c.f. [Ooguri, Vafa ’16]
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5. To what extent does the low lying operator spectrum of a CFT pin down the entire theory? (Existence and uniqueness of UV completion of gravity+matter in AdS?)
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Unlike the “old” bootstrap which aims to construct exact solutions, [Ferrara, Grillo, Gatto ’73, Polyakov, ’74, Mack, ’77, Belavin, Polyakov, Zamolodchikov, ’84, ....]
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Unlike the “old” bootstrap which aims to construct exact solutions, [Ferrara, Grillo, Gatto ’73, Polyakov, ’74, Mack, ’77, Belavin, Polyakov, Zamolodchikov, ’84, …] the “new” bootstrap aims to rule out theories (and constrain known theories). [Ratazzi, Rychkov, Tonni, Vichi ’08, …many more]
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We would like to know: what are the possible spectra of local operators and structure constants?
Conformal Bootstrap
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crossing invariance
(associativity of OPE)
Conformal Bootstrap

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modular invariance
Conformal Bootstrap

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modular invariance

(any spacetime dimension)
Conformal Bootstrap

crossing invariance  (associativity of OPE)  modular invariance
(any spacetime dimension)  (only properly understood in 2D)
Let us begin with a simple example
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Modular invariance of the torus partition function

\[
Z(\tau, \bar{\tau}) = \text{Tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}.
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The modular constraint on the operator spectrum goes much further!

[Hellerman ’09, Friedan-Keller ’13, Qualls-Shapere ’13, Collier-Lin-XY ’16]
Modular Bootstrap
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The torus partition function admits character decomposition:

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We aim to rule out all spectra $\mathcal{I}$ with some hypothetical properties.
Modular crossing equation

\[ \sum_{h, \tilde{h}} d_{h, \tilde{h}} \left[ \chi_h(\tau) \bar{\chi}_{\tilde{h}}(\tilde{\tau}) - \chi_h(-1/\tau) \bar{\chi}_{\tilde{h}}(-1/\tilde{\tau}) \right] = 0 \]
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Strategy: seek a linear functional

\[ \alpha = \sum_{m+n=\text{odd}} a_{m,n} \left. \partial_z^m \partial_{\bar{z}}^n \right|_{z=\bar{z}=0}, \quad \tau \equiv ie^z \]
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Depending on the hypothesis we choose to make on the spectrum, such a linear functional \( \alpha \) may or may not exist. If \( \alpha \) is found, then the modular crossing equation cannot be satisfied, thereby ruling out the hypothetical spectrum.
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In practice, can work with a basis of linear functionals up to a finite derivative order, truncate on the range of spin (must check stability wrt increasing spin truncation), and use semidefinite programming [SDPB by D. Simmons-Duffin] to optimize the bound numerically.
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY ’16]
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![Diagram showing modular bounds and exclusion regions for various Lie groups.](attachment:image.png)
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(bound exists up to c=25 only)

all-spin bound

large c asymptotics?
numerics: < c/9
Modular bounds on the gap in the spectrum (all-spin Virasoro primaries vs scalar Virasoro primaries) [Collier-Lin-XY ’16]

- Scalar bound: (bound exists up to $c=25$ only)
- All-spin bound

Large $c$ asymptotics?
- Numerics: $< c/9$
- Conjecture motivated by 3D gravity: $c/12 + \text{“less than linear in } c\text{”}$
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The 4-punctured sphere is conformally mapped to the pillow geometry \((T^2/Z_2)\), with the identification of moduli:

\[ \tau = i \frac{K(1-z)}{K(z)}, \quad K(z) = {_2F_1}(\frac{1}{2}, \frac{1}{2}; 1; z) \]

\[ q = e^{\pi i \tau} \]
z-plane \quad \text{q-disc}
It is useful to work with the q-expansion of the Virasoro conformal block, which makes manifest various analyticity and positivity properties, and allows for efficient numerical evaluation.
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[Zamolodchikov ’87, Maldacena-Simmons-Duffin-Zhiboedov ’15]
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[Zamolodchikov ’87, Maldacena-Simmons-Duffin-Zhiboedov ’15]

We perform practical computations using Zamolodchikov’s recurrence relations, in which the Virasoro blocks are expressed in terms of residue contributions from poles in its analytic continuation in the weights or in the central charge.
The Crossing equation
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\[
\sum (h, \tilde{h}) = \sum (h', \tilde{h}')
\]

\[
\sum \phi_1 \phi_2 (h, \tilde{h}) \phi_3 \phi_4 = \sum \phi_1 \phi_2 (h', \tilde{h}') \phi_3 \phi_4
\]
The Crossing equation

\[ \sum (h, \tilde{h}) = \sum (h', \tilde{h}') \]

\[ \sum_{i} C_{12i} C_{34i} F_{12|i|34}(z, \tilde{z}) = \sum_{i} C_{14i} C_{32i} F_{14|i|32}(1 - z, 1 - \tilde{z}) \]
For simplicity, restrict to a pair of primaries
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\[ \sum \phi_1 \phi_2 (h, \bar{h}) \quad = \quad \sum \phi_1 \phi_2 (h', \bar{h}') \]

\[ \sum_i C_{12i}^2 \left[ F_{12|i|12}(z, \bar{z}) - F_{12|i|12}(1 - z, 1 - \bar{z}) \right] = 0 \]
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\[ \sum_{i} C_{12i}^2 \left[ F_{12|1|12}(z, \bar{z}) - F_{12|1|12}(1 - z, 1 - \bar{z}) \right] = 0 \]

In a unitarity CFT, the OPE coefficients are real. We can again exploit the positivity of the coefficients of the conformal block expansion using semidefinite programming.
Example of a bound on the OPE gap:
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Assuming *only scalar* Virasoro primaries in OPE
Example of a bound on the OPE gap:

Assuming **only scalar** Virasoro primaries in OPE

[van Rees, unpublished; Collier, Lin, XY, unpublished]
Can do better than just bounding the gaps and OPE coefficients of the first few operators!
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4-point function $G(z, \bar{z}) = \sum_{i} C_{12i}^{2} F_{12|i|12}(z, \bar{z})$
Can do better than just bounding the gaps and OPE coefficients of the first few operators!

4-point function \[ G(z, \bar{z}) = \sum_i C^2_{12i} F_{12|i|12}(z, \bar{z}) \]

The spectral function \[ f(\Delta_*) = \frac{1}{G(\frac{1}{2}, \frac{1}{2})} \sum_{\Delta_i < \Delta_*} C^2_{12i} F_{12|i|12}(1/2, 1/2) \]
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captures distribution of OPE coefficients in scaling dimension.

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Can bound spectral function using semidefinite programming.

Recall crossing equation, in the schematic form

\[ \sum_{\Delta} C^2_{\Delta} F_{\Delta}^{(m,n)} = 0, \quad F_{\Delta}^{(m,n)} \equiv \partial_z^m \partial_{\bar{z}}^n F_{\Delta}|_{z=\bar{z}=-\frac{1}{2}}, \quad m + n \text{ odd} \]
Now consider the inequality

$$\theta(\Delta_\ast - \Delta) F^{(0,0)}_\Delta - y_{0,0} F^{(0,0)}_\Delta + \sum_{m+n \text{ odd}} y_{m,n} F^{(m,n)}_\Delta \geq 0, \quad \forall \Delta \in \mathcal{I}. \tag{\star}$$
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If (\ast) is obeyed by a set of coefficients $y_{0,0}, y_{m,n}$ ($m + n$ odd)
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**If** $(\ast)$ is obeyed by a set of coefficients $y_{0,0}, \ y_{m,n} \ (m + n \text{ odd})$

then multiply by $C^2_\Delta$, and sum over $\Delta$, the coefficients $y_{m,n}$ drop out by virtue of crossing equation, and we end up with

$$f(\Delta_\ast) \geq y_{0,0}$$
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i.e. \( y_{0,0} \) is a lower bound on the spectral function.
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Optimal lower bound achieved by maximizing $y_{0,0}$ subject to $(\star)$. 
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Likewise, optimal upper bound obtained by minimizing \( y_{0,0} \) subject to

\[ \theta(\Delta_* - \Delta)F_{\Delta}^{(0,0)} - y_{0,0}F_{\Delta}^{(0,0)} + \sum_{m+n \text{ odd}} y_{m,n}F_{\Delta}^{(m,n)} \leq 0, \quad \forall \Delta \in \mathcal{I}. \]
Crossing invariance of sphere 4-point function
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Assuming **only scalar** Virasoro primaries (c>1) [Collier-Kravchuk-Lin-XY ’17]
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Upper and lower bounds on spectral function:

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Upper and lower bounds on spectral function:

![Graph showing crossing invariance and bounds on spectral function](image)

Conjecture: the bounds pin down Liouville CFT
Brief recap of Liouville CFT

[Seiberg ’91, Dorn-Otto ’94, Zamolodchikov, ’95, Teschner ’95, Ponsot-Teschner ’99]
Brief recap of Liouville CFT

\[ S_{\text{Liouville}} = \frac{1}{4\pi} \int d^2 z \sqrt{g} \left( g^{mn} \partial_m \phi \partial_n \phi + Q R \phi + 4\pi \mu e^{2b\phi} \right) \]

\[ c = 1 + 6Q^2 \quad \quad Q = b + b^{-1} \]
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Virasoro primary operators take the form

\[ \mathcal{V}_\alpha \sim S(\alpha)^{-\frac{1}{2}} e^{2\alpha \phi} + S(\alpha)^{\frac{1}{2}} e^{2(Q-\alpha)\phi} \]

\[ \phi \rightarrow -\infty \]
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\[ \alpha = \frac{Q}{2} + iP \]
Brief recap of Liouville CFT

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Reflection coefficient:

\[ S(\alpha) = - (\pi \mu \gamma(b^2))^{(Q-2\alpha)/b} \frac{\Gamma(1 - (Q - 2\alpha)/b)\Gamma(1 - (Q - 2\alpha)b)}{\Gamma(1 + (Q - 2\alpha)/b)\Gamma(1 + (Q - 2\alpha)b)} \]
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DOZZ structure constants:

\[ \langle \mathcal{V}_{\alpha_1} \mathcal{V}_{\alpha_2} \mathcal{V}_{\alpha_3} \rangle = \prod_{j=1}^{3} S(\alpha_i)^{-\frac{1}{2}} \left[ \pi \mu \gamma(b^2)b^{2-2\alpha} \right]^{\frac{Q-\sum \alpha_i}{b}} \times \frac{\Gamma_b'(0) \Gamma_b(2\alpha_1) \Gamma_b(2\alpha_2) \Gamma_b(2\alpha_3)}{\Gamma_b(\sum \alpha_i - Q) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_3) \Gamma_b(\alpha_2 + \alpha_3 - \alpha_1) \Gamma_b(\alpha_3 + \alpha_1 - \alpha_2)} \]
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\]

\[
\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}
\]

\[
\log \Upsilon_b(x) = \int_0^\infty dt \ t^{-1} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left[ \frac{(Q-x) x}{2} \right]}{\sinh \frac{x}{2} \sinh \frac{1}{2b}} \right], \ 0 < \text{Re}(x) < \text{Re}(Q)
\]
Direct numerical solution for the scalar-only spectral function from truncated crossing equation:

[Collier, Kravchuk, Lin, XY, '17]
Direct numerical solution for the scalar-only spectral function from truncated crossing equation:

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\[ f_N(\Delta_s) \]

\[ c=8, \ h_\phi = 7/24 \]

---

**DOZZ**

---

numerical solution from crossing
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\[ \Delta_\ast \]

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(No assumption of unitarity here!)
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\[ f_N(\Delta_*) \quad c=8, \ h_\phi = 7/24 \]

- --- DOZZ
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Examples that demonstrate numerical convergence:
A few words on superconformal theories
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In (2,2) SCFT, marginality implies exactly marginality.
[Dixon ’87, Green, Komargodski, Seiberg, Tachikawa ’10]
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They provide abundant examples of interacting, compact, irrational CFTs (along the conformal manifold).
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Bootstrap method allows us to get a handle on the non-BPS operators in SCFTs, by analyzing e.g. the OPE of BPS operators.
Some technical ingredients
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1. Moduli dependence is fed into the crossing equation through chiral ring relations and/or protected BPS correlators.
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e.g. in superconformal NLSM on K3, can determine integrated half BPS 4-point function

\[
\int \frac{d^2z}{|z(1-z)|} \langle \mathcal{O}_i(z, \bar{z}) \mathcal{O}_j(0) \mathcal{O}_k(1) \mathcal{O}_\ell(\infty) \rangle = \left. \frac{\partial^4}{\partial y^i \partial y^j \partial y^k \partial y^\ell} \right|_{y=0} \int_{\mathcal{F}} d^2 \tau \frac{\Theta_\Lambda(y|\tau, \bar{\tau})}{\eta(\tau)^{24}}
\]

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   e.g. in superconformal NLSM on K3, can determine integrated half BPS 4-point function \([\text{Kiritsis-Obers-Pioline, '00, Lin-Shao-Wang-XY, '15}]\)

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2. N=2 Super-Virasoro conformal blocks known for BPS external operators.
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2. N=2 Super-Virasoro conformal blocks known for BPS external operators.

   determined from BPS correlators in N=2 cigar SCFT, related to bosonic Virasoro blocks in simple ways [Chang, Lin, Shao, Wang, XY, ’14]
Example 1: for **K3 CFT**, bounding the gap of non-BPS primaries in the OPE of a pair of 1/2-BPS operators along the moduli space.
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Gap bound saturated by $A_1$ cigar CFT (c=6, N=4 Liouville)

[Lin, Shao, Wang, Simmons-Duffin, XY ’15]
Example 2: (2,2) SCFTs with marginal deformation

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Upper bound on the scaling dimension of the first non-BPS primary in the OPE of a pair of marginal chiral and anti-chiral primaries
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$q = 1, \; \lambda = 0$
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Upper bound on the scaling dimension of the first non-BPS primary in the OPE of a pair of marginal chiral and anti-chiral primaries

$q = 1, \ \lambda = 0$

bound saturated by products of N=2 minimal models

(Landau-Ginzburg model with two superfields at special point on the conformal manifold)
Combine OPE crossing and modular invariance?
Hard to work with a large set of crossing equations simultaneously (one equation for each set of external primaries)
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TMI: don’t necessarily want to know every single structure constant in the CFT. Rather, want to know about their distributions.
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We will study genus two.
- rich enough to capture OPE and modular invariance.
A few technical challenges
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   Plumbing frame:
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   Recursion relations via analytic continuation in central charge.
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3. Need to handle semidefinite programming on functions of three internal weights

   (Don’t have the computer program to do this efficiently yet.)
General Virasoro conformal blocks
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In the plumbing frame, the large $c$ limit is finite.
General Virasoro conformal blocks

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\[
g_c(h^\text{ext}_a; h_i; q_i) = g_\infty(0; 0; q_j) g_{\text{SL}(2)}(h^\text{ext}_a; h_i; q_i) \\
+ \sum_{j} \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} g_{\text{crs}(h_j)}(h^\text{ext}_a; h_j \rightarrow h_j + rs; q_i)
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General Virasoro conformal blocks

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General Virasoro conformal blocks

In the plumbing frame, the large $c$ limit is finite.

$c = \infty$ vacuum block

$g_c(h_a^\text{ext}; \{h_i\}; \{q_i\}) = g_\infty(0; 0; \{q_j\}) g_{SL(2)}(h_a^\text{ext}; \{h_i\}; \{q_i\})$

$$+ \sum_j \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} g_{c_{rs}(h_j)}(h_a^\text{ext}; h_j \to h_j + rs; \{q_i\})$$

global $SL(2)$ block
General Virasoro conformal blocks

In the plumbing frame, the large c limit is finite.

\[ g_c(h^\text{ext}; h_i; q_i) = g_{\infty}(0; 0; q_j) g_{\text{SL}(2)}(h^\text{ext}; h_i; q_i) \]
\[ + \sum_{j} \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} g_{c_{rs}(h_j)}(h^\text{ext}; h_j \rightarrow h_j + rs; q_i) \]
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\[ g_c(\{ h_a^\text{ext} \}; \{ h_i \}; \{ q_i \}) = g_\infty(0; 0; \{ q_j \}) g_{SL(2)}(\{ h_a^\text{ext} \}; \{ h_i \}; \{ q_i \}) + \sum_j \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} g_{crs}(h_j)(\{ h_a^\text{ext} \}; h_j \rightarrow h_j + rs; \{ q_i \}) \]

(captured by 1-loop partition function of 3D gravity on a handlebody) [Giombi-Maloney-XY '08]

residues at poles are determined recursively
General Virasoro conformal blocks

In the plumbing frame, the large $c$ limit is finite.

\[
g_c(\{h_a^{\text{ext}}\}; \{h_i\}; \{q_i\}) = g_{\infty}(0; 0; \{q_j\})g_{\text{SL}(2)}(\{h_a^{\text{ext}}\}; \{h_i\}; \{q_i\}) + \sum_{j} \sum_{r \geq 2, s \geq 1} \frac{Q_{j}^{r,s}}{c - c_{rs}(h_j)}g_{\text{crs}}(h_j)(\{h_a^{\text{ext}}\}; h_j \rightarrow h_j + rs; \{q_i\})
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[Giombi-Maloney-XY '08]

[Zamolodchikov '84]
General Virasoro conformal blocks

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[Giombi-Maloney-XY '08]

[Zamolodchikov '84]
generalizations by [Hadasz, Jaksålski, Suchanek '09] [Cho, Collier, XY '17]
General Virasoro conformal blocks

\[
G_c(\{h_a^{\text{ext}}\}; \{h_i\}; \{q_i\}) = G_{\infty}(0; 0; \{q_j\}) G_{SL(2)}(\{h_a^{\text{ext}}\}; \{h_i\}; \{q_i\}) \\
+ \sum_j \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} G_{crs(h_j)}(\{h_a^{\text{ext}}\}; h_j \rightarrow h_j + rs; \{q_i\})
\]
General Virasoro conformal blocks

\[ G_c(\{h^\text{ext}_a\}; \{h_i\}; \{q_i\}) = G_\infty(0; 0; \{q_j\}) G_{SL(2)}(\{h^\text{ext}_a\}; \{h_i\}; \{q_i\}) \]

\[ + \sum_{j} \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} G_{crs(h_j)}(\{h^\text{ext}_a\}; h_j \rightarrow h_j + rs; \{q_i\}) \]

Allows for efficient computation of arbitrary Virasoro conformal blocks.
General Virasoro conformal blocks

\[ G_{\infty}(0; 0; \{ q_j \}) G_{SL(2)}(\{ h_a^{ext} \}; \{ h_i \}; \{ q_i \}) \]
\[ + \sum_{j} \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs}(h_j)} G_{c_{rs}}(\{ h_a^{ext} \}; h_j \rightarrow h_j + rs; \{ q_i \}) \]

Allows for efficient computation of arbitrary Virasoro conformal blocks.

A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY ’17].
General Virasoro conformal blocks

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A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY ’17].

OPE channel

Necklace channel

Red: numerical result for genus one reflection amplitude in c=1 string theory from the worldsheet
General Virasoro conformal blocks

\[ \mathcal{G}_c (\{h_a^{\text{ext}}\}; \{h_i\}; \{q_i\}) = \mathcal{G}_\infty (0; 0; \{q_j\}) \mathcal{G}_{SL(2)} (\{h_a^{\text{ext}}\}; \{h_i\}; \{q_i\}) + \sum_j \sum_{r \geq 2, s \geq 1} \frac{Q_j^{r,s}}{c - c_{rs} (h_j)} \mathcal{G}_{crs(h_j)} (\{h_a^{\text{ext}}\}; h_j \rightarrow h_j + rs; \{q_i\}) \]

Allows for efficient computation of arbitrary Virasoro conformal blocks.

A recent application is the evaluation of torus 2-point function in Liouville CFT, and upon moduli integration, the genus one 2-point reflection amplitude in c=1 string theory [Balthazar-Rodriguez-XY '17].

OPE channel

Necklace channel

Red: numerical result for genus one reflection amplitude in c=1 string theory from the worldsheet
Blue: matrix model result
Genus two Renyi surface

To make modular invariance manifest, work in a different conformal frame. A convenience choice is the “Renyi frame”.
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To begin with, consider the $\mathbb{Z}_3$ invariant Renyi surface (a genus two surface that is a 3-fold cover of the Riemann sphere branched at four points):
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The Renyi surfaces occupy a 1 complex dimensional locus of the moduli space of genus two Riemann surfaces.

$$
\Omega = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\frac{i_{2F1}(\frac{2}{3}, \frac{1}{3}, 1|1 - z)}{\sqrt{3} 2F1(\frac{2}{3}, \frac{1}{3}, 1|z)}
$$
To make modular invariance manifest, work in a different conformal frame. A convenience choice is the “Renyi frame”.

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The Renyi surfaces occupy a 1 complex dimensional locus of the moduli space of genus two Riemann surfaces.

$$\Omega = \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right) \frac{i_2 F_1(\frac{2}{3}, \frac{1}{3}, 1|1 - z)}{\sqrt{3} F_2(\frac{2}{3}, \frac{1}{3}, 1|z)}$$

The parameter $z$ is the cross ratio of the four branch points on the sphere.
Genus two crossing

$$\sigma_3 \quad \bar{\sigma}_3$$
Genus two crossing
A nontrivial generator of the genus two modular group $Sp(4,\mathbb{Z})$ is the crossing transformation of the four-point function of $\mathbb{Z}_3$ twist fields.
Genus two conformal block

\[ \langle \sigma_3(0)\bar{\sigma}_3(z, \bar{z})\sigma_3(1)\bar{\sigma}'_3(\infty) \rangle = \sum_{i,j,k} C^2_{i,j,k} F_c(h_i, h_j, h_k; z) \mathcal{F}_c(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k; \bar{z}) \]
Genus two conformal block

\[ \langle \sigma_3(0)\bar{\sigma}_3(z, \bar{z})\sigma_3(1)\bar{\sigma}_3(\infty) \rangle = \sum_{i,j,k} C_{ijk}^2 \mathcal{F}_c(h_i, h_j, h_k; z) \mathcal{F}_c(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k; \bar{z}) \]

\[ \mathcal{F}_c(h_1, h_2, h_3; z) = \exp \left[ c\mathcal{F}^{cl}(z) \right] \mathcal{G}_c(h_1, h_2, h_3; z) \]
Genus two conformal block

\[ \langle \sigma_3(0)\bar{\sigma}_3(z, \bar{z})\sigma_3(1)\bar{\sigma}_3(\infty) \rangle = \sum_{i,j,k} C_{ijk}^2 F_c(h_i, h_j, h_k; z) \bar{F}_c(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k; \bar{z}) \]

\[ F_c(h_1, h_2, h_3; z) = \exp \left[ z \mathcal{F}^{cl}(z) \right] G_c(h_1, h_2, h_3; z) \]
Genus two conformal block

\[
\langle \sigma_3(0) \sigma_3(z, \bar{z}) \sigma_3^\prime(1) \sigma_3^\prime(\infty) \rangle = \sum_{i,j,k} C^2_{ijk} F_c(h_i, h_j, h_k; z) \overline{F}_c(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k; \bar{z})
\]

\[
F_c(h_1, h_2, h_3; z) = \exp \left[ c \mathcal{F}_{cl}(z) \right] G_c(h_1, h_2, h_3; z)
\]

conformal anomaly
plumbing frame block
Genus two conformal block

\[
\langle \sigma_3(0)\overline{\sigma}_3(z, \bar{z})\sigma_3(1)\overline{\sigma}_3(\infty) \rangle = \sum_{i,j,k} C_{ijk}^{3} \mathcal{F}_c(h_i, h_j, h_k; z) \overline{\mathcal{F}}_c(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k; \bar{z})
\]

\[
\mathcal{F}_c(h_1, h_2, h_3; z) = \exp \left[ c\mathcal{F}^{cl}(z) \right] \mathcal{G}_c(h_1, h_2, h_3; z)
\]

\[
\mathcal{F}^{cl}(z) = -\frac{2}{9} \log(z) + 6 \left( \frac{z}{27} \right)^2 + 162 \left( \frac{z}{27} \right)^3 + 3975 \left( \frac{z}{27} \right)^4 + 96552 \left( \frac{z}{27} \right)^5 + 2356039 \left( \frac{z}{27} \right)^6 + \cdots
\]
Genus two conformal block

\[
\langle \sigma_3(0)\sigma_3(z, \bar{z})\sigma_3(1)\sigma_3(\infty) \rangle = \sum_{i,j,k} C_{ijk}^2 F_c(h_i, h_j, h_k; z) \Phi_c(h_i, h_j, h_k; \bar{z})
\]

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F_c(h_1, h_2, h_3; z) = \exp \left[ c F^{cl}(z) \right] G_c(h_1, h_2, h_3; z)
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\]

The infinite $c$ limit of the plumbing frame block for the Renyi surface is

\[
G_\infty(h_1, h_2, h_3; z) = \left( \frac{z}{27} \right)^{h_1+h_2+h_3} \left\{ 1 + \left[ \frac{h_1 + h_2 + h_3}{2} + \frac{(h_2 - h_3)^2}{54h_1} + \frac{(h_3 - h_1)^2}{54h_2} + \frac{(h_1 - h_2)^2}{54h_3} \right] z^{1(\infty)} + O(z^3) \right\}
\]
The infinite $c$ limit of the plumbing frame block for the Renyi surface is

$$G_{\infty}(h_1, h_2, h_3|z) = \left( \frac{z}{27} \right)^{h_1+h_2+h_3} \left\{ 1 + \left[ \frac{h_1+h_2+h_3}{2} + \frac{(h_2-h_3)^2}{54h_1} + \frac{(h_3-h_1)^2}{54h_2} + \frac{(h_1-h_2)^2}{54h_3} \right] z + O(z^3) \right\}$$

(Finite $c$ result can be recovered by recursion formula.)
Genus two crossing equation beyond the Renyi surface
Genus two crossing equation beyond the Renyi surface
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\[
(-)^{\sum_{j=1}^{3}(|R_{j}|+|\tilde{R}_{j}|)} \sum_{(h_{i},\tilde{h}_{i})} C_{h_{1},h_{2},h_{3};h_{1},h_{2},h_{3}}^{2} F(h_{1}, h_{2}, h_{3}; R_{1}, R_{2}, R_{3}; w|z)F(\tilde{h}_{1}, \tilde{h}_{2}, \tilde{h}_{3}; \tilde{R}_{1}, \tilde{R}_{2}, \tilde{R}_{3}; \tilde{w}|\tilde{z})
\]

\[
= \sum_{(h_{i},\tilde{h}_{i})} C_{h_{1},h_{2},h_{3};h_{1},h_{2},h_{3}}^{2} F(h_{1}, h_{2}, h_{3}; R_{1}, R_{2}, R_{3}; 1-w|1-z)F(\tilde{h}_{1}, \tilde{h}_{2}, \tilde{h}_{3}; \tilde{R}_{1}, \tilde{R}_{2}, \tilde{R}_{3}; 1-\tilde{w}|1-\tilde{z}).
\]
Genus two crossing equation beyond the Renyi surface

\[ (-1)^\sum_{j=1}^3(|R_j|+|\tilde{R}_j|) \sum_{(h_i, \tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3} \mathcal{F}(h_1, h_2, h_3; R_1, R_2, R_3; w \, z) \mathcal{F}(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3; \tilde{R}_1, \tilde{R}_2, \tilde{R}_3; \tilde{w} \, \tilde{z}) \]

\[ = \sum_{(h_i, \tilde{h}_i)} C^2_{h_1,h_2,h_3;\tilde{h}_1,\tilde{h}_2,\tilde{h}_3} \mathcal{F}(h_1, h_2, h_3; R_1, R_2, R_3; 1-w|1-z) \mathcal{F}(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3; \tilde{R}_1, \tilde{R}_2, \tilde{R}_3; 1-\tilde{w}|1-\tilde{z}). \]
Genus two crossing equation beyond the Renyi surface

\[ (-) \sum_{j=1}^{3} (|R_{j}| + |\bar{R}_{j}|) \sum_{(h_{i}, \bar{h}_{i})} C^{2}_{h_{1}, h_{2}, h_{3}, \bar{h}_{1}, \bar{h}_{2}, \bar{h}_{3}} \mathcal{F}(h_{1}, h_{2}, h_{3}; R_{1}, R_{2}, R_{3}; w|z) \mathcal{F}(\bar{h}_{1}, \bar{h}_{2}, \bar{h}_{3}; \bar{R}_{1}, \bar{R}_{2}, \bar{R}_{3}; \bar{w}|\bar{z}) \]

\[ = \sum_{(h_{i}, \bar{h}_{i})} C^{2}_{h_{1}, h_{2}, h_{3}, \bar{h}_{1}, \bar{h}_{2}, \bar{h}_{3}} \mathcal{F}(h_{1}, h_{2}, h_{3}; R_{1}, R_{2}, R_{3}; 1 - w|1 - z) \mathcal{F}(\bar{h}_{1}, \bar{h}_{2}, \bar{h}_{3}; \bar{R}_{1}, \bar{R}_{2}, \bar{R}_{3}; 1 - \bar{w}|1 - \bar{z}). \]

Modified genus two conformal blocks (with insertions of Virasoro descendants of id)
Some nontrivial bounds relating structure constants of small and large dimension operators can be derived by simply inspecting the first few orders of the expansion of the genus two modular crossing equation around $z=1/2$. 
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e.g. “critical domain” for structure constants in the space of weights
A systematic investigation of the consequences of the genus two modular crossing equation is yet to be performed.

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At least the rule of the game is clear.

Lots of work to do for physicists and mathematicians!